K3 SURFACES OF DEGREE SIX ARISING FROM DESMIC TETRAHEDRA

ALEX DEGTYAREV, IGOR DOLGACHEV, AND SHIGEYUKI KONDŌ

ABSTRACT. We study K3 surfaces of degree 6 containing two sets of 12 skew lines such that each line from a set intersects exactly six lines from the other set. These surfaces arise as hyperplane sections of the cubic line complex associated with the pencil of desmic quartic surfaces introduced by George Humbert and recently studied by the second and third authors. We discuss alternative birational models of the surfaces, compute the Picard lattice and a group of projective automorphisms, and describe rational curves of low degree on the general surface.

1. INTRODUCTION

Three tetrahedra in projective space $\mathbb{P}^3$ are called *desmic* if any two of them are perspective with respect to any vertex of the third one. Equivalently, three tetrahedra are desmic if they can be included in a pencil of quartic surfaces, called a desmic pencil. An irreducible member of a desmic pencil is called a *desmic quartic surface*. It contains 16 lines, the base locus of the pencil, and 12 nodes lying by pairs in the edges of any desmic tetrahedron. Each line passes through 3 nodes, and each node is contained in four lines. A desmic pencil defines an associated desmic pencil such that the original twelve nodes are the vertices of the new desmic tetrahedra. George Humbert, in his study of desmic quartic surfaces [7], showed that the set of lines contained in a quadric passing through the vertices of any two of the desmic tetrahedra in the associated desmic pencil is a cubic line complex $\mathfrak{G}$ in the Grassmannian $G_1(\mathbb{P}^3)$ of lines in $\mathbb{P}^3$; it does not depend on the choice of the pair of tetrahedra used. The complex $\mathfrak{G}$ contains 24 planes, twelve from each family of planes in the Plücker embedding of $G_1(\mathbb{P}^3)$ in $\mathbb{P}^5$.

The subject of the present paper is a transversal hyperplane section X of $\mathfrak{G}$, which is a smooth K3 surface equal to the complete intersection of a quadric and a cubic hypersurface in $\mathbb{P}^4$. In this paper, we call X a Humbert sextic K3 surface (this should not be confused with the Humbert surfaces from the theory of abelian surfaces with special properties of their endomorphism ring, see [13, Chap. IX]). It contains two sets of twelve lines whose incidence relation is an abstract configuration $(12_6, 12_6)$. It will be shown elsewhere by the first author that other sextic K3 surfaces cannot contain such a configuration of lines.

2010 *Mathematics Subject Classification.* 14J28.

Key words and phrases. K3 surfaces, desmic tetrahedra, cubic line complex.

The first author was partially supported by the T'UBİTAK grant 123F111.

Research of the third author is partially supported by JSPS Grant-in-Aid for Scientific Research (A) No.20H00112, (B) No.25K00906.

Projecting from any line, we find a birational model of X isomorphic to a double cover of $\mathbb{P}^2$ branched along a plane sextic B with nodes at the vertices of a complete quadrilateral P, see §3. The curve B is contact (i.e., has an even intersection index at each intersection point, which are all distinct from the nodes) to the diagonals of P, a nodal plane cubic, and six lines in general linear position. Although the ten tangency conditions seem to impose too many constraints on the existence of the curve B, of which we have a 5-dimensional family, in §6. We prove that, in fact, the conditions of the tangency to the diagonals and one line almost imply (modulo some finite combinatorial choices) all other conditions. This was made possible by the computation of the Picard lattice of a general Humbert sextic X: it turns out to be freely generated by 15 lines, see §4, Theorem 4.6.

Among other things dealt with in the paper is a description of smooth rational curves of degree at most 4 (see §2.3 and §7.1) and elliptic pencils (see §7.2) on a very general Humbert sextic K3 surface X. We also discuss the groups of projective and birational automorphisms of X, see §5. Most notably, we show that any projective automorphism of X is induced from one of the Humbert line complex $\mathfrak{G}$; in view of [5], this gives us a complete description of such automorphisms (Theorem 5.2).

We work over the field of complex numbers, however, many of our results are valid assuming only that the ground field is an algebraically closed field of characteristic other than 2 or 3.

2. Desmic pencils and the Humbert cubic line complex

In this section, we introduce Humbert sextic K3 surfaces and discuss the configuration $(12_6, 12_6)$ of lines on them.

2.1. The line complex. Consider three tetrahedra in $\mathbb{P}^3$:

$$\begin{aligned} T_1 &: (x^2-y^2)(z^2-w^2)=0,\\ T_2 &: (x^2-z^2)(y^2-w^2)=0,\\ T_3 &: (w^2-x^2)(z^2-w^2)=0. \end{aligned} \tag{2.1}$$

They are desmic, i.e., belong to the same pencil

$$aT_1+bT_2+cT_3=0,\quad a+b+c=0.$$

Any other member of the pencil is a *desmic quartic surface* with 12 nodes given in (2.2), lying by pairs on the edges of each of the tetrahedra (see [5]). It also contains 16 lines lying by four on each face of each tetrahedron and intersecting the edges at the singular points. This forms a configuration $(12_4, 16_3)$ of nodes and lines.

Consider the following twelve points P_i in $\mathbb{P}^3$:

$$\begin{array}{llll} 1\colon [0,0,0,1], & 2\colon [0,0,1,0], & 3\colon [0,1,0,0], & 4\colon [1,0,0,0],\\ 5\colon [1,1,1,1], & 6\colon [1,-1,1,-1], & 7\colon [1,1,-1,-1], & 8\colon [1,-1,-1,1],\\ 9\colon [1,-1,1,1], & 10\colon [1,1,-1,1], & 11\colon [1,1,1,-1], & 12\colon [-1,1,1,1]. \end{array} \tag{2.2}$$

They lie in pairs on the edges of any of the desmic tetrahedra with the faces Π_j:

$$\begin{array}{llll} \scriptstyle 1:\ x+y=0, & \scriptstyle 2:\ x-y=0, & \scriptstyle 3:\ z+w=0, & \scriptstyle 4:\ z-w=0, \\ \scriptstyle 5:\ y-w=0, & \scriptstyle 6:\ x-z=0, & \scriptstyle 7:\ y+w=0, & \scriptstyle 8:\ x+z=0, \\ \scriptstyle 9:\ y+z=0, & \scriptstyle 10:\ x-w=0, & \scriptstyle 11:\ x+w=0, & \scriptstyle 12:\ y-z=0. \end{array} \tag{2.3}$$

Each of the twelve points lies in six planes. Each plane contains six points.

Each pair of points in the same row of (2.2) lies in two planes from (2.3), and each pair of planes from the same row of (2.3) contains two common points from (2.2). The precise incidence relations are illustrated in Table 1.

TABLE 1. The incidence relation between $12 + 12$ lines

		$x+y$	$x-y$	$z+w$	$z-w$	$y-w$	$x-z$	$y+w$	$x+z$	$y+z$	$x-w$	$x+w$	$y-z$
		1*	2*	3	4	5	6*	7	8*	9*	10	11	12*
$[0,0,0,1]$	1*	●	●				●		●	●			●
$[0,0,1,0]$	2	●	●			●		●			●	●	
$[0,1,0,0]$	3			●	●		●		●		●	●	
$[1,0,0,0]$	4			●	●	●		●		●			●
$[1,1,1,1]$	5		●		●	●	●				●		●
$[1,-1,1,-1]$	6	●		●		●	●			●		●	
$[1,1,-1,-1]$	7		●		●			●	●	●		●	
$[1,-1,-1,1]$	8	●		●				●	●		●		●
$[1,-1,1,1]$	9	●			●		●	●		●	●		
$[1,1,-1,1]$	10		●	●		●			●	●	●		
$[1,1,1,-1]$	11		●	●			●	●				●	●
$[-1,1,1,1]$	12	●			●	●			●			●	●

George Humbert [7] constructed a line complex of degree 3, i.e., a hypersurface $\mathfrak{G}$ in the Grassmannian $G_1(\mathbb{P}^3) \subset \mathbb{P}^5$ cut out by a cubic hypersurface in $\mathbb{P}^5$ whose rays are lines in quadrics passing through the eight points forming any two rows in (2.2). It contains

$$\begin{array}{l} 12\ \alpha\text{-planes of rays through each of the points } P_1, \ldots, P_{12} \text{ from (2.2)}, \\ 12\ \beta\text{-planes of rays lying in each plane } \Pi_1, \ldots, \Pi_{12} \text{ from (2.3)}. \end{array} \tag{2.4}$$

The cubic complex $\mathfrak{G}$ has 34 isolated singular points, 16 of which are lines on a desmic quartic surface, and 18 are the edges of all three tetrahedra.

Definition 2.5. A *Humbert sextic K3 surface* X is a transversal hyperplane section of $\mathfrak{G}$.

Remark 2.6. Thus, we assume that X is cut off $\mathfrak{G}$ by a hyperplane R transversal to $\mathfrak{G}$, but not necessarily to $\mathcal{Q} = G_1(\mathbb{P}^3)$. Note though that, even if R is tangent to $\mathcal{Q}$, none of the 24 planes (2.4) lies in R, as otherwise $\mathfrak{G}$ would contain the tangency point and R would not be transversal to $\mathfrak{G}$.

