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A Supersingular K3 Surface in Characteristic 2 and the Leech Lattice

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1 Introduction

Let k be an algebraically closed field of characteristic 2. Consider $\mathbb{F}_4 \subset k$ and $\mathbb{P}^2(\mathbb{F}_4) \subset \mathbb{P}^2(k)$. Let $\mathcal{P}$ be the set of points and let $\check{\mathcal{P}}$ be the set of lines in $\mathbb{P}^2(\mathbb{F}_4)$. Each set contains 21 elements, each point is contained in exactly 5 lines, and each line contains exactly 5 points. It is known that the group of automorphisms of the configuration $(\mathcal{P}, \check{\mathcal{P}})$ is isomorphic to $M_{21} \cdot D_{12}$, where $M_{21} (\cong \mathrm{PSL}(3, \mathbb{F}_4))$ is a simple subgroup of the Mathieu group M_{24} and D_{12} is a dihedral group of order 12.

In this paper, we prove the following main theorem.

Theorem 1.1. There exists a unique (up to isomorphism) K3 surface over k satisfying the following equivalent properties:

(i) the Picard lattice of X is isomorphic to $U \perp D_{20}$;

(ii) X has a Jacobian quasi-elliptic fibration with one fiber of type $\tilde{D}_{20}$;

(iii) X has a quasi-elliptic fibration with the Weierstrass equation

$$y^2 = x^3 + t^2 x + t^{11}; \tag{1.1}$$

(iv) X has a quasi-elliptic fibration with 5 fibers of type $\tilde{D}_4$ and the group of sections isomorphic to $(\mathbb{Z}/2)^4$;

(v) X contains a set $\mathcal{A}$ of 21 disjoint (-2)-curves and another set $\mathcal{B}$ of 21 disjoint (-2)-curves such that each curve from one set intersects exactly 5 curves from the other set with multiplicity 1;

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(vi) X is birationally isomorphic to the inseparable double cover of $\mathbb{P}^2$ with branch divisor

$$x_0x_1x_2(x_0^3+x_1^3+x_2^3)=0; \tag{1.2}$$

(vii) X is isomorphic to a minimal nonsingular model of the quartic surface with 7 rational double points of type A_3, which is defined by the equation

$$x_0^4+x_1^4+x_2^4+x_3^4+x_0^2x_1^2+x_0^2x_2^2+x_1^2x_2^2+x_0x_1x_2(x_0+x_1+x_2)=0; \tag{1.3}$$

(viii) X is isomorphic to the surface in $\mathbb{P}^2\times\mathbb{P}^2$ given by the equations

$$x_0y_0^2+x_1y_1^2+x_2y_2^2=0, \qquad x_0^2y_0+x_1^2y_1+x_2^2y_2=0. \tag{1.4}$$

The automorphism group Aut(X) contains a normal infinite subgroup generated by 168 involutions and the quotient is a finite group isomorphic to $\mathrm{PGL}(3,\mathbb{F}_4)\cdot 2$. □

Since the group $\mathrm{PSL}(3,\mathbb{F}_4)\cdot 2$ is not a subgroup of the Mathieu group M_{23}, our theorem shows that Mukai's classification of finite groups acting symplectically on a complex K3 surface (see [12]) does not extend to positive characteristic.

2 Supersingular K3 surfaces in characteristic 2

2.1 Known facts

Recall that a *supersingular* K3 *surface* (in the sense of Shioda [16]) is a K3 surface with the Picard group of rank 22. This occurs only if the characteristic of the ground field k is a positive prime p. By a result of Artin [1], the Picard lattice $S_X=\mathrm{Pic}(X)$ of a supersingular K3 surface is a p-elementary lattice (i.e., the discriminant group S_X^*/S_X is a p-elementary Abelian group). The dimension r of the discriminant group over $\mathbb{F}_p$ is even and the number $\sigma=r/2$ is called the *Artin invariant*. We assume that $p=2$. A fundamental theorem of Rudakov and Shafarevich [13, 14] tells that any supersingular K3 surface admits a quasi-elliptic fibration, that is, a morphism $f: X\to\mathbb{P}^1$ whose general fiber is a regular but not smooth geometrically irreducible curve of genus 1. Over an open subset U of the base, each fiber is isomorphic to an irreducible cuspidal cubic, the reducible fibers are Kodaira genus 1 curves of additive type. The closure of the set of cusps of irreducible fibers is a smooth irreducible curve C, the *cusp curve*. The restriction of f to C is a purely inseparable cover of degree 2. It follows that C is isomorphic to $\mathbb{P}^1$, and hence by adjunction is a (-2)-curve on X. A surface with a quasi-elliptic fibration is unirational. Thus, any supersingular K3 surface in characteristic 2 is unirational. This is one of the main results of [13, 14].

In this paper, we study a supersingular K3 surface with the Artin invariant $\sigma = 1$. It follows from the classification of 2-elementary lattices of signature $(1, 21)$ that the Picard lattice of such a K3 surface is unique up to isometries (see [14, Section 1]). Therefore, it is isomorphic to $U \perp D_{20}$. As usual, U denotes the unique even unimodular indefinite lattice of rank 2 and A_n, D_n, or E_n denotes the *negative* definite even lattice defined by the Cartan matrix of type A_n, D_n, or E_n, respectively. Moreover, it is known that any supersingular K3 surface with Artin invariant $\sigma = 1$ is unique up to isomorphisms (see [14, Section 11]).

2.2 The Weierstrass model

Proposition 2.1. Let X be a supersingular K3 surface whose Picard lattice S_X is isometric to $U \perp D_{20}$. Then X has a quasi-elliptic fibration with one singular fiber of type $\tilde{D}_{20}$ and a section. Its affine Weierstrass equation is

$$y^2 + x^3 + t^2 x + t^{11} = 0. \tag{2.1}$$

□

Proof. First we see that the above Weierstrass equation defines a K3 surface whose Picard lattice is isomorphic to $U \perp D_{20}$. Recall (see [14, Section 12] and [8, Chapter 5]) that the Weierstrass model of a quasi-elliptic fibration $f : S \to \mathbb{P}^1$ on a nonsingular projective surface S over an algebraically closed field of characteristic 2 with $\chi(S, \mathcal{O}_S) = n$ is a closed subscheme of the total space of the vector bundle $\mathcal{O}_{\mathbb{P}^1}(2n) \oplus \mathcal{O}_{\mathbb{P}^1}(4n)$ over $\mathbb{P}^1$ given by the equation

$$y^2 + x^3 + a(t_0, t_1)x + b(t_0, t_1) = 0, \tag{2.2}$$

where $a \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4n))$ and $b \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(6n))$. Dividing the equation by t_0^{6n} and making the substitution $x \to x/t_0^{2n}$ and $y \to y/t_0^{3n}$, we get the affine Weierstrass equation

$$y^2 + x^3 + A(t)x + B(t) = 0, \tag{2.3}$$

where $A(t)$ (resp., $B(t)$) is an inhomogeneous polynomial of degree $4n$ (resp., $6n$). Conversely, given an affine Weierstrass equation, we can homogenize it (with respect to t)

by substituting $t = t_1/t_0$, multiplying the equation by t_0^{6n} and making the substitutions $y \to yt_0^{3n}$, $x \to xt_0^{4n}$. The discriminant of the Weierstrass model is a section of $\mathcal{O}_{\mathbb{P}^1}(12n) \otimes \omega_{\mathbb{P}^1}^{\otimes 2} \cong \mathcal{O}_{\mathbb{P}^1}(12n-4)$ defined by

$$\Delta = a(da)^2 + (db)^2. \tag{2.4}$$

In our case, $n = 2$ and

$$a(t_0, t_1) = t_0^6 t_1^2, \qquad b(t_0, t_1) = t_0 t_1^{11}, \qquad \Delta = t_1^{20}. \tag{2.5}$$

It is known (see [8, Corollary 5.5.8]) that the order of vanishing of the discriminant at a point $s \in \mathbb{P}^1$ is equal to $e(F_s) - 2$, where F_s is the fiber over the point s and $e(F_s)$ is its Euler-Poincaré characteristic. Applying this to our situation, we obtain that the fibration has one degenerate fiber F over the point $(1, 0)$ with Euler-Poincaré characteristic equal to 22. It follows from the classification of degenerate fibers of a quasi-elliptic fibration that F is of type $\tilde{D}_{20}$. Obviously, its Picard lattice contains $U \perp D_{20}$, which is generated by the class of components of fibers and a section. Since $\sigma \geq 1$, the Picard lattice isomorphic to $U \perp D_{20}$. Now, the assertion follows from the uniqueness of supersingular K3 surface with the Artin invariant 1. ■

3 Leech roots

In this section, we denote by X the K3 surface whose Picard lattice is isomorphic to $U \perp D_{20}$.

