# On Elements of Prime Order in the Plane Cremona Group over a Perfect Field

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We show that the plane Cremona group over a perfect field $k$ of characteristic $p \geq 0$ contains an element of prime order $\ell \geq 7$ not equal to $p$ if and only if there exists a two-dimensional algebraic torus $T$ over $k$ such that $T(k)$ contains an element of order $\ell$. If $p = 0$ and $k$ does not contain a primitive $\ell$th root of unity, we show that there are no elements of prime order $\ell > 7$ in $\mathrm{Cr}_2(k)$ and all elements of order 7 are conjugate.

## 1  Introduction

The classification of conjugacy classes of elements of prime order $\ell$ in the plane Cremona group $\mathrm{Cr}_2(k)$ over an algebraically closed field $k$ of characteristic 0 has been known for more than a century. The possible orders of elements not conjugate to a projective transformation are $2, 3$, and $5$ (see [6] and historic references there). Much less is known in the case when the field $k$ is not algebraically closed. For example, when $k = \mathbb{Q}$, there are no elements of prime order $\ell > 3$ that are conjugate to projective automorphisms of the plane. However, there exists an element of order 5 in $\mathrm{Cr}_2(\mathbb{Q})$ that acts biregulary on a rational Del Pezzo surface of degree 5. The first example of a birational automorphism of $\mathbb{P}^2_{\mathbb{Q}}$ of order 7 was constructed by J.-P. Serre. It is realized as an automorphism of a

rational Del Pezzo surface of degree 6. He also raised the question about the existence of birational automorphisms of prime order $\ell > 7$. As a byproduct of results of this article, we prove that $Cr_2(\mathbb{Q})$ does not contain elements of prime order $\ell > 7$ and all elements of order 7 are conjugate in $Cr_2(\mathbb{Q})$.

The main result of the article is the following theorem.

**Theorem 1.** Let $k$ be a perfect field of characteristic $p \geq 0$. Then $Cr_2(k)$ contains an element of prime order $\ell > 5$ not equal to $p$ if and only if there exists a two-dimensional algebraic $k$-torus $T$ such that $T(k)$ contains an element of order $\ell$.

Recall that an algebraic $k$-torus is an affine algebraic group $T$ over $k$ that becomes isomorphic to the torus $\mathbb{G}_m^n$ over an algebraic closure $\bar{k}$ of $k$. In dimension two all $k$-tori are rational over $k$ [16], hence $T(k)$ is contained in $Cr_2(k)$. The following result of J.-P. Serre from [15], Theorem 4, gives a necessary and sufficient condition in order that an algebraic $k$-torus contains an element of order $\ell$ coprime to $p$.

**Theorem 2.** Let $k$ be a field of characteristic $p \geq 0$ and $\ell \neq p$ a prime number. Then the following assertions are equivalent:

(i)   there exists a two-dimensional algebraic $k$-torus $T$ such that $T(k)$ contains an element of order $\ell$;

(ii)  the degree of the cyclotomic extension $k(\zeta_\ell)$ over $k$ takes values in the set $\{1, 2, 3, 4, 6\}$. Here $\zeta_\ell$ is a generator of the group of $\ell$th roots of unity in $\bar{k}$.   □

We will also prove the following uniqueness result.

**Theorem 3.** Assume that $k$ is of characteristic 0 and does not contain a primitive $\ell$th root of unity. Then $Cr_2(k)$ does not contain elements of prime order $\ell > 7$ and all elements of order 7 in $Cr_2(k)$ are conjugate to an automorphism of a Del Pezzo surface of degree 6.   □

We are using the same tools applied in [6] for classification of conjugacy classes of finite subgroups of $Cr_2(\mathbb{C})$ adjusted to the case when the ground field is not algebraically closed. As in [6], we show that there is a bijection between the conjugacy classes of finite subgroups $G$ of $Cr_2(k)$ and $G$-equivariant birational isomorphism classes of projective $k$-rational smooth surfaces $X$ equipped with a biregular action of $G$. Then we use an equivariant version of two-dimensional Mori theory of minimal models over $k$. It allows us to describe minimal pairs $(X, G)$.