Convention 2.7. In view of Remark 2.6, the hyperplane R cuts each α-plane corresponding to a point P_i and each β-plane corresponding to a plane Π_i see (2.4), along a line. If $P_i \in \Pi_j$, the corresponding α-plane and β-plane intersect along the line consisting of rays passing through P_i and lying on Π_j. This gives us twelve *α-lines* $L_1, \ldots, L_{12}$ and twelve *β-lines* $M_1, \ldots, M_{12}$ on X forming a configuration $(12_6, 12_6)$; we use the same numbering for the lines as that for the planes, so that their incidence matrix is also given by Table 1.

In the table, each of the 12-tuples α, β is divided into three groups of four, called *quartets*:

$$\alpha = \alpha_1 \cup \alpha_2 \cup \alpha_3, \quad \beta = \beta_1 \cup \beta_2 \cup \beta_3.$$

From now on, we reserve H for the divisor class of a hyperplane section of X, and we identify the α-lines $L_1, \ldots, L_{12} \in \alpha$ and β-lines $M_1, \ldots, M_{12} \in \beta$ (as well as other irreducible curves C with $C^2 < 0$) with their divisor classes.

2.2. **Equation.** One can replace the Plücker coordinates in the Plücker embedding $G_1(\mathbb{P}^3) \hookrightarrow \mathbb{P}^5$ with the Klein coordinates to get the following explicit equation of the Humbert cubic line complex (see [5, the equation (12)]):

$$\begin{aligned} Q &:= x_1^2 + x_2^2 + x_3^2 + y_1^2 + y_2^2 + y_3^2 = 0, \\ F &:= x_1x_2x_3 + \sqrt{-1}y_1y_2y_3 = 0. \end{aligned} \tag{2.8}$$

So, the equation of the Humbert sextic K3 surface in $\mathbb{P}^5$ is obtained by adding an extra equation

$$R = a_1x_1 + a_2x_2 + a_3x_3 + a_4y_1 + a_5y_2 + a_6y_6 = 0. \tag{2.9}$$

The condition that the surface is singular defines a closed subset of $\mathbb{P}^4$ (for example, containing the intersection of two coordinate hyperplanes $a_i = a_j = 0$).

The set $\mathcal{P}$ of the 24 planes is naturally indexed by the elements of the symmetric group $\mathfrak{S}_4 \cong (\mathbb{Z}/2\mathbb{Z})^2 \rtimes \mathfrak{S}_3$: interpreting $(\mathbb{Z}/2\mathbb{Z})^2$ as $\{\pm\sqrt{-1}\}^3/\epsilon_1\epsilon_2\epsilon_3$, the plane corresponding to $\sigma \in \mathfrak{S}_3$ and $(\epsilon_1, \epsilon_2, \epsilon_3) \in (\mathbb{Z}/2\mathbb{Z})^2$ is

$$V(x_1 - \epsilon_1 y_{\sigma(1)}, x_2 - \epsilon_2 y_{\sigma(2)}, x_3 - \epsilon_3 y_{\sigma(3)}). \tag{2.10}$$

The α- and β-planes differ by the sign of σ.

Note that equation (2.8) is invariant with respect to the natural action of the group $\mathfrak{S}_4 \times \mathfrak{S}_4 \cong (\mathbb{Z}/2\mathbb{Z})^4 \rtimes (\mathfrak{S}_3 \times \mathfrak{S}_3)$ (and it is this action that defines the group structure on the set theoretic Cartesian product of $(\mathbb{Z}/2\mathbb{Z})^2$ and $\mathfrak{S}_3$ above). It is also invariant with respect to the transformation g_0 of order 4 defined by

$$(x_1, x_2, x_3, y_1, y_2, y_3) \mapsto (-y_1, -y_2, -y_3, x_1, x_2, x_3).$$

It is proved in [5] that the group $G = (\mathfrak{S}_4 \times \mathfrak{S}_4) \rtimes \mathbb{Z}/2\mathbb{Z}$ of order 1152 is the full group of projective automorphisms of the cubic line complex $\mathfrak{G}$.

2.3. **Lines, conics, quartics.** The K3 surface X is a transversal intersection of a quadric $\mathcal{Q} \cap$ (hyperplane) and a cubic, where $\mathcal{Q} = V(Q)$. A hyperplane section of X is a curve of bidegree $(3, 3)$ on a quadric. It follows from Table 1 that there are $3\binom{4}{2} = 18$ hyperplane sections that contains a *quadrangle* of lines, two from each ruling of the quadric, and an irreducible residual conic. Each of these quadrangles, called *proper*, is determined by its pair of α-lines (or pair of β-lines) and can be characterised by the property that its α-lines (resp. β-lines) are in the same quartet.

Each line belongs to 3 proper quadrangles, indexed by the quartets in the opposite family. Thus, the 24 lines and 18 proper quadrangles form an abstract configuration $(24_3, 18_4)$. Each proper quadrangle determines a pair α_r, β_s of quartets, i.e., a (4×4)-cell in Table 1. Each cell is the union of two proper quadrangles.

There are 16 hyperplane sections that contain a union of 6 lines, three from each family α, β. Its dual graph is a complete bipartite graph $K(3,3)$. For this reason, we call such unions of lines $(3,3)$-*configurations* of lines, or just $(3,3)$-*fragments* (of the full configuration of lines of a given K3 sextic). The α- (resp. β-) lines of each $(3,3)$-configuration are in a bijection with the α- (resp. β-) quartets. The 16 $(3,3)$-configurations and 24 lines form an abstract configuration $(16_6, 24_4)$. It follows that $16H \sim 4\sum_{i=1}^{12}(L_i + M_i)$, hence,

$$\sum_{i=1}^{12}(L_i + M_i) \sim 4H. \tag{2.11}$$

Each $(3,3)$-fragment consists of nine *improper* quadrangles (with all $144 = 16 \times 9$ quadrangles pairwise distinct), which are characterised by the property that their α-lines (resp. β-lines) are in distinct quartets. In more detail, $(3,3)$-configurations on smooth sextic K3 surfaces are discussed in Remark 2.16 below.

For each quadrangle q, we denote by $\sum q$ the sum of the four lines in q. Thus, each quadrangle q gives rise to a residual conic $C_q \in |H - \sum q|$; the latter is irreducible if and only if q is proper. If the α-lines (hence, also β-lines) of two proper quadrangles q_1, q_2 constitute a whole quartet, q_1 and q_2 are disjoint and, hence, constitute two singular fibers of a common elliptic pencil. It follows that $\sum q_1 = \sum q_2$ and $C_1 = C_2$. This common conic is the intersection $\mathcal{Q} \cap H_1 \cap H_2$, where $H_i \subset \mathbb{P}^4$ is the hyperplane spanned by q_i.

Observation 2.12. We conclude that the conics on X (at least those accounted for so far) can be indexed by

1. pairs L_i, L_j of α-lines that are in the same quartet, or
2. pairs M_i, M_j of β-lines that are in the same quartet, or
3. pairs (α_r, β_s) of quartets, i.e., the nine (4×4)-cells in Table 1.

The first two indexing schemes are two-to-one, so we end up with the 9 conics C_{rs}, $1 \le r, s \le 3$, rather than the expected 18.

Indexed as in (3), one has $C_{rs} \cdot L_i = 1$ if and only if $L_i \in \alpha_r$ and $C_{rs} \cdot M_j = 1$ if and only if $M_j \in \beta_s$; otherwise, $C_{rs} \cdot N = 0$ for a line N. Besides, $C_{rs} \cdot C_{uv} = 2$ if $r \ne u$, $s \ne v$ or 0 otherwise. For this reason, (3) is the preferred indexing scheme.

Remark 2.13. One can see the 9 conics and 18 proper quadrangles from (2.8).

The cubic hypersurface $V(F)$ contains nine three-dimensional subspaces $V(x_i, y_j)$, each intersecting the quadric $\mathcal{Q}$ along a quadric surface. Intersecting the latter by $V(R)$, we obtain nine conics C_{ij} contained in X. Each C_{ij} is contained in two hyperplanes $H_{ij}^{\pm} = V(x_i \pm iy_j)$. The hyperplane $H_{ij}^{\pm}$ cuts X along a quadrangle cut out by the planes (2.10) with $\sigma(i) = j$. This gives us the 18 proper quadrangles of lines.

Observation 2.14. If $L_i \in \alpha$ and $M_j \in \beta$ are skew lines from distinct families, the residual curve $Q_{ij} \in |H - L_i - M_j|$ is a smooth rational quartic curve. This gives us $6 \times 12 = 72$ quartics, all distinct. The lines L_i, M_j are singled out via $Q_{ij} \cdot L_i = Q_{ij} \cdot M_j = 3$ whereas $Q_{ij} \cdot N \in \{0, 1\}$ for any other line N.

In §7.1 below we assert that, apart from the 24 lines, 9 conics, and 72 quartics described in this section, a general Humbert sextic has no smooth rational curves of degree up to 4. There are infinitely many other smooth rational curves, see §5.