3.1 The Leech lattice

We follow the notation and the main ideas from [2, 3, 4, 5, 6, 11]. First, we embed the Picard lattice $S_X \cong U \perp D_{20}$ in the lattice $L = \Lambda \perp U \cong II_{1,25}$, where Λ is the *Leech lattice* and U is the hyperbolic plane. We denote each vector $x \in L$ by (λ, m, n), where $\lambda \in \Lambda$, and $x = \lambda + mf + ng$, with f, g being the standard generators of U, that is, $f^2 = g^2 = 0$, and $\langle f, g \rangle = 1$. Note that $r = (\lambda, 1, -1 - \langle \lambda, \lambda \rangle/2)$ satisfies $r^2 = -2$. Such vectors will be called *Leech roots*. We denote by $\Delta(L)$ the set of all Leech roots. Recall that Λ is an even negative definite unimodular lattice of rank 24, realized as a certain subgroup in $\mathbb{R}^{24} = \mathbb{R}^{\mathbb{P}^1(\mathbb{F}_{23})}$ equipped with inner product $\langle x, y \rangle = -x \cdot y/8$ (in group theory one often changes the sign to the opposite). For any subset A of $\Omega = \mathbb{P}^1(\mathbb{F}_{23})$, let ν_A denote the vector $\sum_{i \in A} e_i$, where

$\{e_\infty, e_0, \dots, e_{22}\}$ is the standard basis in $\mathbb{R}^{24}$. A *Steiner system* $S(5,8,24)$ is a set consisting of 8-element subsets of Ω such that any 5-element subset belongs to a unique element of $S(5,8,24)$. An 8-element subset in $S(5,8,24)$ is called an *octad*. Then Λ is defined as a lattice generated by the vectors $\nu_\Omega - 4\nu_\infty$ and $2\nu_K$, where K belongs to the Steiner system $S(5,8,24)$. Let $W(L)$ be the subgroup generated by reflections in the orthogonal group $O(L)$ of L. Let $P(L)$ be a connected component of

$$P(L) = \{x \in \mathbb{P}(L \otimes \mathbb{R}) : \langle x, x\rangle > 0\}. \tag{3.1}$$

Then $W(L)$ acts naturally on $P(L)$. A fundamental domain of this action of $W(L)$ is given by

$$\mathcal{D} = \{x \in P(L) : \langle x, r\rangle > 0,\ r \in \Delta(L)\}. \tag{3.2}$$

It is known that $O(L)$ is a split extension of $W(L)$ by $\mathrm{Aut}(\mathcal{D})$ (see [4, 6]).

Lemma 3.1. Let X be the K3 surface whose Picard lattice S_X is isomorphic to $U \perp D_{20}$. Then there is a primitive embedding of S_X in L such that the orthogonal complement $S_X^\perp$ is generated by some Leech roots and isomorphic to the root lattice D_4. □

Proof. Consider the following vectors in Λ:

$$\begin{aligned} &X = 4\nu_\infty + \nu_\Omega, \qquad Y = 4\nu_0 + \nu_\Omega, \qquad Z = 0, \\ &T = (x_\infty, x_0, x_1, x_{k_2}, \dots, x_{k_{22}}) = (3,3,3,-1,-1,-1,-1,-1,1,\dots,1), \end{aligned} \tag{3.3}$$

where $K = \{\infty, 0, 1, k_2, \dots, k_6\}$ is an octad. The corresponding Leech roots

$$x = (X,1,2), \qquad y = (Y,1,2), \qquad z = (0,1,-1), \qquad t = (T,1,2) \tag{3.4}$$

generate a root lattice R isomorphic to D_4. Obviously, R is primitive in L.

Let S be the orthogonal complement of R in L. Then it is an even lattice of signature $(1,21)$ and the discriminant group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ (the discriminant groups of S and R are isomorphic). Since such a lattice is unique, up to isometry, we obtain that $S \cong U \perp D_{20}$. ■

3.2 (42 + 168) Leech roots

We fix an embedding of S_X into L as in Lemma 3.1. Let $P(X)$ be the positive cone of X, that is, a connected component of

$$\{x \in \mathbb{P}(S_X \otimes \mathbb{R}) : \langle x, x\rangle > 0\}, \tag{3.5}$$

which contains the class of an ample divisor. Let $\mathcal{D}(X)$ be the intersection of $\mathcal{D}$ and $P(X)$. It is known that $\mathrm{Aut}(\mathcal{D}(X)) \cong M_{21} \cdot D_{12}$, where $M_{21} \cong \mathrm{PSL}(3, \mathbb{F}_4)$ and D_{12} is a dihedral group of order 12 (see [2, Section 8, Example 5]). Each hyperplane bounding $\mathcal{D}(X)$ is the one perpendicular to a negative norm vector in $S_X \otimes \mathbb{Q}$ which is the projection for a Leech root. There are two possibilities for such Leech roots r:

(i) r and R ($\cong D_4$) generate a root lattice isomorphic to $D_4 \perp A_1$;

(ii) r and R generate a root lattice isomorphic to D_5.

Lemma 3.2. There are exactly 42 Leech roots which are orthogonal to R. □

Proof. Let R' be the sublattice of L generated by vectors x, y, z from (3.4). Obviously, R' is isomorphic to the root lattice A_3. By Conway [3, 5], the Leech roots orthogonal to R' are

$$4v_\infty + 4v_0, \qquad v_\Omega - 4v_k, \qquad 2v_{K'}, \tag{3.6}$$

where K' contains $\{\infty, 0\}$ and $k \in \{1, 2, \dots, 22\}$.

Among these Leech roots, the followings are orthogonal to vector $t \in R$:

$$4v_\infty + 4v_0, \qquad v_\Omega - 4v_k, \qquad 2v_{K'}, \tag{3.7}$$

where $K' \in S(5, 8, 24)$, $K' \cap K = \{\infty, 0\}$, or $K' \cap K = \{\infty, 0, 1, *\}$, and $k \in K \setminus \{\infty, 0, 1\}$. Obviously, the number of these roots is equal to 42. ■

Remark 3.3. In [17, Table I, page 219], Todd listed the 759 octads of the Steiner system $S(5, 8, 24)$. The octads in the proof of Lemma 3.2 correspond to the following $\{E_\alpha, L_\beta\}$ in Todd's table. Here, we assume that

$$K = \{\infty, 0, 1, 2, 3, 5, 14, 17\}. \tag{3.8}$$

Then

$$\begin{aligned}
&E_1 = \{\infty, 0, 1, 2, 4, 13, 16, 22\}, && E_2 = \{\infty, 0, 1, 2, 6, 7, 19, 21\},\\
&E_3 = \{\infty, 0, 1, 2, 8, 11, 12, 18\}, && E_4 = \{\infty, 0, 1, 2, 9, 10, 15, 20\},\\
&E_5 = \{\infty, 0, 1, 3, 4, 11, 19, 20\}, && E_6 = \{\infty, 0, 1, 3, 6, 8, 10, 13\},\\
&E_7 = \{\infty, 0, 1, 3, 7, 9, 16, 18\}, && E_8 = \{\infty, 0, 1, 3, 12, 15, 21, 22\},\\
&E_9 = \{\infty, 0, 1, 4, 5, 7, 8, 15\}, && E_{10} = \{\infty, 0, 1, 4, 6, 9, 12, 17\},\\
&E_{11} = \{\infty, 0, 1, 4, 10, 14, 18, 21\}, && E_{12} = \{\infty, 0, 1, 5, 6, 18, 20, 22\},\\
&E_{13} = \{\infty, 0, 1, 5, 9, 11, 13, 21\}, && E_{14} = \{\infty, 0, 1, 5, 10, 12, 16, 19\},\\
&E_{15} = \{\infty, 0, 1, 6, 11, 14, 15, 16\}, && E_{16} = \{\infty, 0, 1, 7, 10, 11, 17, 22\},\\
&E_{17} = \{\infty, 0, 1, 7, 12, 13, 14, 20\}, && E_{18} = \{\infty, 0, 1, 8, 9, 14, 19, 22\},\\
&E_{19} = \{\infty, 0, 1, 8, 16, 17, 20, 21\}, && E_{20} = \{\infty, 0, 1, 13, 15, 17, 18, 19\},\\
&L_1 = \{\infty, 0, 4, 6, 8, 16, 18, 19\}, && L_2 = \{\infty, 0, 4, 6, 13, 15, 20, 21\},\\
&L_3 = \{\infty, 0, 4, 7, 9, 10, 13, 19\}, && L_4 = \{\infty, 0, 4, 7, 11, 12, 16, 21\},\\
&L_5 = \{\infty, 0, 4, 8, 10, 12, 20, 22\}, && L_6 = \{\infty, 0, 4, 9, 11, 15, 18, 22\},\\
&L_7 = \{\infty, 0, 6, 7, 8, 9, 11, 20\}, && L_8 = \{\infty, 0, 6, 7, 10, 12, 15, 18\},\\
&L_9 = \{\infty, 0, 6, 9, 10, 16, 21, 22\}, && L_{10} = \{\infty, 0, 11, 12, 13, 16, 19, 22\},\\
&L_{11} = \{\infty, 0, 7, 8, 13, 18, 21, 22\}, && L_{12} = \{\infty, 0, 7, 15, 16, 19, 20, 22\},\\
&L_{13} = \{\infty, 0, 8, 9, 12, 13, 15, 16\}, && L_{14} = \{\infty, 0, 8, 10, 11, 15, 19, 21\},\\
&L_{15} = \{\infty, 0, 9, 12, 18, 19, 20, 21\}, && L_{16} = \{\infty, 0, 10, 11, 13, 16, 18, 20\}.
\end{aligned} \tag{3.9}$$