The authors believe that, granted time and patience, the methods of [6] and [15] can be used for classification of the conjugacy classes of all finite subgroups of $Cr_2(k)$.

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## 2    Minimal Rational $G$-Surfaces

Let $k$ be a field and $\bar{k}$ be its algebraic closure. A *geometrically rational surface $X$* is a smooth projective surface over $k$ such that $\bar{X}$ is birationally isomorphic to $\mathbb{P}^2_{\bar{k}}$. Here and below, for any scheme $Z$ over $k$ we denote by $\bar{Z}$ the scheme $Z \times_k \operatorname{Spec} \bar{k}$. A geometrically rational surface is called *$k$-rational* if it is $k$-birational to $\mathbb{P}^2_k$.

The following result is a culmination of several results due to V. Iskovskikh and Yu. Manin. Its modern proof based on the theory of elementary links can be found in [7], Section 4, p. 642.

**Theorem 4.**    A minimal geometrically rational surface $X$ over a perfect field $k$ is $k$-rational if and only if the following two conditions are satisfied:

(i)    $X(k) \neq \emptyset$;

(ii)    $d = K_X^2 \geq 5$. $\hfill\square$

Let $G$ be a finite subgroup of $\operatorname{Aut}_k(X)$. For any extension $k'/k$, the group $G$ acts naturally on $X_{k'} = X \times_k \operatorname{Spec} k'$ and hence can be considered as a subgroup of $\operatorname{Aut}_{k'}(X_{k'})$.

A *geometrically rational $G$-surface* is a pair $(X, G)$, where $X$ is a geometrically rational surface over $k$ and $G$ is a finite subgroup of $\operatorname{Aut}_k(X)$. A geometrically rational $G$-surface $(X, G)$ is called minimal if any $G$-equivariant birational $k$-morphism $X \to X'$ to a geometrically rational $G$-surface $X'$ is a $k$-isomorphism. In the case when $G = \{1\}$, a minimal surface is just a $k$-minimal surface in the sense of the theory of minimal models. If $G \neq \{1\}$, a minimal rational $G$-surface is not necessarily a $k$-minimal surface.

For lack of reference, we will prove the following theorem that is essentially due to S. Mori.

**Theorem 5.**    Let $(X, G)$ be a minimal geometrically rational $G$-surface over a perfect field $k$. Then one of the following two cases occurs:

(i) $X$ is a Del Pezzo surface with $\mathrm{Pic}(X)^G \cong \mathbb{Z}$ of degree $d := K_X^2 \neq 7$;

(ii) $X$ admits a $G$-equivariant conic bundle structure with $\mathrm{Pic}(X)^G \cong \mathbb{Z}^2$ generated, after tensoring with $\mathbb{Q}$, by $K_X$ and the class of a fiber. □

Recall that a conic bundle structure on a geometrically rational surface $X$ is a morphism $\phi : X \to C$, where $C$ is a smooth genus 0 curve (not necessary isomorphic to $\mathbb{P}_k^1$) and all fibers over closed points $ts$ are isomorphic to reduced conics over the residue field $k(t)$ of $t$. A nonsmooth fiber over a closed point $t$ becomes isomorphic to a bouquet of two $\mathbb{P}^1$'s over a quadratic extension of the residue field $k(t)$ of $t$. If $c$ denotes the number of singular geometric fibers of $\phi$, then $d := K_X^2 = 8 - c$.

Let us first introduce the main attributes of Mori theory. These are the space $N(X) = \mathrm{Pic}(X) \otimes \mathbb{R}$, the cone $\overline{NE}(X) \subset N(X)$ spanned by effective divisor classes in $\mathrm{Pic}(X)$ and the part $\overline{NE}(X)_{K_X \geq 0}$ of $\overline{NE}(X)$ defined by the inequality $K_X \cdot D \geq 0$.