Observation 2.15. We use the `digraph` package in `GAP` [6] to compute the group $G = \mathrm{Sym}(\Gamma)$ of symmetries of the dual adjacency graph Γ of lines on X: one has $|G| = 1152 = 16 \times 72$, and the group is generated by the involution $L_i \leftrightarrow M_i$, $i = 1, \ldots, 12$, and two permutations

$$L_i \mapsto L_{\sigma_1(i)}, \quad M_i \mapsto M_{\sigma(i)}, \quad \sigma = (1, 9, 4, 12)(2, 10, 3, 11)(5, 8, 6, 7),$$

and

$$L_i \mapsto L_{\sigma(i)}, \quad \sigma = (1, 10, 8, 2, 11, 6)(3, 9, 7)(4, 12, 5),$$
$$M_i \mapsto M_{\sigma(i)}, \quad \sigma = (1, 2, 3)(5, 12)(6, 9, 8, 10, 7, 11).$$

The group acts transitively on the set of the $(3, 3)$-fragments, and the stabilizer of a $(3, 3)$-fragment q is isomorphic to the full group $\mathrm{Sym}(q) = (\mathfrak{S}_3 \times \mathfrak{S}_3) \rtimes \mathbb{Z}/2$. In particular, G is transitive. Alternatively, G induces the full automorphism group $(\mathfrak{S}_3 \times \mathfrak{S}_3) \rtimes \mathbb{Z}/2$ on the set of the (4×4)-cells in Table 1. As an abstract graph, each cell c is the disjoint union of two quadrangles and the stabilizer of c in G maps two-to-one onto the index 2 subgroup of $\mathrm{Sym}(c) = (\mathfrak{D}_8 \times \mathfrak{D}_8) \rtimes \mathbb{Z}/2$ that is not mixing α- and β-lines.

A posteriori one can easily verify that $\mathrm{Sym}(\Gamma)$ is indeed as claimed: the three permutations indicated do belong to $\mathrm{Sym}(\Gamma)$ and, given Table 1, it is immediate that any $g \in \mathrm{Sym}(\Gamma)$ fixing pointwise a certain $(3, 3)$-fragment q is the identity.

Remark 2.16. The configuration of lines on any smooth sextic K3 surface $X \subset \mathbb{P}^4$ (not necessarily the one considered in this paper) has the following $(3, 3)$-*property*: given five distinct lines A_1, A_2, A_3 and B_1, B_2, such that $A_i \cdot B_k = 1$ for all i, k, there is a unique sixth line B_3 such that $q = \{A_1, \ldots, B_3\}$ constitute a $(3, 3)$-configuration. Furthermore, any line on X that is not in q intersects exactly one line in q. Arithmetically, B_3 is found from

$$A_1 + \cdots + B_3 = H \quad \text{in} \quad \mathrm{Pic}(X).$$

Geometrically, once the residual conic C above splits, its two components are in the two distinct rulings of the quadric. Note that we do not even need to assume beforehand that the A-lines or B-lines are pairwise disjoint. Should there be $n > 0$ extra intersection points, the class

$$e = H - (A_1 + A_2 + A_3 + B_1 + B_2) \in \mathrm{Pic}(X)$$

would have $e^2 = 2n - 2$ and $e \cdot H = 1$. If $n = 1$, then, since H is ample, e must be the class of an irreducible curve of arithmetic genus one, and $|H|$ restricted to the curve has a base point, contradicting [11, Theorem 3.1]. If $n \geq 2$, the sublattice $\mathbb{Z}H + \mathbb{Z}e$ is positive definite, contradicting the Hodge index theorem.

In particular, it follows also that two lines B_1, B_2 cannot meet more than three common lines A_i. A similar argument shows that two *intersecting* lines B_1, B_2 can meet at most one common line.

3. Double plane model

Let us consider the projection

$$f\colon X \to \mathbb{P}^2$$

of X with center at some α-line or β-line, say $L = L_1$. It is given by the linear system $|H - L|$. Its restriction to L is a hyperplane in $|\mathcal{O}_L(3)| \cong \mathbb{P}^3$, which has no base points; hence, f is a regular map.

We split β into the complementary subsets

$$\beta_1^* = \{M_1, M_2, M_6, M_8, M_9, M_{12}\}, \quad \bar{\beta}_1^* = \{M_3, M_4, M_5, M_7, M_{10}, M_{11}\}$$

of the lines that, respectively, intersect or are disjoint from L; the former are marked with a $*$ in Table 1. (The subscript 1 refers to the chosen line $L = L_1$.)

The standard formula for the canonical class of a double cover shows that the branch curve B of f is of degree 6. The lines $M_i \in \beta_1^*$ intersecting L are blown down to the nodes $p_1, \ldots, p_6$ of B.

It follows from Table 1 that

- M_2, M_6, M_{12} intersect L_5, L_{11},
- M_1, M_6, M_9 intersect L_6, L_9,
- M_2, M_8, M_9 intersect L_7, L_{10},
- M_1, M_8, M_{12} intersect L_8, L_{12},
- M_1, M_2 intersect L_2,
- M_6, M_8 intersect L_3,
- M_9, M_{12} intersect L_4.

Since $(H-L)\cdot L_j = 1$ for any $j \neq 1$, the images $f(L_j)$ are lines in the plane. Each of the pairs (L_5, L_{11}), (L_6, L_9), (L_7, L_{10}), (L_8, L_{12}) is mapped to the same line passing through three of the nodes. Thus, their images form a complete quadrilateral with vertices $p_1, \ldots, p_6$ and sides

$$\ell_{236} = \langle p_2, p_3, p_6 \rangle, \quad \ell_{135} = \langle p_1, p_3, p_5 \rangle,$$
$$\ell_{245} = \langle p_2, p_4, p_5 \rangle, \quad \ell_{146} = \langle p_1, p_4, p_6 \rangle.$$

The remaining α-lines L_2, L_3, L_4 are mapped to the diagonals

$$\ell_{12} = \langle p_1, p_2 \rangle, \quad \ell_{34} = \langle p_3, p_4 \rangle, \quad \ell_{56} = \langle p_5, p_6 \rangle.$$

The lines L_i, $i = 2, 3, 4$, are in the same quartet with L_1, thus giving rise to conics C_{1i} (see Observation 2.12). These conics are mapped to the diagonals, i.e.,

$$f^{-1}(\ell_{12}) = M_1 + M_2 + L_2 + C_{12},$$
$$f^{-1}(\ell_{34}) = M_6 + M_8 + L_3 + C_{13},$$
$$f^{-1}(\ell_{56}) = M_9 + M_{12} + L_4 + C_{14}.$$

For the remaining lines $M_i \in \bar{\beta}_1^*$, since $(H-L)\cdot M_i = 1$, the images

$$\ell_1 = f(M_3), \quad \ell_2 = f(M_4), \quad \ell_3 = f(M_5),$$
$$\ell_4 = f(M_7), \quad \ell_5 = f(M_{10}), \ \ \ell_6 = f(M_{11})$$

are lines. The quartic Q_{1i} (see Observation 2.14) is mapped to the same line as M_i. Since all β-lines are skew, none of these lines ℓ_k passes through any of the nodes $p_1, \dots, p_6$. Each line splitting under the cover, cuts out an even divisor $2d$ on the branch curve B. In other words, it is a *tritangent* (or *contact line*) to B.

Now, let us look at the image of L under the projection. Since $(H-L)\cdot L = 3$, the image of L is a singular irreducible cubic K.

We have $f^{-1}(K) \in |3H-3L|$; since $(3H-4L)\cdot M_i = -1$ for $M_i \in \beta_1^*$, this cubic K passes through all nodes $p_1, \dots, p_6$ and

$$f^{-1}(K) = M_1 + M_2 + M_6 + M_8 + M_9 + M_{12} + L + L' \in |3H - 3L|,$$

where $H \cdot L' = 8$ and $L'^2 = -2$. Since L is the image of the line L_1, following the classical terminology, we say that the node of K is *apparent*, i.e., it is resolved under the double cover. The cubic is tangent to the branch curve B at three smooth points (which may collide), so that L intersects L' at the pre-images of these three points and at the two points corresponding to the branches at the node of K.

Remark 3.1. As is well known, the condition that a double cover $\pi\colon X \to \mathbb{P}^2$ branched along a nodal sextic curve B has a quartic birational model with an extra node is the existence of a contact conic that passes through the nodes of the sextic. The proper transform of the conic under the double cover splits into the union of two curves $C_1 + C_2$ and the linear system $|\pi^*\mathcal{O}_{\mathbb{P}^2}(1) + C_1|$ maps X to a quartic surface blowing down C_1 to a node.

Along these lines, the condition that X admits a birational sextic model with a line is the existence of a contact cubic K with an apparent node; this model is smooth if and only if K passes through all nodes of B. The proper transform of K splits into the union $L + L'$ of smooth rational curves and the linear system $|\pi^*\mathcal{O}_{\mathbb{P}^2}(1) + L|$ maps X birationally onto a surface of degree 6; the image of L is a line. The ramification curve $\bar{B}$ of the cover belongs to $|L + L'|$.

4. The Picard lattice

Let $\operatorname{Pic}(X)$ be the Picard lattice of a general Humbert sextic K3 surface. In this section, we show that the 24 lines L_i, M_j generate a primitive sublattice S of rank 15. We give a $\mathbb{Z}$-basis of S and compute the discriminant quadratic form of S, upon which we conclude that $S = \operatorname{Pic}(X)$.

4.1. **Hyperbolic bipartite graphs.** Given a graph Γ with n vertices, we denote by $\mathbb{Z}\Gamma$ the quadratic lattice of rank n with the Gram matrix $G = A - 2\mathbb{I}_n$, where A is the adjacency matrix of Γ and $\mathbb{I}_n$ is the identity matrix of size n. For a lattice L, we let

$$\operatorname{rad} L := L^\perp$$

be the radical of L, the kernel of the map $\mathbb{Z}\Gamma \to (\mathbb{Z}\Gamma)^\vee$ defined by the Gram matrix. We often abbreviate $L/\operatorname{rad} L$ to L/rad.

We take for Γ the bipartite graph with the bipartition (α, β) and the adjacency relation defined by Table 1.