The remaining 6 Leech roots correspond to

$$4\nu_\infty + 4\nu_0, \qquad \nu_\Omega - 4\nu_k, \tag{3.10}$$

where $k \in \{2, 3, 5, 14, 17\}$.

Lemma 3.4. There are exactly 168 Leech roots r such that r and R generate a root lattice D_5. □

Proof. We count the number of such Leech roots r with $\langle r, t\rangle = 1$. Such r corresponds to one of the following vectors in Λ (see the proof of Lemma 3.2):

$$\nu_\Omega - 4\nu_k \quad (k \notin K), \quad 2\nu_{K'} \quad \left(K' \in S(5, 8, 24),\ \left|K \cap K'\right| = 4,\ 1 \notin K'\right). \tag{3.11}$$

Obviously, the number of vectors of the first type is 16. Since the number of octads containing fixed 4 points is 5, the number of vectors of the second type is 40. Thus, we have the desired number $56 \times 3 = 168$. ■

3.3 The 42 smooth rational curves

Let

$$w = (0, 0, 1) \tag{3.12}$$

be the *Weyl vector* in L (characterized by the property that $\langle w, \lambda \rangle = 1$ for each Leech root). Since the Leech lattice does not contain (-2)-vectors, $\langle w, r \rangle \neq 0$ for any (-2)-vector r in $L = \Lambda \perp U$. Consider its orthogonal projection w' to $R^{\perp} \otimes \mathbb{Q}$. Easy computation gives

$$w' = w + 5z + 3x + 3y + 3t \in R^{\perp},$$
$$\langle w', w' \rangle = 14. \tag{3.13}$$

Fix an isometry from S_X to $R^{\perp}$ and let h be the divisor class corresponding to w'. The above property of w implies that $\langle h, r \rangle \neq 0$ for any (-2)-vectors r in S_X. Composing the embedding with reflections in (-2)-vectors in S_X, we may assume that h is an ample divisor class. Each of the 42 Leech vectors from Lemma 3.2 defines a vector v from $\mathrm{Pic}(X)$ with self-intersection -2. Since $\langle h, v \rangle = 1$, by Riemann-Roch, we obtain that v is the divisor class of a curve R_v with $R_v^2 = -2$. Since h is ample, each R_v is an irreducible curve, and hence isomorphic to $\mathbb{P}^1$.

Thus, the 42 Leech roots in Lemma 3.2 define 42 smooth rational curves in X.

Lemma 3.5. Let X be a K3 surface over a field of characteristic 2. The following properties are equivalent:

(i) the Picard lattice S_X of X is isomorphic to $U \perp D_{20}$;

(ii) X has a quasi-elliptic fibration with 5 singular fibers of type $\tilde{D}_4$ and 16 disjoint sections;

(iii) there are two families $\mathcal{A}$ and $\mathcal{B}$ each consisting of 21 disjoint smooth rational curves. Each member in one family meets exactly five members in another family. The set $\mathcal{A} \cup \mathcal{B}$ generates S_X. □

Proof. (i)⇒(ii) Fix an isometry from S_X to $R^{\perp}$. Consider the five disjoint (-2)-curves K_k corresponding to

$$v_{\Omega} - 4v_k, \quad \big(k \in \{2, 3, 5, 14, 17\}\big). \tag{3.14}$$

Then $|2K_k + R_1 + R_2 + R_3 + R_4|$ gives a genus 1 fibration with five singular fibers of type $\tilde{D}_4$, where R_i correspond to the vectors E_i in Remark 3.3. Moreover, the curves corresponding to the vectors L_i are 16 disjoint sections of this fibration. The fibration satisfies the Rudakov-Shafarevich criterion of quasi-elliptic fibration [14, Section 4].

(ii)⇒(i) It follows from the Shioda-Tate formula [15] that the discriminant of S_X is equal to 4.

(ii)⇒(iii) We take for $\mathcal{A}$ the set of sections and multiple components of fibers. We take for $\mathcal{B}$ the set of nonmultiple components of reducible fibers and the cusp curve. The curves from the set $\mathcal{A}$ correspond to the Leech roots $\{E_\alpha, 4\nu_\infty + 4\nu_0\}$, and the curves from $\mathcal{B}$ correspond to the Leech roots $\{L_\alpha, \nu_\Omega - 4\nu_k\}$. The cusp curve C corresponds to the Leech root $4\nu_\infty + 4\nu_0$ (see Remark 3.7). It follows from the Shioda-Tate formula that the set $\mathcal{A} \cup \mathcal{B}$ generates S_X.

(iii)⇒(ii) Take $R_0 \in \mathcal{A}$ and 4 curves R_1, R_2, R_3, R_4 from $\mathcal{B}$ which intersect R_0. Then

$$F = 2R_0 + R_1 + R_2 + R_3 + R_4 \tag{3.15}$$

defines a quasi-elliptic fibration with a fiber F of type $\tilde{D}_4$. Let N be the fifth curve from $\mathcal{B}$ which intersects R_0. It is easy to see that the curves from $\mathcal{A} \cup \mathcal{B} \setminus \{N\}$ which do not intersect the curves R_1, R_2, R_3, R_4 form four more reducible fibers of type $\tilde{D}_4$. The remaining curves give 16 disjoint sections. ■

Lemma 3.6. Let X be the K3 surface whose Picard lattice is isomorphic to $U \perp D_{20}$ and let $\mathcal{A}$, $\mathcal{B}$ be the sets described in Lemma 3.5. Let h be the divisor class corresponding to the projection of the Weyl vector $w = (0, 0, 1) \in \Lambda \perp U$ to $R^\perp \otimes \mathbb{Q}$. Then

$$h = \frac{1}{3} \sum_{E \in \mathcal{A} \cup \mathcal{B}} E. \tag{3.16}$$

□

Proof. Observe that the right-hand side h' of (3.16) intersects each curve R_i with multiplicity 1. Since the curves R_i generate S_X, $h - h'$ is orthogonal to S_X, and hence is equal to zero. ■

Remark 3.7. Note that the curve C from the proof of Lemma 3.5 is the cusp curve C' of the quasi-elliptic fibration. In fact, since it intersects the double components of the reducible fibers, it is a 2-section. Since the components F_i of fibers and C' generate a sublattice of finite index in S_X, we must have $C \equiv C' + \sum_i n_i F_i$ for some rational coefficients n_i. Intersecting both sides with each F_j, we get $\sum_i n_i (F_i \cdot F_j) = 0$. Since the intersection matrix of irreducible components is semidefinite, this implies that $\sum_i n_i F_i = mF$, where

F is the class of a fiber and $m \in \mathbb{Q}$. But now $(C - C')^2 = -4 - 2(C \cdot C') = (mF)^2 = 0$ gives $C \sim C'$. Since a smooth rational curve does not move, we get $C = C'$.