Let $N(\bar{X}), \overline{NE}(\bar{X})$ and $\overline{NE}(\bar{X})_{K_{\bar{X}} \geq 0}$ be the similar objects for the surface $\bar{X}$. The Galois group $\Gamma_k = \mathrm{Gal}(\bar{k}/k)$ of $k$ acts naturally on $\bar{X}$ and hence acts on $N(\bar{X})$ and $\overline{NE}(\bar{X})$. The action factors through the action of a finite quotient of $\Gamma_k$ and we have ([12, Proposition 2.6])

$$N(\bar{X})^{\Gamma_k} = N(X), \quad \overline{NE}(\bar{X})^{\Gamma_k} = \overline{NE}(X), \quad \overline{NE}(\bar{X})_{K_{\bar{X}} \geq 0}^{\Gamma_k} = \overline{NE}(X)_{K_X \geq 0},$$
$$\overline{NE}(X) = \overline{NE}(X)_{K_X \geq 0} + \mathbb{R}_{\geq 0}[\ell_1] + \cdots + \mathbb{R}_{\geq 0}[\ell_s], \tag{1}$$

where $\ell_1, \ldots, \ell_s$ are $k$-irreducible curves such that $\bar{\ell}_i$ is equal to the $\Gamma_k$-orbit of an extremal curve, i.e. a smooth rational curve $\bar{E}_i$ such that $0 > K_{\bar{X}} \cdot \bar{E}_i \geq -3$.

By Theorem 2.7 from [12] one of the following cases occurs:

(i) There exists $\bar{\ell}_i$ equal to a disjoint sum of $(-1)$-curves on $\bar{X}$. In this case there exists a birational $k$-morphism $f : X \to Y$ onto a smooth surface $Y$ such that $f(\ell_i)$ is a point.

(ii) rank $\mathrm{Pic}(X) = 2$, $\ell_1^2 = 0$ and there exists a conic bundle structure on $X$ with the class of a fiber over a closed point equal to the divisor class of $\ell_1$.

(iii) $\mathrm{Pic}(X) \cong \mathbb{Z}$ and $X$ is a $k$-minimal Del Pezzo surface of degree $d \neq 7$.

Applying the averaging operator $\sum_{g \in G} g$ to (1), we obtain

$$\overline{NE}(X)^G = \overline{NE}(X)_{K_X \geq 0}^G + \mathbb{R}_{\geq 0} \left[ \sum_{g \in G} g(\ell_1) \right] + \cdots + \mathbb{R}_{\geq 0} \left[ \sum_{g \in G} (g(\ell_s)) \right]. \tag{2}$$

Write $\mathbb{R}_{\geq 0}[\sum_{g \in G} g(\ell_i)] = \mathbb{R}_{\geq 0}[L_i]$, where $L_i$ is a reduced $G$-orbit of $\ell_i$. Let $L$ be one of the orbits that generate an extremal ray in $\overline{NE}(X)^G$ (since $K_X$ is not nef, it exists).

From now on we assume that rank $\mathrm{Pic}(X)^G \geq 2$, otherwise the assertion of Theorem 5 is obvious. Note that the case $d = 7$ is excluded for the obvious reason that $\bar{X}$ contains three $(-1)$-curves forming a chain, the middle one is $\Gamma_k \times G$-invariant.

The proof of Lemma 2.5 from [12], shows that $L^2 \leq 0$ since otherwise $[L]$ lies in the interior of $\overline{\mathrm{NE}}(X)^G$. Since $\mathbb{R}_{\geq 0}[L]$ is an extremal ray in $\overline{\mathrm{NE}}(X)^G$, the divisor $\bar{L}$ is a $\Gamma_k \times G$-orbit of an irreducible component $\bar{E}$ of $\bar{\ell}$.

Assume $\bar{E}^2 = 0$ as in case (ii) from above. Then $\bar{L}^2 \leq 0$ implies that $\bar{L}$ is the disjoint sum of irreducible smooth rational curves with self-intersection 0. By the Hodge index theorem, they are proportional in $\overline{\mathrm{NE}}(\bar{X})$ (otherwise the signature of the intersection form on $\mathrm{Pic}(\bar{X})$ is not equal to $(1, r)$) and the contraction of $\Gamma_k \times G$-invariant extremal ray $\mathbb{R}_{\geq 0}[\bar{L}]$ defines a $G$-equivariant $k$-map to some $k$-curve $C$ with geometrically connected fibers (see [12, 2.5.1]).