Let

$$S := \mathbb{Z}\Gamma/\operatorname{rad}.$$

With the geometric applications in mind, we refer to the vertices of Γ as lines. We reiterate that S readily contains the "6-polarization" $H = \sum q$, where q is any of the $(3,3)$-fragments (see Remark 2.16).

Proposition 4.1. *The lattice S is of rank 15 and freely generated by the lines*

$$(L_2, L_3, L_4, L_5, L_6, L_7, L_8, L_{11}, M_1, M_2, M_3, M_6, M_8, M_9, M_{12}). \tag{4.2}$$

The discriminant group $S^\vee/S$ of S is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{\oplus 4} \oplus \mathbb{Z}/16\mathbb{Z}$.

Proof. The Gram matrix G of $\mathbb{Z}\Gamma$ can be written in the form

$$\begin{bmatrix} -2\mathbb{I}_{12} & P \\ P & -2\mathbb{I}_{12} \end{bmatrix},$$

where P is the incidence matrix from Table 1. We can compute its integral Smith normal form to check the assertions about the rank and the discriminant group. We also compute the intersection matrix of the sublattice spanned by the lines from (4.2) and check that its rank equals 15 and its discriminant group coincides with that of S. ☐

Remark 4.3. It is easily seen that $\operatorname{rad}(\mathbb{Z}\Gamma + \mathbb{Z}H)$ is generated, over $\mathbb{Z}$, by the classes of the form $H - \sum q$ (see Remark 2.16), where q is one of the 16 $(3,3)$ fragments in Γ. Furthermore, one can find a free basis for $\operatorname{rad} \mathbb{Z}\Gamma$ consisting of vectors of the form $\sum q' - \sum q''$.

Remark 4.4. By brute force (starting with a discrete graph on 12 vertices and adding 12 more vertices one-by-one), up to isomorphism there are but six bipartite graphs Γ_i of type $(12_6, 12_6)$ that define a hyperbolic lattice S_i; we let $S_1 = S$ as above. It is remarkable that all these lattices are of rank 15, admit a vector $h \in S_i \otimes \mathbb{Q}$ such that $h \cdot v = 1$ for each vertex of the graph, and have $h^2 = 6$. If $S_i \neq S$, the graph violates the $(3,3)$-property of Remark 2.16 and, hence, cannot be realized as the full graph of lines on a smooth sextic K3 surface. For example, see L_1, L_3, M_1, M_3, M_4 in Figure 1.

In fact, it can be shown that only the graphs $\Gamma_1 = \Gamma$ and Γ_2 defined by the intersection matrix in Figure 1 can be realized as a part of the graph of lines on a K3 surface, smooth or singular, of degree 6 (if $i = 1$) or degree 4 (if $i = 1, 2$). In all three cases, a general surface is smooth and Γ_i is its full graph of lines.

4.2. **The Picard lattice of X.** We have established that the Picard lattice $\operatorname{Pic}(X)$ of X contains a sublattice S of rank 15 spanned by the 24 lines. The following proposition computes the discriminant form $q\colon \operatorname{discr}(S) \to \mathbb{Q}/2\mathbb{Z}$ on the discriminant group $\operatorname{discr}(S) = S^\vee/S \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 4} \oplus \mathbb{Z}/16\mathbb{Z}$.

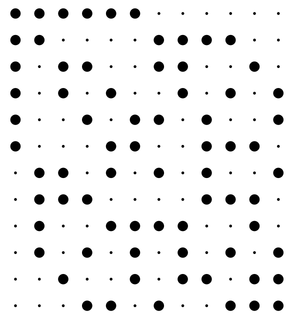


FIGURE 1. The graph Γ_2

Proposition 4.5. *The discriminant quadratic form* q *on* $\operatorname{discr}(S)$ *is defined by the following Gram matrix:*

$$\begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \oplus \begin{bmatrix} \frac{3}{16} \end{bmatrix},$$

where, as usual, the diagonal entries are considered defined $\operatorname{mod} 2\mathbb{Z}$ *whereas the others are defined* $\operatorname{mod} \mathbb{Z}$*.*

Proof. It is straightforward that, in the basis (4.2) introduced in Proposition 4.1, the five vectors $\gamma_1, \ldots, \gamma_5 \in S \otimes \mathbb{Q}$ given by

$$\begin{aligned} 2\gamma_1 &= [\ 0\ \ 0\ \ 1\ \ 1\ \ 1\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ ], \\ 2\gamma_2 &= [\ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 1\ \ 0\ \ 1\ \ 1\ \ 0\ \ 0\ ], \\ 2\gamma_3 &= [\ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 0\ \ 1\ \ 0\ \ 0\ \ 1\ \ 1\ \ 0\ ], \\ 2\gamma_4 &= [\ 1\ \ 1\ \ 1\ \ 1\ \ 0\ \ 0\ \ 0\ \ 1\ \ 1\ \ 1\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0\ ], \\ 16\gamma_5 &= [\ 2\ \ 2\ \ 4\ \ 2\ \ 10\ \ 12\ \ 2\ \ 1\ \ 1\ \ 7\ \ 1\ \ 2\ \ 12\ \ 12\ \ 10\ ] \end{aligned}$$

belong to $S^\vee$. The Gram matrix of these vectors is as in the statement and, since the matrix is nondegenerate, modulo S they generate a group of the correct size $256 = |\operatorname{discr}(S)|$. □

Theorem 4.6. *Let* X *be a Humbert sextic K3 surface with the Picard number* 15*. Then,* $\operatorname{Pic}(X) \cong S$ *and the transcendental lattice* $T(X)$ *is isomorphic to*

$$T \cong \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & -1 \\ 0 & -1 & -6 \end{bmatrix} \oplus \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}.$$

Proof. By the assumption, $\operatorname{Pic}(X)$ contains S as a sublattice of finite index. The overlattices of S of finite index correspond to isotropic subgroups of $\operatorname{discr}(S)$. A non-trivial isotropic subgroup would contain an element of order 2 and, up to the action of the group $\operatorname{Sym}(\Gamma)$ (see Observation 2.15), there are but two such elements, namely, γ_1 and $8\gamma_5$. They are represented by the rational vectors

$$\delta_1 = \tfrac{1}{2}(M_6 - M_8 + M_9 - M_{12}), \quad \delta_2 = \tfrac{1}{2}(L_5 - L_6 + L_7 - L_8),$$

with $\delta_i^2 = -2$ and $\delta_i \cdot H = 0$. If $\delta_i \in \operatorname{Pic}(X)$, then $\pm\delta_i$ would be represented by a smooth rational curve contracted by $|H|$ and X would be singular.

It is immediate that $\operatorname{discr}(T) = -\operatorname{discr}(S)$ and the signature $\sigma(T) = -3$, i.e., T is in the genus of $T(X)$. Due to [9, Theorem 1.14.2], this particular genus consists of a single isomorphism class; hence, $T(X) \cong T$. □

Corollary 4.7. *Let Y be a K3 surface and $\phi\colon \mathbb{Z}\Gamma/\operatorname{rad} \to \operatorname{Pic}(Y)$ an isomorphism. Then Y is isomorphic to a Humbert sextic X so that the classes of lines in $\operatorname{Pic}(Y)$ are the images of the vertices of Γ.*

Proof. It suffices to show that the moduli space of lattice S polarized K3 surfaces is irreducible. By [2, Proposition 5.6], the latter follows from the fact that $T(Y)$ contains an admissible 2-isotropic vector. □

Remark 4.8. The original construction of a Humbert sextic surface depends on 5 parameters: the choice of a hyperplane section of the Humbert cubic complex $\mathfrak{G}$. This agrees with the fact that the Picard number of a general Humbert sextic equals $20 - 5 = 15$.

The sublattice $U(2) \oplus U(2)$ (the last two summands in T) contains a primitive sublattice $U(2) \oplus \langle 4 \rangle$ isomorphic to the transcendental lattice of a minimal resolution of the Kummer surface of the self-product of an elliptic curve. As we know, the latter is birationally isomorphic to a Desmic quartic surface. Thus, we expect that the closure of the family of Humbert sextic surfaces contains a one-parameter family of sextic surfaces (probably singular) birational to desmic quartic surfaces. Unfortunately, we were not able to find this family explicitly.

5. Automorphisms of X

In this section, we discuss the automorphism of general and some special representatives of the Humbert family.

5.1. **Projective automorphisms.** We start with asserting that a general Humbert sextic X has no automorphisms induced from $\operatorname{PGL}(5, \mathbb{C})$.

Lemma 5.1. *Any projective automorphism σ of a Humbert sextic K3 surface X extends to a projective automorphism of the Humbert line complex $\mathfrak{G}$.*

Proof. Recall that $\mathfrak{G}$ is cut off the quadric $\mathcal{Q}$ by a cubic hypersurface. Assume that $X = \mathfrak{G} \cap R$ for a hyperplane R. Then X lies in $\mathcal{Q}_0 = \mathcal{Q} \cap R$ and, hence, we have $\sigma(\mathcal{Q}_0) = \mathcal{Q}_0$, as otherwise, X would be contained in the quartic surface $\mathcal{Q}_0 \cap \sigma(\mathcal{Q}_0)$. Letting $\mathcal{Q} = V(q)$ for a quadratic form q, we conclude that (an appropriately scaled lift to $\mathbb{C}^5$ of) σ is an automorphism of the restriction $q|_R$. By Witt's extension theorem, it extends to an automorphism $\tilde{\sigma}$ of q, so that $\tilde{\sigma}(\mathcal{Q}) = \mathcal{Q}$.