Remark 3.8. A quasi-elliptic fibration with 5 fibers of type $\tilde{D}_4$ has 16 sections if and only if the Picard lattice has discriminant equal to 4, and hence is isomorphic to $U \perp D_{20}$. This follows immediately from the Shioda-Tate formula. Also, if X has a quasi-elliptic fibration with 5 fibers of type $\tilde{D}_4$ and 16 sections, then the 16 sections are automatically disjoint (note that m-torsion sections on an elliptic or a quasi-elliptic surface over a field of characteristic dividing m are not necessarily disjoint). Hence, the surface contains 42 smooth rational curves with the intersection matrix as described in Lemma 3.5. We now show this. Let C be the cuspidal curve of the fibration and let $F_i = 2E_0(i) + E_1(i) + \cdots + E_4(i)$, $i = 1, \ldots, 5$, be the reducible fibers of the fibration. The divisor $D = 2C + E_0(1) + \cdots + E_0(4)$ is nef and satisfies $D^2 = 0$. Thus it defines a genus 1 pencil $|D|$. No sections intersect D, so they are contained in fibers of the fibration. Also, the components $E_1(5)$, $E_2(5)$, $E_3(5)$, $E_4(5)$ do not intersect D, and hence are contained in fibers of $|D|$. We have 16 sections, so there exists $E_i(5)$, say, $E_1(5)$, such that it intersects at least four of the sections. By inspection of the list of possible reducible fibers of a genus 1 fibration, we find that $E_1(5)$ together with 4 disjoint sections form a fiber of type $\tilde{D}_4$. Now, we have 12 remaining sections, which intersect one of the components $E_i(5)$, $i = 2, 3, 4$. Again, we may assume that $E_2(5)$ intersects at least four of the remaining 12 sections, and hence, we find another fiber of type $\tilde{D}_4$. Continuing in this way, we obtain that the 16 sections together with the curves $E_1(5)$, $E_2(5)$, $E_3(5)$, $E_4(5)$ form 4 fibers of type $\tilde{D}_4$. In particular, they are disjoint.

3.4 The 168 divisors

Let r be a Leech root as in Lemma 3.4. The projection r' of r into $S_X \otimes \mathbb{Q}$ is a (-1)-vector. We can directly see that each r' meets exactly 6 members in each family $\mathcal{A}$ and $\mathcal{B}$ stated in Lemma 3.5. For example, if we use the same notation as in Remark 3.3 and take $(\nu_\Omega - 4\nu_4, 1, 1)$ as r, then r meets exactly twelve Leech roots corresponding to

$$E_1,\ E_5,\ E_9,\ E_{10},\ E_{11},\ 4\nu_\infty + 4\nu_0,\ L_1,\ L_2,\ L_3,\ L_4,\ L_5,\ L_6. \tag{3.17}$$

We remark that these twelve curves are mutually disjoint. We set

$$l = \frac{1}{7}\Big(2h + \sum_{R \in \mathcal{A}} R\Big). \tag{3.18}$$

Then $l \in S_X$, $l^2 = 2$, $l \cdot R = 0$ for any $R \in \mathcal{A}$, $l \cdot R = 1$ for any $R \in \mathcal{B}$, and

$$2r' = 2l - (R_1 + \cdots + R_6), \tag{3.19}$$

where $R_1, \ldots, R_6$ are (-2)-curves in $\mathcal{A}$ which meet r'. Each r' defines an isometry

$$s_{r'} : x \longrightarrow x + 2(r' \cdot x)r' \tag{3.20}$$

of S_X, which is nothing but the reflection with respect to the hyperplane perpendicular to r'.

Consider a quasi-elliptic fibration on X with five reducible fibers of type $\tilde{D}_4$ with 16 sections. Let C be the cusp curve and let F be the class of a fiber. We now take as $\mathcal{A}$ the set of 21 curves consisting of C and simple components of fibers, and as $\mathcal{B}$ the set consisting of 16 sections and multiple components of fibers. Then, we can easily see that $l = C + F$ and

$$\begin{aligned} s_{r'}(l) &= 5l - 2(R_1 + \cdots + R_6),\\ s_{r'}(R_i) &= 2l - (R_1 + \cdots + R_6) + R_i. \end{aligned} \tag{3.21}$$

4 Explicit constructions

4.1 Inseparable double covers

Let $\mathcal{L}$ be a line bundle on a nonsingular surface Y over a field of characteristic 2, and $s \in H^0(Y, \mathcal{L}^2)$. The pair $(\mathcal{L}, s)$ defines a double cover $\pi : Z \to Y$ which is given by local equations $z^2 = f(x, y)$, where (x, y) is a system of local parameters and $f(x, y) = 0$ is the local equation of the divisor of zeroes of the section s. Replacing s with $s + t^2$, where $t \in H^0(Y, \mathcal{L})$, we get an isomorphic double cover. The singular locus of Z is equal to the zero locus of the section $ds \in H^0(Y, \Omega^1_Y \otimes \mathcal{L}^2)$. It is locally given by the common zeroes of the partials of $f(x, y)$. The canonical sheaf of Z is isomorphic to $\pi^*(\omega_Y \otimes \mathcal{L})$. All of these facts are well known (see, e.g., [8, Chapter 0]).

We consider the special case when $Y = \mathbb{P}^2$ and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^2}(3)$. The section s is identified with a homogeneous form $F(x_0, x_1, x_2)$ of degree 6. We assume that $F_6 = 0$ is a reduced plane curve of degree 6. The expected number N of zeroes of the section dF_6 is equal to the second Chern number $c_2(\Omega^1_{\mathbb{P}^2}(6))$. The standard computation gives $c_2 = 21$. So, if $N = 21$, the partial derivatives of any local equation of F_6 at a zero x of ds generate the maximal ideal $\mathfrak{m}_x$. This easily implies that the cover Z has 21 ordinary double points. So, a minimal resolution X of Z has 21 disjoint smooth rational curves.

4.2 The double plane model

To construct our surface X as in Theorem 1.1(vi), we take the inseparable cover corresponding to the plane sextic $F_6 = 0$ with dF_6 vanishing at the 21 points of $\mathbb{P}^2$ defined over $\mathbb{F}_4$. Consider the sextic defined by the equation

$$F_6 = x_0^4 x_1 x_2 + x_1^4 x_0 x_2 + x_2^4 x_0 x_1 = 0. \tag{4.1}$$

Its partial derivatives are

$$x_1^4 x_2 + x_2^4 x_1, \qquad x_0^4 x_2 + x_2^4 x_0, \qquad x_1^4 x_0 + x_0^4 x_1. \tag{4.2}$$

Since $a_i^4 = a_i$ for $a \in \mathbb{F}_4$, we see that all partials vanish at any point in $\mathbb{P}^2(\mathbb{F}_4)$. Thus, the exceptional divisor of a minimal resolution X of the corresponding double cover $X' \to \mathbb{P}^2$ is a set $\mathcal{A}$ of 21 disjoint smooth rational curves. Let $\Sigma(\mathcal{P})$ be the blowup of the set $\mathcal{P}$. Then X is isomorphic to the normalization of the base change $X' \times_{\mathbb{P}^2} \Sigma(\mathcal{P})$. Now let $\check{\mathcal{P}}$ be the set of 21 lines on $\mathbb{P}^2$ defined over $\mathbb{F}_4$. Their equations are $a_0x_0 + a_1x_1 + a_2x_2 = 0$, where $(a_0, a_1, a_2) \in \mathcal{P}$. The 21 lines are divided into three types: three lines which are components of the sextic $F_6 = 0$; 9 lines which intersect the sextic at three of its double points; and 9 lines which intersect the sextic at two points of its double points. In the last case, each line is a cuspidal tangent line of the cubic $x_0^3 + x_1^3 + x_2^3 = 0$. The proper inverse transform $\bar{l}$ of any line in $\Sigma(\mathcal{P})$ is a smooth rational curve with self-intersection -4, which is either disjoint from the proper inverse transform of the sextic, or is tangent to it, or is contained in it. In each case, the preimage of $\bar{l}$ in X is a (-2)-curve taken with multiplicity 2. Let $\mathcal{B}$ be the set of such curves. Since each line $\check{p} \in \check{\mathcal{P}}$ contains exactly 5 points from $\mathcal{P}$, and each point $p \in \mathcal{P}$ is contained in exactly 5 lines from $\check{\mathcal{P}}$, the sets $\mathcal{A}$ and $\mathcal{B}$ satisfy property (v) from Theorem 1.1. Finally, notice that the map $X \to \mathbb{P}^2$ is given by the linear system $|l|$, where l is defined by (3.18).

