AUTOMORPHIC FORMS AND QUASIHOMOGENEOUS SINGULARITIES

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In this paper we announce several results related to a variant and a generalization of a construction for normal singularities with a $\mathbf{C}^*$-action (see [3]).

1. Definition. Let U be a homogeneous complex manifold, Γ a group of analytic automorphisms of the manifold U, and L a one-dimensional vector (or line) Γ-bundle on U (see [8]). An automorphic form with weight k with respect to L is a cross section $\varphi \in H^0(U, L^{\otimes k})^\Gamma$ of k-th-order tensors for the Γ-bundle L which is invariant with respect to the natural action of Γ. The graded $\mathbf{C}$-algebra $A(L) = \bigoplus_{k \in \mathbf{Z}} H^0(U, L^{\otimes k})^\Gamma$ will be called the algebra of automorphic forms with respect to L.

THEOREM 1. We shall assume that the triple (U, Γ, L) is admissible, i.e., that the following assumptions hold:

A1. There is a normal subgroup of finite index $\Gamma' \subset \Gamma$ which acts freely and discretely on U.

A2. The factor space U/Γ is a compact analytic space.

A3. For some subgroup $\Gamma' \subset \Gamma$ satisfying A1, the factor L/Γ' determines a positive (in the sense of Kodaira) line bundle over the manifold U/Γ'.

Under these assumptions, the algebra of automorphic forms A(L) is a normal C-algebra of finite type and dimension dim U + 1, with nonnegative grading.

2. Definition. The affine algebraic manifold X over the algebraically closed field k is said to be a quasi-cone if the one-dimensional algebraic torus G_m acts effectively on X and there is a unique point $x_0 \in X$ which belongs to the closure of every orbit. The point x_0 is called the vertex of the quasi-cone X.

PROPOSITION. Let X be an affine algebraic manifold over k. The following are equivalent:

1) X is a quasi-cone;

2) the coordinate ring k[X] has nonnegative grading and $k[X]_0 \simeq k$;

3) there is a closed inclusion j: $X \to k^n$ such that j(X) is invariant with respect to the action of G_m on k^n where the action is defined by the formula $(x_1, \ldots, x_n) \to (x_1 t^{q_1}, \ldots, x_n t^{q_n})$, with $t \in G_m(k)$ and $q_1, \ldots, q_n$ being positive integers;

4) there is an inclusion j: $X \to k^n$ such that the ideal giving j(X) is generated by weighted-homogeneous polynomials with positive rational weights [4].

The proof of this proposition is based on standard arguments about the actions of algebraic tori on affine manifolds (cf. [9]).

THEOREM 2. Let (U, Γ, L) be an admissible triple. Then the affine algebraic manifold Spec A(L) is a normal quasi-cone with vertex x_0 defined by the maximal ideal $A(L)_+ = \bigoplus_{i>0} A(L)_i$. Conversely, each normal two-dimensional quasi-cone is isomorphic to the manifold Spec A(L) for some admissible triple (U, Γ, L).

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TABLE 1

Q_{16}	(3, 3, 9)	2	E_{18}	(3, 3, 5)	2
Q_{17}	(2, 4, 13)	3	E_{19}	(2, 4, 7)	3
Q_{18}	(2, 3, 21)	5	E_{20}	(2, 3, 11)	5
Z^1_{17}	(3, 3, 7)	2	U_{16}	(5, 5, 5)	2
Z^1_{18}	(2, 4, 10)	3	$W_{1,0}$	(2, 2, 3, 3)	1
			$J_{3,0}$	(2, 2, 2, 3)	1
Z^1_{19}	(2, 3, 16)	5	$Q_{2,0}$	(2, 2, 2, 5)	1
S_{16}	(3, 5, 7)	2	$S_{1,0}$	(2, 2, 3, 4)	1
S_{17}	(2, 7, 10)	3	$U_{1,0}$	(2, 3, 3, 3)	1
W_{17}	(3, 5, 5)	2	$Z^1_{1,0}$	(2, 2, 2,4)	1
W_{18}	(2, 7, 7)	3			

While the first part of this theorem follows immediately from the preceding results, the proof of the second part is very specailized and uses the idea of a "singular Seifert bundle" from [9].

Definition. A singularity in this article is the jet (Y, y) of the analytic space Y at the point y. A singularity will be called a normal singularity if Y is normal at the point y. Isomorphism of singularities means an analytic isomorphism between the corresponding jets. A singularity is called quasi-homogeneous if it is isomorphic to the jet of some quasi-cone at its vertex.

COROLLARY. Each admissible triple (U, Γ, L) determines a normal quasi-homogeneous singularity S(L). Each normal two-dimensional quasi-homogeneous singularity is isomorphic to a singularity of the form S(L).

3. Examples. 1) Let G be a finite subgroup of the group LG(n + 1, C), Γ its image under the canonical homomorphism $\varphi\colon GL(n+1, C) \to PL(n, C)$, m the order of the subgroup G ∩ Ker φ. The bundle $L = H^{\otimes m}$, where H corresponds to the hyperplane cross section of $P^n(C)$, is a Γ-bundle with respect to the natural action of Γ on $P^n(C)$. The triple ($P^n(C)$, Γ, L) is admissible, and the corresponding singularity S(L) is isomorphic to the factor-singularity (C^{n+1}/G, 0), where 0 is the image of the coordinate origin.

When n = 1 and G ⊂ SL(2, C) the singularity obtained in this way is a Klein singularity (in other terms it is a double rational singularity, a platonic singularity, a singularity of type A, D, E; see [4], §9).

2) Let U be a bounded homogeneous region in C^n, Γ a discrete group of analytic automorphisms of U with compact factor U/Γ. Each Γ-bundle over U is given by the trivial bundle U × C with Γ-action (z, α) → (g(z), h(g; z) α), specified by the automorphicity factor $h \in Z^1(\Gamma, \mathcal{O}(U)^*)$. In particular, the automorphicity factor is defined as $h = J^{-1}$, where J(g; z) is the Jacobian of g ∈ Γ at the point z. The well-known results of Borel [5] and Kodaira [7] show that the triple (U, Γ, J) is admissible. The quasi-homogeneous singularity associated with it is called canonical and is denoted by S(Γ).

In particular, let $U = \{z \in C \mid |z| < 1\}$, Γ the Fuchsian group of the first kind with signature (0, m; n_1, . . ., n_m). When m = 3 the singularities S(Γ) were called canonical triangular singularities in [3], and in that article there were listed those singularities which occurred in C^3 (the 14 unimodular singularities of Arnol'd). If r is a positive integer relatively prime to each of the n_i, then there is not more than one automorphicity factor h with $h^r = J^{-1}$. When such a factor exists (the appropriate conditions can be obtained through explicit calculation of the group of the cohomologies $H^2(\Gamma, Z)$; see [6]), we denote the singularity corresponding to the triple (U, Γ, h) by S(Γ, r). The sets (n_1, . . ., n_m) and factors r for the level surfaces of the bimodal critical points of Arnol'd are given in Table 1 (notation from [2]).

3) Arnol'd's [1] parabolic two-dimensional singularities can be obtained from the appropriate automorphicity factor for the lattice Γ in C.

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