By Remark 7.1 below, σ preserves as a set the "original" 24 lines $L_1, \ldots, M_{12}$ on X. Recall that each line in the Grassmannian $\mathcal{Q} = G_1(\mathbb{P}^3)$ is contained in exactly one α-plane and exactly one β-plane. For $N = L_1, \ldots, M_{12}$, denote by π_N^+ (resp. π_N^-) the plane of the same (resp. opposite) type α or β as N (as defined in Convention 2.7). In other words, π_N^+ are the 24 planes (2.4) whereas π_N^- are some "wrong" planes most likely not even contained in $\mathfrak{G}$. Since σ respects (i.e., simultaneously preserves or simultaneously reverses) the type of lines (see Observation 2.15) and $\tilde{\sigma}$ respects the type of planes, we have $\tilde{\sigma}(\pi_N^+) = \pi_N^\epsilon$ for a *constant* $\epsilon = \epsilon(\tilde{\sigma}) = \pm 1$. We consider separately the following two cases.

Case 1: the restriction $q|_R$ is degenerate (necessarily of corank 1).

Since the line N lies in the cone $\mathcal{Q}_0$ *not passing through its vertex*, exactly one of $\pi_N^\pm$ lies in $\mathcal{Q}_0 \subset R$ and, by Remark 2.6, it is π_N^-. This property clearly distinguishes the two planes and we have $\tilde{\sigma}(\pi_N^+) = \pi_N^+$ for each $N = L_1, \ldots, M_{12}$. Therefore, $\mathfrak{G} \cap \tilde{\sigma}(\mathfrak{G})$ contains the 24 planes (2.4) and we conclude that $\tilde{\sigma}(\mathfrak{G}) = \mathfrak{G}$: indeed, otherwise $\mathfrak{G} \cap \tilde{\sigma}(\mathfrak{G})$ would be a surface of degree 18.

Case 2: the restriction $q|_R$ is non-degenerate.

This time, there are two extensions $\tilde{\sigma}$: they differ by a "reflection" against R (a choice of sign ± 1 in the direction q-orthogonal to R). Since the reflection itself is type reversing, a unique extension $\tilde{\sigma}$ can be *chosen* so that $\tilde{\sigma}(\pi_N^+) = \pi_N^+$, upon which concludes the proof in the same manner as in the previous case. □

Theorem 5.2. *The subgroup of* $\operatorname{Aut}(\mathbb{P}^4)$ *that leaves a general Humbert sextic* X *invariant is trivial.*

Proof. By Lemma 5.1, the group of projective automorphisms of a Humbert sextic K3 surface $X = \mathfrak{G} \cap R$ coincides with the group $\operatorname{Aut}_R(\mathfrak{G})$ of projective automorphisms of $\mathfrak{G}$ leaving the hyperplane R invariant. Since $\operatorname{Aut}(\mathfrak{G})$ is finite (see [5] and §2.2), the points $R \in \check{\mathbb{P}}^5$ invariant under the natural action of $\operatorname{Aut}(\mathfrak{G})$ or a non-trivial subgroup thereof constitute a proper Zariski closed set. □

Remark 5.3. In general, if $\operatorname{rk} S < 20$, the group of projective automorphisms of a very general lattice S-polarized K3 surface X is computed as the pull-back $\rho^{-1}(\pm \operatorname{id})$ under the natural homomorphism

$$\rho\colon O_H(S) \to O(\operatorname{discr} S),$$

where $O_H(S) \subset O(S)$ is the (finite) subgroup preserving the polarization $H \in S$. (If, as in Theorem 5.2, $\operatorname{rk} S$ is odd, the somewhat vague "very general" can be replaced with the requirement $\operatorname{Pic}(X) = S$.) Since, in our case, S is generated by lines, $O_H(S) = \operatorname{Sym}\Gamma$, whereupon, using Proposition 4.5 (and the proof thereof), Observation 2.15, and GAP [6], we arrive at $\rho^{-1}(\pm \operatorname{id}) = \{\operatorname{id}\}$.

5.2. **Birational automorphisms.** In spite of Theorem 5.2, we claim that the full group $\operatorname{Aut}(X)$ of automorphisms of X is infinite. For example, in §7.2 below we find quite a few elliptic fibrations on X. Some admit a section E so that the divisor classes of E and of the irreducible components of the fibers span a positive corank sublattice of $\operatorname{Pic}(X)$. By the Shioda–Tate formula [8, Chapter 11, Corollary 3.4], the Mordell–Weyl group of translation automorphisms of X along the fibers of the elliptic fibration is infinite (e.g., see Remark 7.2 below).

Another approach would be using the classification of the Picard lattices of the algebraic K3 surfaces with finite automorphism group found in [10]: our lattice $S = \operatorname{Pic}(X)$ is not on the list.

Besides, we have 24 involutions $\tau_N\colon X \to X$ each of which is the covering transformation of the projection $X \to \mathbb{P}^2$ from a line N on X (see §3). For any pair of lines, the two covering involutions generate an infinite dihedral group. We do not know whether these 24 involutions generate the whole group $\mathrm{Aut}(X)$.

Remark 5.4. In fact, any smooth sextic K3 surface with at least two lines has infinite group of birational automorphisms. Furthermore, it can be shown that, with very few exceptions, the involutions defined by a pair of distinct lines generate an infinite dihedral group. Proofs and details will appear in [1].

5.3. **Anti-symplectic involutions and cubic surfaces.** In principle, Lemma 5.1 and the description of $\mathrm{Aut}(\mathfrak{G})$ found in [5] (see §2.2) let us find all Humbert sextic K3 surfaces admitting a non-trivial projective automorphism. Below we confine ourselves to a maximal stratum with an anti-symplectic involution. Other examples are mentioned in Remark 7.1 below.

Example 5.5. Let $a_1 = a_2 = 1$ in (2.8), (2.9). Then, X admits an anti-symplectic involution $\gamma\colon x_1 \leftrightarrow x_2$ with X^γ a smooth hyperplane section $C = V(x_1 - x_2)$. Let $s = x_1x_2$, $t = x_1 + x_2$. Equations (2.8), (2.9) can be rewritten in the form

$$\begin{gathered} t^2 - 2s + x_3^2 + y_1^2 + y_2^2 + y_3^2 = 0, \\ sx_3 + \sqrt{-1}y_1y_2y_3 = 0, \\ t + a_3x_3 + a_4y_1 + a_5y_2 + a_6y_3 = 0. \end{gathered} \tag{5.6}$$

These equations define a surface in $\mathbb{P}(1^4, 2)$, where we weight s with degree 2 and other coordinates with degree 1. Projecting from the point $(0 : 0 : 0 : 0 : 1)$, we obtain that Y is isomorphic to the cubic surface in $\mathbb{P}^3$ given by an equation

$$F = \big((a_3x_3 + a_4y_1 + a_5y_2 + a_6y_3)^2 + x_3^2 + y_1^2 + y_2^2 + y_3^2\big)x_3 + 2\sqrt{-1}y_1y_2y_3.$$

The plane $V(x_3)$ is a tritangent plane of Y. Recall that any line ℓ on the cubic surface Y is contained in one of the plane sections of Y containing one of the lines $V(x_3, y_i)$. Plugging in $x_3 = ky_i$ in the equation of the cubic surface, we find that the residual conic in the plane $V(x_3 - ky_i)$ is given by a symmetric 3×3 matrix whose entries are homogeneous polynomials in k (of different degrees).

Computing the determinant of this matrix, we find that the parameters k defining singular residual conics are the zeros of the polynomial

$$k(k^2 + 1)\big((a_3^2 + a_5^2 + a_6^2 + 1)k^2 - 2(\sqrt{-1}a_5a_6 - a_4a_3)k + a_4^2 + 1\big) = 0. \tag{5.7}$$

The parameter $k = 0$ corresponds to the tritangent plane $V(x_3)$ containing the residual conic equal to the union of the two lines $\ell_j \neq \ell_1$. We choose the parameters $k = \pm\sqrt{-1}$. The branch curve B of the cover $X \to Y$ is cut out by the quadric given by the equation

$$t^2 - 4s = (a_3x_3 + a_4y_1 + a_5y_2 + a_6y_3)^2 + 2(x_3^2 + y_1^2 + y_2^2 + y_3^2) = 0.$$

Plugging in the equation $x_3 = \sqrt{-1}y_1$, we find

$$(a_3x_3 + a_4y_1 + a_5y_2 + a_6y_3)^2 + 2(y_2^2 + y_3^2) = 0,$$

and plugging in $F = 0$, we get

$$(a_3x_3 + a_4y_1 + a_5y_2 + a_6y_3)^2 + y_2^2 + y_3^2 + 2y_2y_3 = 0.$$

Together this gives $(y_2 + y_3)^2 = 0$. This shows that each line on Y contained in the plane $V(x_3 - \sqrt{-1}y_1)$ is tangent to the branch curve B. Similarly, we find that all 12 lines in Y contained in the planes $V(x_3 - \sqrt{-1}y_i)$ are tangent to B.

Remark 5.8. The set of 24 lines in Y lying in the planes $V(x_3 \pm ky_i)$, where $k \neq 0$ satifies equation (5.7) is equal to the union of two sets of 12 lines, one of which is a double-six of lines. By checking the intersections of the 12 lines corresponding to the parameters $k = \pm\sqrt{-1}$, we ontain that our set is complementary to the double-six.