4.3 A switch

A *switch* is an automorphism of X which interchanges the sets $\mathcal{A}$ and $\mathcal{B}$. We show that it exists. Consider the pencil of cubic curves generated by the cubics $x_0x_1x_2 = 0$ and $x_0^3 + x_1^3 + x_2^3 = 0$. Its base points are

$$\begin{gathered}(1,1,0), \quad (1,a,0), \quad (1,a^2,0), \quad (1,0,1), \quad (1,0,a), \\ (1,0,a^2), \quad (0,1,1), \quad (0,1,a), \quad (0,1,a^2).\end{gathered} \tag{4.3}$$

After blowing up the base points, we obtain a rational elliptic surface V with 4 reducible fibers of type $\tilde{A}_2$. The 12 singular points of the four fibers are the preimages of the points in $\mathcal{P}$ not appearing in (4.3). The surface X is the inseparable cover of the blowup of V at these points. It has an elliptic fibration with 4 fibers of type $\tilde{A}_5$. The base of this fibration is an inseparable double cover of the base of the elliptic fibration on V. Now we find that the Mordell-Weil group is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus (\mathbb{Z}/3\mathbb{Z})^2$. Nine of these sections are curves from the set $\mathcal{A}$ of (-2)-curves corresponding to points in $\mathbb{P}^2(\mathbb{F}_4)$. They correspond to the base points (4.3). Another 9 sections are from the set $\mathcal{B}$ of (-2)-curves corresponding to lines in $\mathbb{P}^2(\mathbb{F}_4)$. They correspond to the lines dual to the points (4.3), that is, the lines $ax_0 + bx_1 + cx_2 = 0$, where (a, b, c) is one of (4.3). Fix a zero section s_0 represented by, say, an $\mathcal{A}$-curve. Let $F_i = \sum_{k\in\mathbb{Z}/6} E_k(i)$, $i = 1, 2, 3, 4$, be the reducible fibers such that $E_k(i) \cdot E_{k+1}(i) = 1$ and s_0 intersects $E_0(i)$. It is easy to see that the 9 sections from $\mathcal{A}$ intersect the components $E_0(i)$, $E_2(i)$, $E_4(i)$, each component is intersected by 3 sections. The $\mathcal{B}$-sections intersect the components $E_1(i)$, $E_3(i)$, $E_5(i)$, again each component is intersected by 3 sections. Also, the components $E_0(i)$, $E_2(i)$, $E_4(i)$ are $\mathcal{B}$-curves and the components $E_1(i)$, $E_3(i)$, $E_5(i)$ are $\mathcal{A}$-curves. Thus, we obtain that $\mathcal{A} \cup \mathcal{B}$ consists of 18 sections and 24 components of fibers. Now, consider the automorphism T of the surface X defined by the translation by the 2-torsion section s_1. Obviously, s_1 intersects the components $E_3(i)$. We see that T interchanges the sets $\mathcal{A}$ and B. Also, note that if s_0 corresponds to (a_0, a_1, a_2), then s_1 is a $\mathcal{B}$-curve corresponding to the line $a_0x_0 + a_1x_1 + a_2x_2 = 0$. To be more specific, let s_0 correspond to the point $p = (1, 1, 0)$. Then s_1 corresponds to the line $l : x_0 + x_1 = 0$. Indeed l is the cuspidal tangent of $F : x_0^3 + x_1^3 + x_2^3 = 0$ at the point p. After we blow up p, the two sections s_0 and s_1 intersect at one point of the fiber represented by F. But this could happen only for the 2-section since the 3-torsion sections do not intersect (we are in characteristic 2). Similarly, we see that T sends a section represented by a point (a, b, c) to the section represented by the dual line $ax_0 + bx_1 + cx_2 = 0$. Also, it is easy to see that T transforms a fiber component corresponding to a point to the component corresponding to the dual line. Thus T is a switch.

We remark that we can directly find a configuration of four fibers of type $\tilde{A}_5$ and 18 sections on 42 smooth rational curves $\mathcal{A} \cup \mathcal{B}$. This implies the existence of such elliptic fibration (and hence a switch) without using the above double plane construction.

4.4 Mukai's model

In a private communication to the second author, Mukai suggested that the surface X is isomorphic to a surface in $\mathbb{P}^2 \times \mathbb{P}^2$ defined by the equations

$$x_0^2y_0 + x_1^2y_1 + x_2^2y_2 = 0, \qquad x_0y_0^2 + x_1y_1^2 + x_2y_2^2 = 0. \tag{4.4}$$

To prove this, we consider the linear system $|h|$, where h is the divisor class representing the projection w' of the Weyl vector $w \in L$. Recall that h is an ample divisor satisfying $h \cdot R = 1$ for any $R \in \mathcal{A} \cup \mathcal{B}$. Consider the quasi-elliptic fibration $|F|$ as in Lemma 3.5. We may assume that the set $\mathcal{A}$ consists of nonmultiple components of fibers and the cusp curve C, and the set $\mathcal{B}$ consists of 16 sections, and 5 multiple components of fibers.

Let $D_1 = 2R_0+R_1+R_2+R_3+R_4$ be a reducible member of $|F|$ and let $S_1, \dots, S_4$ be four sections intersecting R_1. Consider the quasi-elliptic pencil $|F'| = |2R_1 + S_1 + S_2 + S_3 + S_4|$. The set $(\mathcal{B} \setminus \{R_0\}) \cup \{C\}$ is the set of irreducible components of reducible members of $|F'|$. The curve $C' = R_0$ is its cuspidal curve. Now we check that, for any $R \in \mathcal{A} \cup \mathcal{B}$,

$$(h - C - F) \cdot R = \left(C' + F'\right) \cdot R. \tag{4.5}$$

Since the curves R generate the Picard group of X, we see that

$$h = (C + F) + \left(C' + F'\right). \tag{4.6}$$

The linear system $|C+F|$ defines a degree 2 map $\pi_1 : X \to \mathbb{P}^2$ which blows down the curves from the set $\mathcal{A}$ and maps the curves from the set $\mathcal{B}$ to lines. The linear system $|C' + F'|$ defines a degree 2 map $\pi_2 : X \to \mathbb{P}^2$ which blows down the curves from the set $\mathcal{B}$ and maps the curves from $\mathcal{A}$ to lines. Let $\phi : X \to \mathbb{P}^8$ be the map defined by the linear system $|h|$. Using (4.6), we easily see that ϕ maps X isomorphically onto a surface contained in the Segre variety $s(\mathbb{P}^2 \times \mathbb{P}^2)$. We identify X with a surface in $\mathbb{P}^2 \times \mathbb{P}^2$. The restriction of the projections $p_1, p_2 : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^2$ are the maps π_1, π_2 defined by the linear systems $|C + F|$ and $|C'+F'|$. Let l_1, l_2 be the standard generators of $\mathrm{Pic}(\mathbb{P}^2 \times \mathbb{P}^2)$ and $[X] = a l_1^2 + b l_1 \cdot l_2 + c l_2^2$ be the class of X in the Chow ring of $\mathbb{P}^2 \times \mathbb{P}^2$. Intersecting $[X]$ with l_1^2 and l_2^2, we get $a = c = 2$. Since $h^2 = 14$, we get

$$14 = [X] \cdot \left(l_1 + l_2\right)^2 = \left(2l_1^2 + b l_1 \cdot l_2 + 2l_2^2\right) \cdot \left(l_1^2 + 2l_1 \cdot l_2 + l_2^2\right) = 4 + 2b. \tag{4.7}$$

Thus

$$[X] = 2l_1^2 + 5l_1 \cdot l_2 + 2l_2^2 = \left(2l_1 + l_2\right) \cdot \left(l_1 + 2l_2\right). \tag{4.8}$$

This shows that X is a complete intersection of two hypersurfaces V_1, V_2 of bidegree $(2,1)$ and $(1,2)$.

The image of each curve from the set $\mathcal{A}$ (resp., $\mathcal{B}$) in $\mathbb{P}^2 \times \mathbb{P}^2$ is a line contained in a fiber of the projection p_1 (resp., p_2). The fibers of the projection $p_1 : V_1 \to \mathbb{P}^2$ define a linear system $|L|$ of conics in $\mathbb{P}^2$. Let V be the Veronese surface parametrizing double lines. The intersection $|L| \cap V$ is either a subset of a conic, or is the whole $|L|$. Since $|L|$ contains at least 21 fibers which are double lines that are not on a conic, we see that $|L| \subset V$. Thus all fibers of $p_1 : V_1 \to \mathbb{P}^2$ are double lines. This implies that the equation of V_1 can be chosen in the form

$$A_0 y_0^2 + A_1 y_1^2 + A_2 y_2^2 = 0, \tag{4.9}$$

where the coefficients are linear forms in x_0, x_1, x_2. It is easy to see that the linear forms must be linearly independent (otherwise, X is singular). Thus, after a linear change of the variables x_0, x_1, x_2, we may assume that $A_i = x_i,\ i = 0, 1, 2$.