Assume that we are in the case (i) from above, i.e. $\bar{E}$ is a $(-1)$-curve. Let $\bar{L} = \bar{E}_1 + \cdots + \bar{E}_m$, where $\{\bar{E}_1, \ldots, \bar{E}_m\}$ is the orbit of $\bar{E} = \bar{E}_1$ with respect to $\Gamma_k \times G$. We have

$$0 \geq \bar{L}^2 = m\bar{E}_1 \cdot (\bar{E}_1 + \cdots + \bar{E}_m) = m\left(-1 + \sum_{i=2}^{m} \bar{E}_1 \cdot \bar{E}_i\right). \tag{3}$$

If $\bar{L}^2 < 0$, then $\sum_{i=2}^{m} \bar{E}_1 \cdot \bar{E}_i = 0$. Replacing $\bar{E}_1$ with some other $\bar{E}_i$, we obtain that $\bar{L}$ is a disjoint sum of $(-1)$-curves. Thus $L$ can be $G$-equivariantly contracted contradicting the minimality of $(X, G)$.

So we may assume that $\bar{L}^2 = 0$ and $\sum_{i=2}^{m} \bar{E}_1 \cdot \bar{E}_i = 1$. Without loss of generality, $\bar{E} \cdot \bar{E}_2 = 1, \bar{E} \cdot \bar{E}_i = 0, i > 2$. Write $\bar{E}_2 = g(\bar{E})$ for some $g \in \Gamma_k \times G$. Then $\bar{E} \cdot \bar{E}_2 = g^{-1}(\bar{E}) \cdot \bar{E} = 1$, hence $g^{-1}(\bar{E}) = \bar{E}_2 = g(\bar{E})$ and $g^2$ leaves $\bar{E}$ invariant. If $m = 2$, we obtain that the linear system $|\bar{E}_1 + \bar{E}_2|$ defines a $G$-invariant $k$-map onto $\mathbb{P}^1$. Assume $m \geq 3$. Replacing $\bar{E}_1$ with $\bar{E}_3$ in (3), and repeating the argument, we may assume that $\ell$ is equal to the disjoint sum of $\frac{m}{2}$-bouquets $B_i$ of two $(-1)$-curves, and each $g \in \Gamma_k \times G$ either leaves a bouquet invariant or sends it to another bouquet. Since $B_i^2 = 0, B_i \cdot K_X = -2$, and $B_i \cdot B_j = 0, i \neq j$, by the Hodge index theorem, the divisors $B_i$ represent the same divisor class. The linear system $|nB_i|$, for some $n > 0$, defines a $G$-invariant $k$-map to a curve $C$ with geometrically connected fibers.

It remains to be shown that a $G$-invariant $k$-map $f : X \to C$ constructed as above is a conic bundle structure and also rank $\mathrm{Pic}(X)^G = 2$. The divisor class $L$ defines a $G$-equivariant extremal ray. Thus, its contraction from above has the relative Picard rank equal to 1, defining a relatively minimal conic bundle. This implies that rank $\mathrm{Pic}(X)^G = 2$.

**Remark 1.**    The assumption that the ground field $k$ is perfect is not essential. We used it only to simplify the notations (see [12, Theorem 2.7], where $k$ is not assumed to be perfect).                                                                                                                    □

The following lemma is well known. Its proof in [6, Lemma 3.5] can be extended to the case of an arbitrary ground field $k$ by referring to [9], where one can find a proof of the existence of a $G$-equivariant resolution of surfaces. In fact, in Section 2 of his article, Lipman proves that the known existence of any resolution of a normal surface $X$ implies that the sequence

$$X = X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow X_n \leftarrow \cdots ,$$

where each morphism $X_{i+1} \to X_i$ is the composition of the blow-up of all singular points and the normalization, terminates in a nonsingular surface $Y$. Obviously, the composition map $Y \to X$ is $G$-equivariant for any finite group $G$ acting on $X$.