Remark 5.9. Let $\mathcal{M}$ be the moduli space of pairs (F, C), where F is a smooth cubic surface and C is a smooth curve in $|-2K_F|$. It is a variety of dimension 13, a projective bundle over the moduli space of smooth cubic surfaces. Taking the double cover of F branched along C, we obtain a finite map to the moduli space $\mathcal{M}'$ of K3 surfaces polarized by the lattice $\mathrm{I}^{1,6}(2) \cong \operatorname{Pic}(F)(2)$. We find it amazing that our 5-dimensional moduli space of Humbert sextic K3 surfaces intersects $\mathcal{M}'$ along a four-dimensional subvariety. We believe, but could not prove it, that the pre-image of this subvariety in $\mathcal{M}$ is of dimension 4 and projected surjectively onto the moduli space of cubic surfaces.

5.4. **Anti-symplectic involutions via lattices.** Constructed in Example 5.5 is a 4-parameter family of Humbert sextics X, each having 12 extra conics. It follows that, for X general, the corank 1 space $\operatorname{Pic}(X) \otimes \mathbb{Q}$ is generated over $S \otimes \mathbb{Q}$ by any of these conics. Each conic can be regarded as a vector $c \in \operatorname{Hom}(\mathbb{Z}\Gamma + \mathbb{Z}H, \mathbb{Z})$; as such, our "symmetric" conics have the following properties:

- $c(H) = 2$ and $c(N) = 0$ or 1 for each line, i.e., vertex N of Γ;
- $c(N) = 1$ for exactly four α-lines and exactly four β-lines;
- c annihilates $\operatorname{rad}(\mathbb{Z}\Gamma + \mathbb{Z}H)$, cf. Remark 4.3.

Apart from the nine old conics, there is a single G-orbit of such vectors c, and we take for one of the new conics C the one intersecting

$$L_5, L_8, L_{10}, L_{12}, M_6, M_7, M_9, M_{11},$$

see the first row in Figure 2 below. (We change the indexing from §5.3 for a better looking matrix.) Then, we take

$$\bar{S} = (\mathbb{Z}\Gamma + \mathbb{Z}H + \mathbb{Z}C)/\mathrm{rad} \tag{5.10}$$

for the new Picard lattice. A straightforward computation in the spirit of §4 and §7.1 below shows that

- $\bar{S}$ is indeed the Picard lattice of a smooth sextic K3 surface X;
- no non-trivial finite index extension of $\bar{S}$ has this property; hence, the new family constructed is indeed the one in §5.3;
- X has the 24 old lines, 9 old conics, and 12 new conics.

The new conics are depicted in Figure 2. The divisor classes of six of them are shown in the figure, and others are found recursively: given a new conic C', four more are $H - C' - L_i - M_j$, where $L_i \cdot C = M_j \cdot C = L_i \cdot M_j = 1$. (A more conceptual explanation of this phenomenon is found further in this section, where some of the conics on $\bar{X}$ are interpreted as lines on a cubic surface.) Note that the new conics appear in pairs indistinguishable by their intersection with the lines. They are ordered so that $\{\text{odd}\}$, $\{\text{even}\}$ constitute a double-six, see the bottom right corner of Figure 3 below for the intersection matrix (divided by 2).

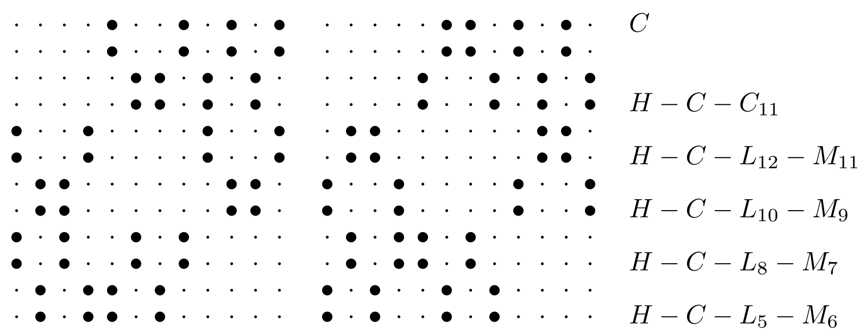


FIGURE 2. The twelve new conics

Remark 5.11. There are several 4-parameter families of Humbert sextics with the same set of lines and a few extra conics, no longer symmetric (cf. also Remark 7.1 below concerning extra lines). However, we only consider the one described.

Let $\bar{\Gamma}$ be the colored graph of lines and conics on $\bar{X}$. Using the `digraph` package in `GAP` [6], we can compute the group $\bar{G} = \operatorname{Sym}(\bar{\Gamma})$. We have $|\bar{G}| = 192$; this group acts transitively on the set of lines and on that of the new conics, and the action of $\bar{G}$ on the set of old conics has two orbits, one being $\{C_{11}, C_{22}, C_{33}\}$. The action of $\bar{G}$ on the lines (which almost determines its action on the conics) is the set-wise stabilizer of the collection (shown in Figure 2)

$$\big\{\{N \in \Gamma \,|\, N \cdot C = 1\} \;\big|\; C \text{ is a new conic}\big\}$$

under $G = \operatorname{Sym}(\Gamma)$, see Observation 2.15. The kernel of this action is generated by an involution interchanging the two Schläfli double-sixes, see below.

Next, we argue as explained in Remark 5.3 and conclude that $\operatorname{Aut}_H \bar{X} = \mathbb{Z}/2$. The generator γ of this group is an anti-symplectic projective involution of X: it acts on the lines and old conics via

$$L_{2i} \leftrightarrow M_{2i-1}, \quad L_{2i-1} \leftrightarrow M_{2i}, \quad C_{rs} \leftrightarrow C_{sr},\ r \neq s,$$

leaving invariant C_{rr}, $r = 1, 2, 3$, and all new conics.

Remark 5.12. A posteriori we can consider the involution $\gamma_T = -r_v \colon T \to T$ (see Proposition 4.5), where r_v is the reflection defined by an appropriate square (-4) vector $v \in T \cap 2T^\vee$ (e.g., the difference of the two generators of one of the two $U(2)$-summands). Then we take $\bar{T} = v^\perp$ for the new transcendental lattice, so that the new Picard lattice is an index 2 extension $\bar{S} \supset S \oplus \mathbb{Z}v$. In more detail this approach will be discussed by the first author elsewhere.

Yet another "systematic" way would be to analyze, in the spirit of Remark 4.4, the realizations of Γ as the graph of lines on a 2-polarized K3 surface. This time we would have to allow singularities, arriving at a single 4-parameter family of 6-nodal double planes. As explained in the next paragraph, this model is merely the cubic surface $\bar{X}/\gamma$ with a sextuple of disjoint lines coming from invariant conics on $\bar{X}$ contracted.

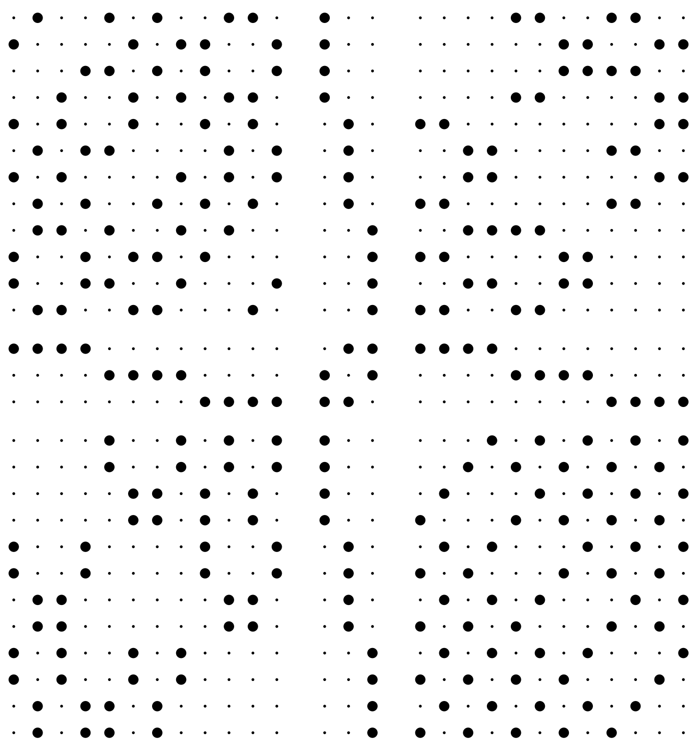


FIGURE 3. The 27 invariant divisors

Thus, we have twelve invariant pairs (split conics) $L_i + \gamma(L_i)$, $i = 1, \ldots, 12$, three invariant old conics C_{rr}, $r = 1, 2, 3$, and twelve invariant new conics. The intersection matrix of these $12 + 3 + 12 = 27$ invariant divisors obtained (upon division by 2) is shown in Figure 3, and one can readily recognize the 27 lines on the smooth cubic surface $F = \bar{X}/\gamma$. The three lines in the middle lie in a tritangent plane, and the two sextuples $\{2i\}$, $\{2i+1\}$, $i = 8, \ldots, 13$ (the even/odd ones of the last twelve) constitute a Schläfli double-six. Any of these two sextuples can be blown down to obtain the double plane model of Remark 5.12.