Now consider a switch $T \in \mathrm{Aut}(X)$ constructed in Section 4.3. Obviously, it interchanges the linear systems $|C+F|$ and $|C'+F'|$, and hence is induced by the automorphism s of $\mathbb{P}^2 \times \mathbb{P}^2$, which switches the two factors. This shows that $s(V_1) = V_2$, hence the equation of V_2 is

$$y_0 x_0^2 + y_1 x_1^2 + y_2 x_2^2 = 0. \tag{4.10}$$

We remark that the curves

$$(x_0, x_1, x_2) = (a_0, a_1, a_2), \quad a_0 y_0^2 + a_1 y_1^2 + a_2 y_2^2 = 0, \quad (a_0, a_1, a_2) \in \mathbb{P}^2(\mathbb{F}_4) \tag{4.11}$$

and their images under the switch form 42 smooth rational curves on $X = V_1 \cap V_2$ satisfying Theorem 1.1(v).

4.5 The quartic model

Consider the quartic curve in $\mathbb{P}^2$ defined by the equation

$$\begin{aligned} F_4(x_0, x_1, x_2) &= x_0^4 + x_1^4 + x_2^4 + x_0^2 x_1^2 + x_0^2 x_2^2 + x_1^2 x_2^2 + x_0^2 x_1 x_2 + x_0 x_1^2 x_2 + x_0 x_1 x_2^2 \\ &= 0. \end{aligned} \tag{4.12}$$

This is a unique quartic curve defined over $\mathbb{F}_2$, which is invariant with respect to the projective linear group $\mathrm{PGL}(3, \mathbb{F}_2) \cong \mathrm{PSL}(2, \mathbb{F}_7)$ (see [9]). Let Y be the quartic surface in $\mathbb{P}^3$ defined by the equation

$$x_3^4 + F_4(x_0, x_1, x_2) = 0. \tag{4.13}$$

Clearly, the group $\mathrm{PGL}(3,\mathbb{F}_2)$ acts on Y by projective transformations leaving the plane $x_3 = 0$ invariant.

Taking the derivatives, we find that Y has 7 singular points,

$$\begin{aligned}(x_0,x_1,x_2,x_3) = {} & (1,1,0,1),(1,0,0,1),(0,1,0,1),\\ & (0,0,1,1),(0,1,1,1),(1,0,1,1),(1,1,1,1).\end{aligned} \tag{4.14}$$

We denote by $\{P_i\}_{1\le i\le 7}$ the set of these singular points. Each singular point is locally isomorphic to the singular point $z^4 + xy = 0$, that is, a rational double point of type A_3. Let X be a minimal resolution of Y. We claim that X is isomorphic to our surface. Note that the points in $\mathbb{P}^2$ whose coordinates are the first three coordinates of singular points (4.14) are the seven points of $\mathbb{P}^2(\mathbb{F}_2)$. Let $a_0x_0 + a_1x_1 + a_2x_2 = 0$ be one of the seven lines of $\mathbb{P}^2(\mathbb{F}_2)$. The plane $a_0x_0 + a_1x_1 + a_2x_2 = 0$ in $\mathbb{P}^3$ intersects Y doubly along a conic which passes through 3 singular points of Y. For example, the plane $x_0 + x_1 + x_2 = 0$ intersects Y along the conic given by the equations

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_0x_1 + x_0x_2 + x_1x_2 = x_0 + x_1 + x_2 = 0. \tag{4.15}$$

We denote by $\{C_i'\}_{1\le j\le 7}$ the set of these conics and by $I(j)$ the set of indices i with $P_i \in C_j'$.

Let $R_1^{(i)} + R_2^{(i)} + R_3^{(i)}$ be the exceptional divisor of a singular point P_i. We assume that $R_2^{(i)}$ is the central component. It is easy to check that the proper inverse transform C_j of each conic C_j' in X intersects $R_2^{(i)}$ with multiplicity 1, if $P_i \in C_j'$, and does not intersect other components. This easily gives

$$h = 2C_j + \sum_{i\in I(j)} \left(R_1^{(i)} + 2R_2^{(i)} + R_3^{(i)}\right), \tag{4.16}$$

where h is the preimage in X of the divisor class of a hyperplane section of Y, and we identify the divisor classes of (-2)-curves with the curves.

Observe now that X contains a set $\mathcal{A}$ of 21 disjoint smooth rational curves, seven curves C_i and 14 curves $R_1^{(i)}, R_3^{(i)}$. Each of the seven curves $R_2^{(i)}$ intersects exactly 5 curves from the set $\mathcal{A}$ (with multiplicity 1). We will exhibit the additional 14 smooth rational curves which together with these 7 curves form a set $\mathcal{B}$ of 21 disjoint (-2)-curves such that $\mathcal{A}\cup\mathcal{B}$ satisfies Theorem 1.1(v).

To do this, we take a line in $\mathbb{P}^2(\mathbb{F}_2)$, for example, $x_1 = 0$. Then there are exactly 4 points on $\mathbb{P}^2(\mathbb{F}_2)$ not lying on this line

$$(x_0,x_1,x_2) = (0,1,0),(1,1,0),(0,1,1),(1,1,1). \tag{4.17}$$

The plane $H : x_1 + x_3 = 0$ in $\mathbb{P}^3$ passes through the 4 singular points of the quartic surface Y

$$P_1 = (0,1,0,1), \qquad P_2 = (1,1,0,1), \qquad P_3 = (0,1,1,1), \qquad P_4 = (1,1,1,1) \tag{4.18}$$

and intersects Y along a quartic curve Q' given by the equations

$$x_0^4 + x_2^4 + x_0^2x_1^2 + x_1^2x_2^2 + x_2^2x_0^2 + x_0x_1x_2(x_0 + x_1 + x_2) = x_1 + x_3 = 0. \tag{4.19}$$

It splits into the union of 2 conics

$$\begin{aligned} Q_1' &: x_0^2 + ax_2^2 + x_0x_1 + ax_1x_2 = x_1 + x_3 = 0, \\ Q_2' &: x_0^2 + a^2x_2^2 + x_0x_1 + a^2x_1x_2 = x_1 + x_3 = 0. \end{aligned} \tag{4.20}$$

Let C' be one of the 7 conic curves C_i''s corresponding to the line $x_1 = 0$. It is given by the equations

$$x_0^2 + x_2^2 + x_3^2 + x_0x_2 = x_1 = 0. \tag{4.21}$$

Then H meets C' at $q_1 = (1,0,a,0),\ q_2 = (1,0,a^2,0)$. Note that Q_1' passes through $P_1,\dots,P_4,q_1$ and Q_2' passes through $P_1,\dots,P_4,q_2$. Each singular point P_i of the quartic Y is locally isomorphic to $z^4 + xy = 0$ and the local equation of the quartic Q' at this point is $z = 0$. This easily shows that the proper inverse transform of Q' in X consists of two smooth rational curves Q_1 and Q_2, each intersects simply the exceptional divisor at one point lying in different components of $R_1^{(i)} \cup R_3^{(i)}$. Thus Q_1 (or Q_2) intersects exactly five curves from $\mathcal{A}$, 4 curves $R_j^{(i)}$ ($j = 1$ or 3, $i = 1,\dots,4$) and the proper inverse transform of C' in X. Also, it is clear that 14 new quartic curves obtained in this way neither intersect each other, after we resolve the singularities of the quartic, nor intersect the curves $R_2^{(i)}$. This proves the claim.

Remark 4.1. The configuration $\mathcal{C}$ of 14 curves $C_i, R_2^{(i)}$ is isomorphic to the configuration of points and lines in $\mathbb{P}^2(\mathbb{F}_2)$. The group $PGL(3,\mathbb{F}_2)$ acts on the surface X via its linear action in $\mathbb{P}^3$ leaving the hyperplane $x_3 = 0$ fixed. Its action on the configuration $\mathcal{C}$ is isomorphic to its natural action on lines and points.