**Lemma 6.**    Let $G$ be a finite subgroup of $\mathrm{Cr}_2(k)$, then there exists a $k$-rational nonsingular surface $X$, an injective homomorphism $\rho : G \to \mathrm{Aut}_k(X)$ and a birational $G$-equivariant $k$-map $\phi : X \to \mathbb{P}^2_k$ such that

$$G = \phi \circ \rho(G) \circ \phi^{-1}.$$

□

The $G$-surface $(X, \rho(G))$ can be obviously replaced by a minimal rational $G$-surface, and in this way we establish a natural bijective correspondence between the conjugacy classes of finite subgroups $G$ of $\mathrm{Cr}_2(k)$ and birational isomorphism classes of minimal nonsingular $k$-rational $G$-surfaces $(X, G)$ (see Theorem 3.6 in [6]).

In this note, we are interested in the case $k$ is any perfect field and surfaces $X$ are $k$-rational. Since $k$ is perfect, the surface $\bar{X}$ is a nonsingular surface over $\bar{k}$, so that we can apply the theory of nonsingular rational surfaces over an algebraically closed field.

It follows from above that we may assume that an automorphism $\sigma$ of prime order $\ell$ acts on a $k$-rational surface $X$ making the pair $(X, \langle \sigma \rangle)$ a minimal $\langle \sigma \rangle$-surface. We say that $\sigma$ *acts minimally* on $X$.

## 3    Elements of Finite Order in Reductive Algebraic Groups

Here we give a short exposition of some of Serre's results from [15] that are relevant to our discussion.

For any integer $N$ and a prime number $\ell$, we denote by $\nu_\ell(N)$ the largest $n$ such that $\ell^n$ divides $N$. For any finite group $A$ we set $\nu_\ell(A)$ to be equal to $\nu_\ell(|A|)$.

The first historical result on elements of finite order in linear groups is the following theorem of Minkowski [11].

**Theorem 7.** Let $n$ be an integer $\geq 1$ and $\ell$ be a prime number. Define

$$M(n, \ell) = \left[\frac{n}{\ell - 1}\right] + \left[\frac{n}{\ell(\ell - 1)}\right] + \left[\frac{n}{\ell^2(\ell - 1)}\right] + \cdots$$

Let $G$ be a finite subgroup of $\mathrm{GL}_n(\mathbb{Q})$. Then $\nu_\ell(G) \leq M(n, \ell)$ and there exists an $\ell$-subgroup $G$ of $\mathrm{GL}_n(\mathbb{Q})$ with $\nu_\ell(G) = M(n, \ell)$. □

We will be interested in a similar result for the group $\mathrm{PGL}_n(k) = \mathrm{GL}_n(k)/k^*$. We cannot apply the previous theorem since an element of finite order does not necessary lift to an element of finite order in $\mathrm{GL}_n(k)$. However, we can apply the following result of Serre from [15].

*From now on $\ell$ will denote a prime odd number not equal to the characteristic of $k$.*

Let $\zeta_\ell$ be a generator of the group $\mu_\ell(\bar{k})$ of $\ell$th roots of unity in $\bar{k}$. Set

$$t_\ell = [k(\zeta_\ell) : k], \quad m_\ell = \sup\{d \geq 1 : \zeta_{\ell^d} \in k(\zeta_\ell)\}.$$

Let $P$ be the prime field contained in $k$. By Galois Theory,

$$t_\ell = [P(\zeta_\ell) : P(\zeta_\ell) \cap k].$$

If $\mathrm{char}\, k = 0$, we obtain that $t_\ell$ divides $\ell - 1$. If $k$ is of characteristic $p > 0$, then $P(\zeta_\ell) \cong \mathbb{F}_{p^s}$, where $s$ is the order of $p$ in $\mathbb{F}_\ell^*$. Thus $t_\ell$ divides $s$.