6. Back to the double plane model

As we observed before, the family of Humbert sextic K3 surfaces depends on 5 parameters. However, their double plane model seems to require $18+1+3+6 = 28$ conditions on plane sextics (depending on 27 parameters) to obtain a family of sextics with 6 nodes, admitting a contact split nodal cubic, six contact lines, and tangent to the diagonals of the complete quadrilateral. Here, we solve this puzzle

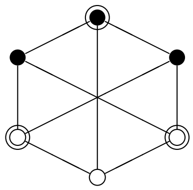
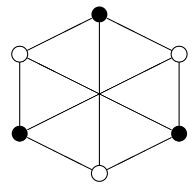
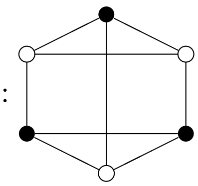
(3):  (4):  (5): 

FIGURE 4. The intersection of M_3' with the pre-images of the diagonals

by proving that the splitting of the three diagonals and one line is almost enough to obtain the double cover isomorphic to a Humbert sextic K3 surface.

We start with a complete quadrilateral ℓ_{236}, ℓ_{135}, ℓ_{245}, ℓ_{146} and a sextic curve B with nodes at the six vertices $p_1, \dots, p_6$. First, we require that each diagonal $\ell_{12}, \ell_{13}, \ell_{23}$ should be tangent to B at some point q_i distinct from all points p_i. A straightforward computation in the spirit of §4 shows that there are two families of such sextics, which differ by the proper transforms of the diagonals: either

(1) the triangle of the diagonals lifts to a hexagon, as in Figure 4, left, or
(2) the triangle of the diagonals splits into two triangles, as in Figure 4, right.

Remark 6.1. The two families can be described geometrically. Let $f\colon Y \to \mathbb{P}^2$ be the minimal resolution of singularities of the double cover branched along B. We factor f as $Y \to F' \to \mathbb{P}^2$, where F' is the weak del Pezzo surface of degree 3 obtained by blowing up the points $p_1, \dots, p_6$. The anti-canonical model of F' is obtained by blowing down the proper transforms of the diagonals ℓ_{ij}. It is isomorphic to the 4-nodal cubic surface F. The proper transforms of the diagonals can be identified with the sum T of three lines m_1, m_2, m_3 cut out by a tritangent plane Π of F. The proper transform B' of B is mapped to the intersection of F with a quadric surface not passing through the nodes of F. Counting constants, we find that there are two families of quadrics, both of dimension 6, intersecting each line m_i at one point with multiplicity two. In each family, one is required to impose three conditions on quadrics.

One family is defined by the condition that a quadric Q intersects the tritangent plane along a conic tangent to the lines m_i. The other family is defined by the condition that Q is tangent to the tritangent plane along a line ℓ. In the latter case, the pencil of planes containing ℓ defines a pencil of cubic curves on F. It is equal to the proper transform of the pencil of cubic curves in the plane with base points $p_1, \dots, p_6$ and the points of tangency of B with the diagonals.

The triangle T of lines on the cubic surface F is a reducible curve of arithmetic genus one. Its pre-image in the double cover is a double cover branched along the Cartier divisor $D = 2(q_1 + q_2 + q_3) \in |\mathcal{O}_T(2)|$, where $Q \cap T = \{q_1, q_2, q_c\}$. Thus, we have two cases, resulting in two families of sextic curves B.

Case (1): $Q \cap \Pi$ is a smooth conic tangent to T at q_1, q_2, q_3.

The double cover is not trivial, i.e., it is not equal to the union of two curves mapped isomorphically to T under the covering. This corresponds to the first two pictures in Figure 4. Its restriction over the open curve $T \setminus \{q_1, q_2, q_3\}$ corresponds to a subgroup of index two of its fundamental group.

The pre-image of T is a reducible curve with the dual graph shown in Figure 4, left. Here and below, the action of the deck translation is the central symmetry.

Case (2): $Q \cap \Pi$ is a double line.

The cover is trivial, i.e., it defines the trivial cover of $T \setminus \{q_1, q_2, q_c\}$. The dual graph of the pre-image of T is given in Figure 4, right.

Since we need to be able to choose pairwise disjoint pull-backs of the diagonals, we concentrate on Case (1), where the triangle of the diagonas does not split under the cover, i.e., the proper transform of the union of the diagonals is a hexagon (with the three long diagonals, cf. Figure 4, left) of (-2)-curves on Y.

Let $M_1', M_2', M_6', M_8', M_9', M_{12}' \subset Y$ be the proper transforms of the exceptional curves of the blow-up, choose three disjoint sides L_2', L_3', L_4' of the hexagon (e.g., the circled vertices in Figure 4, left), and let $L_5', L_{11}'; L_6', L_9'; L_7', L_{10}'; L_8', L_{12}'$ be the (-2)-curves that are mapped, in pairs, by the double cover $f\colon Y \to \mathbb{P}^2$ to the four sides of the quadrilateral. They are all (-2)-curves on Y.

Now, we invoke one more condition that there exists a tritangent line ℓ of B not passing through its nodes. The tritangent ℓ splits under the double cover into the union of two (-2)-curves intersecting at three points; denote them by M_3', M_3''. It can be shown that there are two irreducible families of pairs (B, ℓ), both depending on 5 parameters: either

(3) B is as in Case (1) and has six tritangents, or
(4) B is as in Case (1) and has two tritangents.

In terms of ℓ itself only, the two families differ by the intersection of M_3' with the sides of the hexagon, see the black vertices in Figure 4, left and center (whereas the other pull-back M_3'' is represented by the white vertices); in Case (3), the other pairs are obtained by rotation.

Thus, we have fifteen (-2)-curves

$$M_1', M_2', M_6', M_8', M_9', M_{12}', L_5', L_6', L_7', L_8', L_2', L_3', L_4', L_{11}', M_3'.$$

Here, we choose one full pair L_5', L_{11}' while keeping only one chosen line from the other three pairs. Now, once the pairwise intersections of M_3', L_2', L_3', L_4' have been arranged, it is immediate to check that, *under the appropriate choice of the components* $(L_5', L_{11}'), L_6', L_7', L_8'$, the intersection matrix of these curves coincides with that of the curves constituting the basis (4.2) on a general Humbert sextic K3 surface X. (Indeed, since the pull-backs of the four sides of the quadrilateral are disjoint from each other and from those of the diagonals, we merely index them according to Table 1.) Therefore, these curves span a lattice isomorphic to $\operatorname{Pic}(X)$, and it remains to apply Corollary 4.7.

Remark 6.2. For completeness, in Case (2) there is a single 5-parameter family of pairs (B, ℓ):

(5) B is as in Case (2) and has four tritangents.

A pull-back M_3' of ℓ intersects those of the diagonals as shown in Figure 4, right.

Whereas the existence of (3) is guaranteed by Proposition 4.1, the existence of (4) and (5), as well as the very fact that there are but three families needs proof, which will appear elsewhere. It is also worth mentioning that, in all five cases (1)–(5), i.e., B itself or a pair (B, ℓ), a general curve B has quite a few contact cubics with an apparent node. However, only in Case (3) there is a (unique) such cubic passing through all six nodes of B, so that the corresponding sextic K3 surface $X \subset \mathbb{P}^4$ is smooth (see Remark 3.1).

Remark 6.3. One can compare the specialty of the plane sextic B with respect to the tangency conditions with another plane sextic curve, known as the *Humbert plane sextic* of genus 5. It has five cusps and is tangent to any line connecting a pair of cusps as well as the unique conic passing through the cusps. The double cover of $\mathbb{P}^2$ branched along the Humbert sextic is birationally isomorphic to the Kummer quartic surface associated with a nonsingular curve of genus 2 [3, Remark 8.6.9]. As in our case, the 16 tangency condition would wrongly imply that such a curve should not exist.

7. Elliptic pencils

We keep the notation $\Gamma = \Gamma_1$, $G = \mathrm{Sym}(\Gamma)$, $S = \mathbb{Z}\Gamma/\mathrm{rad}$, etc. from §4.2, and we consider a Humbert sextic X that is general in the sense that $\mathrm{Pic}(X) = S$.

7.1. **Rational curves.** To find smooth rational curves on a polarized K3 surface, we use the well known description of the nef cone and Vinberg's algorithm [14] for computing the fundamental polyhedra. As a step, the vectors of a given square in a definite lattice are found by the Lenstra–Lenstra–Lovász lattice basis reduction algorithm, which is implemented as `ShortestVectors` in `GAP` [6]. One can also use the algorithm from [12].

We find that on X there are

- 24 lines (a single G-orbit),
- 9 conics (a single G-orbit),
- no twisted cubics, and
- 72 rational quartics (also a single G-orbit).

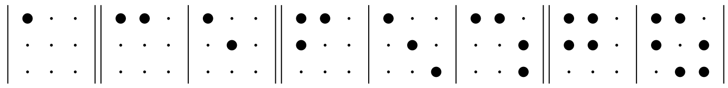
Thus, all lines are those constituting the original configuration Γ and all conics and quartics are those described in Observation 2.12 and Observation 2.14, respectively; we use the notation C, Q and the indexing introduced therein.

Taking this two steps further, we find that there are

- $816 = 48 + 192 + 576$ (three G-orbits) rational quintics and
- $720 = 144 + 288 + 288$ (also three G-orbits) rational sextics,

so that it hardly makes sense to study these or higher degree curves in detail.