5 Automorphisms of X

In this section, we describe the group of automorphisms of the surface X. First, we exhibit some automorphisms of X and then prove that they generate the group $\mathrm{Aut}(X)$.

5.1 The group $\mathrm{PGL}(3,\mathbb{F}_4)$

Consider the double-plane model of X with the branch curve $F_6 = 0$ as in (4.1). The group $G = \mathrm{PSL}(3,\mathbb{F}_4)$ acts naturally on the plane $\mathbb{P}^2(\mathbb{F}_4)$. For any $g \in \mathrm{GL}(3,\mathbb{F}_4)$, let $P_g = F_6(g(x))$. Let

$$g(x_0,x_1,x_2) = (a_0x_0 + a_1x_1 + a_2x_2, b_0x_0 + b_1x_1 + b_2x_2, c_0x_0 + c_1x_1 + c_2x_2). \quad (5.1)$$

After substituting, we obtain that

$$P_g(x) = \alpha_0 x_0^4 x_1 x_2 + \alpha_1 x_1^4 x_0 x_2 + \alpha_2 x_2^4 x_0 x_1 + A_g^2, \quad (5.2)$$

where A_g is a cubic polynomial, and

$$\begin{aligned} \alpha_0 &= a_0^4(b_1c_2 + b_2c_1) + b_0^4(a_1c_2 + a_2c_1) + c_0^4(a_1b_2 + a_2b_1), \\ \alpha_1 &= a_1^4(b_0c_2 + b_2c_0) + b_1^4(a_0c_2 + a_2c_0) + c_1^4(a_2b_0 + a_0b_2), \\ \alpha_2 &= a_2^4(b_1c_0 + b_0c_1) + b_2^4(a_1c_0 + a_0c_1) + c_2^4(a_1b_0 + a_0b_1). \end{aligned} \quad (5.3)$$

Since $x^4 = x$ for all $x \in \mathbb{F}_4$, we see that

$$\alpha_0 = \alpha_1 = \alpha_2 = \det(g). \quad (5.4)$$

Thus, the map $T_g : (z,x) \to (\det(g)^2 z + A_g(x), g(x))$ is an automorphism of the double plane. It is easy to verify that

$$A_{g'g}(x) = \det(g')^2 A_g(x) + A_{g'}(g(x)). \quad (5.5)$$

This implies that the map $g \mapsto T_g$ defines an action of $\mathrm{GL}(3,\mathbb{F}_4)$ on the double plane. Obviously, it factors through an action of $\mathrm{PGL}(3,\mathbb{F}_4)$. Since $G = \mathrm{PSL}(3,\mathbb{F}_4)$ is simple, the action of G is faithful. We can easily see that $\mathrm{PGL}(3,\mathbb{F}_4)$ is also faithful.

Note that the induced actions of G on the sets $\mathcal{A}$ and $\mathcal{B}$ is isomorphic to the actions of G on points and lines in $\mathbb{P}^2(\mathbb{F}_4)$.

5.2 The 168 Cremona transformations

Let $P_1,\ldots,P_6$ be 6 points in $\mathbb{P}^2(\mathbb{F}_4)$ such that no three among them are colinear. Since each smooth conic over $\mathbb{F}_4$ contains exactly 5 points (it is isomorphic to $\mathbb{P}^1_{\mathbb{F}_4}$), we see that the set $\mathcal{P} = \{P_1,\ldots,P_6\}$ is not on a conic. This allows us to define a unique involutive quintic

Cremona transformation Φ given by the linear system of curves of degree 5 defined over $\mathbb{F}_4$ with double points at the points of $\mathcal{P}$ (see [7, Book IV, Chapter VII, Section 4]). Let C_i be the conic containing the set $\mathcal{P} \setminus \{P_i\}$, $i = 1, \dots, 6$. The transformation Φ blows down each conic C_i to a point Q_i of $\mathbb{P}^2(\mathbb{F}_4)$. Let $B : F_6 = 0$ be the branch curve (4.1) of the double cover $X \to \mathbb{P}^2$. By adding to F_6 the square of a cubic form, we may assume that each point from $\mathcal{P}$ is a double point of the curve B. Since Φ^{-1} (a line) is a quintic with double points at $\mathcal{P}$, the image $B' = \Phi(B)$ of B is a curve of degree $5 \cdot 6 - 4 \cdot 6 = 6$. Each point Q_i is a double point of B'. Let $F_6'(y_0, y_1, y_2) = 0$ be the equation of B' and let Φ be given by homogeneous polynomials of degree 5

$$(y_0, y_1, y_2) = (f_0(x_0, x_1, x_2), f_1(x_0, x_1, x_2), f_2(x_0, x_1, x_2)). \tag{5.6}$$

Then

$$F_6'(y_0, y_1, y_2) = F_6(x_0, x_1, x_2) \prod_{i=1}^{6} q_i(x_0, x_1, x_2)^2, \tag{5.7}$$

where $q_i(x_0, x_1, x_2) = 0$ are the equations of the exceptional conics C_i. Taking the partials, we find

$$\sum_{i=0}^{2} \frac{\partial F_6'}{\partial y_i} \frac{\partial f_i}{\partial x_j} = q^2 \frac{\partial F_6}{\partial x_j}, \quad j = 0, 1, 2, \tag{5.8}$$

where $q = \prod_{i=1}^{6} q_i$. Let $P \in \mathbb{P}^2(\mathbb{F}_4) \setminus \mathcal{P}$. We know that the partials of F_6 vanish at P. Since the determinant of the Jacobian matrix $(\partial f_i / \partial x_j)$ of Φ is invertible outside the locus $q = 0$, we obtain that the partials of F_6' vanish at $\Phi(P)$. This shows that the partials of F_6' vanish at all points of $\mathbb{P}^2(\mathbb{F}_4)$. Using the same argument as in Section 5.1, we find that $F_6' = \alpha F_6 + F_3^2$. Since we chose Φ so that Φ^2 is the identity, we get $\alpha^2 = 1$, and hence $\alpha = 1$. Now, we can define a birational transformation of the double plane by the formula $\tilde{\Phi}(z, x) = (z + F_3, \Phi(x))$. This birational automorphism extends to a regular automorphism (since X is a minimal model).

Next, we show that the number of sets $\mathcal{P}$ is equal to 168. We say that a subset of $\mathbb{P}^2(\mathbb{F}_4)$ is *independent* if no three points from it are colinear. Let N_k be the number of independent subsets of $\mathbb{P}^2(\mathbb{F}_4)$ of cardinality k. We have

$$N_1 = 21, \qquad N_2 = \frac{20N_1}{2}, \qquad N_3 = \frac{16N_2}{3}, \qquad N_4 = \frac{9N_3}{4},$$
$$N_5 = \frac{2N_4}{5}, \qquad N_6 = \frac{N_5}{6} = \frac{21 \cdot 20 \cdot 16 \cdot 9 \cdot 2}{6!} = 168. \tag{5.9}$$

Finally, we remark that the above 168 automorphisms act on the Picard lattice S_X as reflections with respect to 168 (-4)-vectors stated in Section 3.4. This follows from (3.21).

We denote by N the normal subgroup of Aut(X) generated by 168 involutions.

5.3 The automorphism group

It is known that the natural map from Aut(X) to $O(S_X)$ is injective (see [14, Section 8, Proposition 3]). Moreover, Aut(X) preserves the ample cone, and hence Aut(X) is a subgroup of the factor group $O(S_X)/W(S_X)^{(2)}$, where $W(S_X)^{(2)}$ is the group generated by (-2)-reflections. By the argument in [11, Lemma 7.3], we can see that, for any isometry g in $O(S_X)$ preserving an ample class, there exists an automorphism $\varphi \in N$ such that $g \circ \varphi \in \mathrm{Aut}(\mathcal{D}(X))$. This implies that $O(S_X)/W(S_X)^{(2)}$ is a subgroup of a split extension of N by $\mathrm{Aut}(\mathcal{D}(X))$. Recall that $\mathrm{Aut}(\mathcal{D}(X)) \cong M_{21} \cdot D_{12}$ (see Section 3.2). Here $M_{21} = \mathrm{PSL}(3, \mathbb{F}_4)$. The Frobenius automorphism of $\mathbb{F}_4$ gives an involution on 21 lines and 21 points in $\mathbb{P}^2(\mathbb{F}_4)$. This involution induces an isometry ι of S_X because 42 smooth rational curves generate S_X (Lemma 3.6). The dihedral group D_{12} is generated by ι, a switch, and an automorphism of order 3 induced from a projective transformation of $\mathbb{P}^2(\mathbb{F}_4)$ given by

$$\begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.10}$$

We will show that ι cannot be represented by an automorphism of X. To do this, we consider a quasi-elliptic fibration with 5 fibers of type $\tilde{D}_4$ from Lemma 3.5. Then ι fixes 3 fibers of type $\tilde{D}_4$ and switches the remaining 2 fibers. This does not happen if ι is realized as an automorphism, since otherwise ι will induce a nontrivial automorphism of $\mathbb{P}^1$ with 3 fixed points.