Remark 7.1. We emphasize that these counts, as well as the smoothness of the conics and quartics hold for a general member of the family only. In fact, as long as lines are concerned, it is the nine conics C_{rs} in Observation 2.12 that solely control the smooth degenerations of Humbert sextics. Beyond the 24 original lines, any other line is a component of one of these conics. Furthermore, the strata with a fixed intersection graph of lines are labeled by the sets of conics that split (or rather the $(\mathfrak{S}_3 \times \mathfrak{S}_3) \rtimes \mathbb{Z}/2$ orbits thereof, see Observation 2.15). There are eight strata



$\{1\}$ $\{1\}$ $\mathbb{Z}/2$ $\mathbb{Z}/2$ $\mathfrak{S}_3$ $(\mathbb{Z}/2)^2$ $(\mathbb{Z}/2)^2$ $\mathfrak{D}_{12}$

itemized by the Picard rank $\rho = 16, 17, 18, 19$. (Here, the grid represents the (4×4)-cells in Table 1 that index the conics, and the $\bullet$'s stand for the conics that split. Distinct values of ρ are separated by $\|$'s. For each stratum, we indicate the group of projective automorphisms of a general representative, cf. Theorem 5.2 and §5.4; it is computed as explained in Remark 5.3. Symplectic automorphisms preserve α and β whereas anti-symplectic ones interchange $\alpha \leftrightarrow \beta$.) Proof will appear in [1]. Degenerate Humbert sextics have many more conics, twisted cubics, and quartics; still, only the original nine conics may split, at most six at a time.

7.2. **Elliptic pencils with a reducible singular fiber.** We are mostly interested in the elliptic pencils on X with at least one reducible singular fiber made of lines. Such fibers are induced subgraphs of Γ isomorphic to an affine Dynkin diagram.

All such subgraphs are listed below, where, for each type, we indicate the total number of subgraphs followed by that itemized by the G-orbits. Marked with a * are (orbits of) pencils admitting a section, which can always be chosen a line.

- $\tilde{A}_3$: $162 = 18^* + 144^*$ (two orbits);
- $\tilde{A}_5$: $1056 = 192^* + 288^* + 576^*$ (three orbits);
- $\tilde{A}_7$: $1512 = 72 + 144^* + 144 + 288^* + 288 + 576^*$ (six orbits);
- $\tilde{A}_{11}$: 48 (one orbit);
- $\tilde{D}_4$: $360 = 72 + 288^*$ (two orbits);
- $\tilde{D}_5$: $720 = 144^* + 576^*$ (two orbits);
- $\tilde{D}_6$: $5184 = 2 \times 144 + 3 \times 288 + 4 \times 576^* + 576 + 1152^*$ (11 orbits);
- $\tilde{D}_8$: $1440 = 288 + 2 \times 576$ (three orbits);
- $\tilde{E}_6$: $3840 = 2 \times 96 + 192^* + 4 \times 576^* + 2 \times 576$ (nine orbits);
- $\tilde{E}_7$: $12672 = 3 \times 576^* + 7 \times 576 + 4 \times 1152^* + 2 \times 1152$ (16 orbits);
- $\tilde{E}_8$: $4608 = 4 \times 1152$ (four orbits).

We analyze but a few interesting cases, leaving the rest to the reader. Note that the list of smooth rational curves of degree up to 6 lets us detect reducible fibers in the pencils of fiber degree up to 12, i.e., all except $\tilde{E}_7$ or $\tilde{E}_8$.

The $\tilde{A}_3$-type subgraphs are the quadrangles discussed in §2.3. If q is a proper quadrangle, apart from q the pencil $\mathcal{P}(q)$ has three reducible fibers: another proper quadrangle q' and two $\tilde{A}_1$-type fibers made of two conics each. (We use the Dynkin diagram notation since homologically we cannot detect the degenerations $\mathrm{I}_2 \to \mathrm{II}$ or $\mathrm{I}_3 \to \mathrm{III}$; most likely they do occur in some special members of the family.) The two quadrangles q, q' constitute a single cell (α_r, β_s), see §2.3, and the four conics are C_{uv}, $u \neq r$, $v \neq s$, see Observation 2.12(3).

Remark 7.2. In particular, the Mordell–Weil group group of a pencil $\mathcal{P}(q)$ has rank 5, proving that $\operatorname{Aut}(X)$ is infinite, cf. the discussion in §5.2.

If q is an improper quadrangle, the pencil $\mathcal{P}(q)$ has two more reducible fibers: another improper quadrangle q' and an $\tilde{A}_2$-type fiber made of a conic C and two lines L_i, M_j. Thus, we also have an involution $q \leftrightarrow q'$ on the set of improper quadrangles; it does *not* preserve $(3,3)$-fragments. The pencil is determined by the pair L_i, M_j of intersecting lines: the conic is $C = C_{rs}$, where $\alpha_r \ni L_i$, $\beta_s \ni M_j$, and q, q' are the complementary quadrangles in the two $(3,3)$-fragments sharing (L_i, M_j) as a common corner.

Representatives of the three orbits of the type $\tilde{A}_5$ (hexagonal) fibers and the other reducible fibers of the corresponding pencils are as follows:

- $\tilde{A}_5 = (L_1, M_1, L_2, M_5, L_4, M_9)$: $(L_3, \mathbf{5}_2) + (L_{11}, \mathbf{5}_1)$;
- $\tilde{A}_5 = (L_1, M_1, L_2, M_5, L_5, M_6)$: $(L_7, \mathbf{5}_3) + (M_3, \mathbf{5}_3) + (C_{33}, Q_{62})$;
- $\tilde{A}_5 = (L_1, M_1, L_2, M_5, L_5, M_{12})$: $(L_3, M_3, Q_{12,2}) + (L_7, \mathbf{5}_2)$,

where $(\cdot)$ is an I_*-type fiber and $\mathbf{d}_n$ is a certain representative of the n-th orbit of degree d curves. In particular, we have a "natural" expression of the quintics in terms of lines. (All sextics appear in the singular fibers of octagonal pencils.)

An example of a longest cycle $\tilde{A}_{11}$ is

$$L_1, M_1, L_6, M_5, L_{10}, M_{10}, L_3, M_4, L_7, M_7, L_{11}, M_{12},$$

and the corresponding pencil has no other reducible fibers. The divisor class of a fiber lies in $4 \cdot S^\vee$, and all lines and conics are 4-fold sections.

The $\tilde{D}_4$-type subgraphs constituting the shorter, 72-element orbit are described as follows. Pick a quartet α_s and a β-line M_j (or vice versa, with α and β reversed). Each quartet α_u, $u \neq s$, has two lines that intersect M_j and, together with M_j, the four lines obtained constitute the $\tilde{D}_4$ in question. For example, L_1, L_2, L_6, L_8, M_1 starting from (α_3, M_1). The elliptic pencil has two other reducible fibers:

- the complementary $\tilde{D}_4$-fragment L_3, L_4, L_5, L_7, M_4, so that all eight α-lines constitute $\alpha \smallsetminus \alpha_s$, and
- an I_4-type fiber $L_{10}, C_{32}, L_{11}, C_{33}$.

In the latter, in the invariant terms, the two conics are C_{sv}, $\beta_v \not\ni M_j$, and the two lines are those from α_s that are disjoint from M_j.

Certainly, one can construct numerically effective isotropic classes by combining smooth rational curves of higher degrees. Some of this combinations appear above (most notably, pairs of conics), but the list is far from complete.

References

[1] A. Degtyarev, S. Rams, *Lines on sextic $K3$-surfaces with simple singularities*, To appear, 2025.

[2] I. Dolgachev. *Mirror symmetry for lattice polarized K3 surfaces*. Algebraic geometry, **4**, J. Math. Sci. **81**, 2599–2630 (1996)

[3] I. Dolgachev, *Classical Algebraic Geometry*, Cambridge Univ. Press 2012.

[4] I. Dolgachev, A. Duncan, *Automorphisms of cubic surfaces in positive characteristic*. Izv. Ross. Akad. Nauk Ser. Mat. **83** (2019), no. 3, 15–92.

[5] I. Dolgachev, S. Kondō, *Desmic quartic surfaces in arbitrary characteristic*, arXiv:2505.01033.

[6] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.13.0*, 2024.

[7] G. Humbert, *Sur la surface desmique du quatriéme ordre*, J. Math. Pures et Appl., **7** (1891), 353–398.

[8] D. Huybrechts, *Lectures on K3 surfaces*, Cambridge Studies in Advanced Mathematics, vol. 158, Cambridge University Press, Cambridge, 2016.
[9] V. V. Nikulin, *Integral symmetric bilinear forms and some of their applications*, Math. USSR-Izv., **14** (1980), 103–167.
[10] V. V. Nikulin, *Factor groups of groups of the automorphisms of hyperbolic forms with respect to subgroups generated by* 2*-reflections*, J. Soviet Math., **22** (1983), 1401–1475.
[11] B. Saint-Donat, *Projective models of K-3 surfaces*, Amer. J. Math., **96** (1972), 602–639.
[12] I. Shimada, *Projective models of the supersingular K3 surface with Artin invariant 1 in characteristic 5*. J. Algebra, **403** (2014), 273–299.
[13] G. van der Geer, *Hilbert modular surfaces*, Springer-Verlag 1987.
[14] È. B. Vinberg, *The groups of units of certain quadratic forms*, Mat. Sb. (N.S.) **87(129)** (1972), 18–36.

BILKENT UNIVERSITY, DEPARTMENT OF MATHEMATICS, 06800 ANKARA, TURKEY
Email address: degt@fen.bilkent.edu.tr

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, 525 EAST UNIVERSITY AVENUE, ANN ARBOR, MI 48109-1109 USA
Email address: idolga@umich.edu

GRADUATE SCHOOL OF MATHEMATICS, NAGOYA UNIVERSITY, NAGOYA, 464-8602 JAPAN
Email address: kondo@math.nagoya-u.ac.jp