Thus, we conclude that

$$\mathrm{Aut}(X) \cong N \cdot \mathrm{PGL}\left(3, \mathbb{F}_4\right) \cdot 2, \tag{5.11}$$

where the involution 2 is generated by a switch.

Corollary 5.1. The finite group $\mathrm{PGL}(3, \mathbb{F}_4) \cdot 2$ is maximal in the following sense. Let G be a finite group of automorphisms of X, then G is conjugate to a subgroup of $\mathrm{PGL}(3, \mathbb{F}_4) \cdot 2$. □

Proof. Note that G fixes the vector

$$\tilde{h} = \sum_{g \in G} g^*(h), \tag{5.12}$$

which is nonzero because h is an ample class and G is an automorphism group. The vector $\tilde{h}$ is conjugate to a vector in $\mathcal{D}(X)$ under N (see Section 5.3). This means that G is conjugate to a subgroup of $\mathrm{Aut}(\mathcal{D}(X))$. Now the assertion follows. ■

Conjecture 5.2. Let X be a K3 surface over an algebraically closed field k of characteristic p admitting the group $\mathrm{PGL}(3, \mathbb{F}_4) \cdot 2$ as its group of automorphisms. Then $p = 2$ and X is isomorphic to the surface X from Theorem 1.1. □

6 Proof of Theorem 1.1

By definition, X satisfies property (i) (see Section 2.1). By Proposition 2.1, X satisfies properties (ii) and (iii). By Lemma 3.5, X satisfies properties (iv) and (v). Property (vi) follows from Section 4.2. Property (vii) was proven in Section 4.5 and property (viii) in Section 4.4. The group of automorphisms was computed in Section 5. The uniqueness follows from the fact that the Artin invariant σ is equal to 1.

It remains to prove the equivalence of properties (i)–(viii).

(ii)⇒(i). The components of fibers and the zero section define the sublattice $U \perp D_{20} \subset \mathrm{Pic}(X)$ of rank 22. If the assertion is false, Pic(X) would be unimodular. However, there are no even unimodular lattices of rank 22 of signature $(1, 21)$.

(i)⇒(ii) An isotropic vector f from U can be transformed with the help of (-2)-reflections into the class of a fiber of a genus 1 fibration. Let $e_1, \ldots, e_{20}$ be a positive root basis of D_{20}. Without loss of generality, we may assume that e_1 is effective. This will imply that all e_i's are effective. Since $f \cdot e_i = 0$ for all i's, the irreducible components of e_i's are contained in fibers of the genus 1 fibration. Since the rank of the subgroup of Pic(X) generated by irreducible components of fibers is at most 20, we see that all e_i's are irreducible (-2)-curves. They are all contained in one fiber which must be of type $\tilde{D}_{20}$. If f' is an isotropic vector from U with $\langle f, f' \rangle = 1$, the class of $f' - f$ gives the class of a section. A theorem from [14, Section 4] implies that the fibration is quasi-elliptic.

(i)⇔(iv) follows from Lemma 3.5.

(iv)⇔(v) follows from Lemma 3.5.

(ii)⇔(iii) follows from Proposition 2.1.

(vi)⇔(i) follows from Section 4.2.

(vii)⇔(i) follows from Section 4.5.

(i)⇔(viii) follows from Section 4.4 and the uniqueness of X.

This finishes the proof of Theorem 1.1.

Remark 6.1. If we take a sublattice $A_2 \perp A_2$ in $U \perp \Lambda$, its orthogonal complement is isomorphic to $U \perp E_8 \perp E_6 \perp E_6$. This lattice is isometric to the Picard lattice of the supersingular K3 surface in characteristic 3 with the Artin invariant 1. It is known that this K3 surface is isomorphic to the Fermat quartic surface [16, Example 5.2]. Using the same method as in this paper, we can see that the projection of the Weyl vector is the class of a hyperplane section of the Fermat quartic surface, 112 lines on the Fermat quartic surface can be written in terms of Leech roots and the projective automorphism group $PGU(4, \mathbb{F}_3)$ of the Fermat quartic surface appears as a subgroup of $Aut(\mathcal{D})$.

Remark 6.2. A lattice is called *reflective* if its reflection subgroup is of finite index in the orthogonal group. The Picard lattice $U \perp D_{20}$ is reflective. This was first pointed out by Borcherds [2] and it is the only known example (up to scaling) of an even reflective lattice of signature $(1, 21)$.

Esselmann [10] determined the range of possible ranks r of even reflective lattices of signature $(1, r-1)$ as $1 \leq r \leq 20$ or $r = 22$. This is the same as the possible ranks of the Picard lattices of K3 surfaces. Of course, the rank $r = 22$ occurs only when the characteristic is positive.

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References

[1] M. Artin, *Supersingular* K3 *surfaces*, Ann. Sci. École Norm. Sup. (4) **7** (1974), 543–567.

[2] R. Borcherds, *Automorphism groups of Lorentzian lattices*, J. Algebra **111** (1987), no. 1, 133–153.

[3] J. H. Conway, *Three lectures on exceptional groups*, Finite Simple Groups (Proc. Instructional Conf., Oxford, 1969), Academic Press, London, 1971.

[4] ———, *The automorphism group of the* 26*-dimensional even unimodular Lorentzian lattice*, J. Algebra **80** (1983), no. 1, 159–163.

[5] J. H. Conway and N. J. A. Sloane, *Sphere Packings*, *Lattices and Groups*, 3rd ed., Grundlehren der Mathematischen Wissenschaften, vol. 290, ch. 10, Springer-Verlag, New York, 1999.

[6] ———, *Sphere Packings*, *Lattices and Groups*, 3rd ed., Grundlehren der Mathematischen Wissenschaften, vol. 290, ch. 27, Springer-Verlag, New York, 1999.

[7] J. L. Coolidge, *A Treatise on Algebraic Plane Curves*, Dover Publications, New York, 1959.

[8] F. R. Cossec and I. V. Dolgachev, *Enriques Surfaces. I*, Progress in Mathematics, vol. 76, Birkhäuser Boston, Massachusetts, 1989.

[9] L. E. Dickson, *A fundamental system of invariants of the general modular linear group with a solution of the form problem*, Trans. Amer. Math. Soc. **12** (1911), no. 1, 75–98.

[10] F. Esselmann, *Über die maximale Dimension von Lorentz-Gittern mit coendlicher Spiegelungsgruppe* [*On the maximal dimension of Lorentzian lattices with cofinite reflection group*], J. Number Theory **61** (1996), no. 1, 103–144 (German).

[11] S. Kondō, *The automorphism group of a generic Jacobian Kummer surface*, J. Algebraic Geom. **7** (1998), no. 3, 589–609.

[12] S. Mukai, *Finite groups of automorphisms of* K3 *surfaces and the Mathieu group*, Invent. Math. **94** (1988), no. 1, 183–221.

[13] A. N. Rudakov and I. R. Shafarevich, *Supersingular* K3 *surfaces over fields of characteristic* 2, Izv. Akad. Nauk SSSR Ser. Mat. **42** (1978), no. 4, 848–869, reprinted in I. R. Shafarevich, *Collected Mathematical Papers*, Springer-Verlag, Berlin, 1989, pp. 614–632.

[14] ———, *Surfaces of type* K3 *over fields of finite characteristic*, Current Problems in Mathematics, Vol. 18, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1981, pp. 115–207, reprinted in I. R. Shafarevich, *Collected Mathematical Papers*, Springer-Verlag, Berlin, 1989, pp. 657–714.

[15] T. Shioda, *On elliptic modular surfaces*, J. Math. Soc. Japan **24** (1972), 20–59.

[16] ———, *Supersingular* K3 *surfaces*, Algebraic Geometry (Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978), Lecture Notes in Mathematics, vol. 732, Springer, Berlin, 1979, pp. 564–591.

[17] J. A. Todd, *A representation of the Mathieu group* M_{24} *as a collineation group*, Ann. Mat. Pura Appl. (4) **71** (1966), 199–238.

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