For example, when $k = \mathbb{Q}$, we have $t_\ell = \ell - 1$ and $m_\ell = 1$. If $k = \mathbb{F}_q$, then $t_\ell$ is equal to the order of $q$ in $\mathbb{F}_\ell^*$ and $m_\ell = \nu_\ell(q^{\ell-1} - 1)$.

The following is a special case of Theorem 6 from [15].

**Theorem 8.** Let $A$ be a finite subgroup of $\mathrm{PGL}_{n+1}(k)$. For any $\ell > 2$,

$$\nu_\ell(A) \leq \sum_{2 \leq s \leq n+1, t_\ell | s} (m_\ell + \nu_\ell(s)),$$

(if the index set is empty, $\nu_\ell(A) = 0$). □

**Corollary 9.** Assume $t_\ell \geq n + 2$. Then $\mathrm{PGL}_{n+1}(k)$ does not contain elements of prime order $\ell$. □

For example, if $k$ is of characteristic zero and $m_\ell(k) = \{1\}$ (e.g. $k = \mathbb{Q}$), then $t_\ell = \ell - 1$ and we get

$$n \geq \ell - 2$$

if $\mathrm{PGL}_{n+1}(k)$ contains an element of order $\ell$. For example, $\mathrm{PGL}_{n+1}(k)$ contains an element of order 7 only if $n \geq 5$.

On the other hand, if $k = \mathbb{F}_2$ and $\ell = 7$, then $t_\ell = 3$ and it is known that $\mathrm{PGL}_3(k)$ is isomorphic to a simple group of order 168 and it contains an element of order 7.

The next result of Serre [15], Theorems 4 and 4′, concerns elements of finite order in an algebraic $k$-torus.

**Theorem 10.**   Let $T$ be an algebraic $k$-torus and $A$ be a finite subgroup of $T(k)$. Then

$$\nu_\ell(A) \leq m_\ell \left[ \frac{\dim T}{\phi(t_\ell)} \right],$$

where $\phi$ is the Euler function. Assume $m_\ell < \infty$ (e.g. $k$ is finitely generated over its prime subfield). For any $n \geq 1$ there exists an $n$-dimensional $k$-torus $T$ and a finite subgroup $A$ of $T(k)$ such that $\nu_\ell(A) = m_\ell[\frac{\dim T}{\phi(t_\ell)}]$. □

Since this will be of importance to us, let us give Serre's construction proving the last assertion.

Let $T$ be an algebraic $k$-torus of dimension $d$ over an arbitrary field $k$. By definition, it is an affine algebraic group over $k$ that becomes isomorphic to the group $\mathbb{G}_m^d$ over some finite extension $E$ of $k$. One can always choose $E$ to be a separable Galois extension, it is called a *splitting field* of $T$.

We denote by $M = \mathrm{Hom}(\mathbb{G}_m^d, \mathbb{G}_m)$ the group of rational characters of $\mathbb{G}_m^d$, so that $\mathbb{G}_{m,E}^d \cong \mathrm{Spec}\ \mathbb{Z}[M]$. A $d$-dimensional $k$-torus $T$ split over a Galois extension $E/k$ with Galois group $\Gamma$ defines a structure of a $\Gamma$-module on $M$ such that $T \cong \mathrm{Spec}\ E[M]^\Gamma$. In this way the category of $d$-dimensional $k$-tori split over $E$ is anti-equivalent to the category of free abelian groups equipped with a structure of a $\Gamma$-module [16].

The group algebra $\mathbb{Z}[\Gamma]$ equipped with the natural structure of a $\mathbb{Z}[\Gamma]$-module $M$ (the regular representation of $\Gamma$) defines an algebraic $k$-torus $R_{E/k}(\mathbb{G}_m)$ of dimension equal to $[E : k]$. It represents the functor on the category of $k$-algebras defined by $L \to (L \otimes_k E)^*$.

Now let us take $E = k(\zeta_\ell)$ and let $\Gamma$ be the Galois group of $E/k$. It is a cyclic group of order $t = t_\ell$. Let

$$x^t - 1 = \Phi_t(x)\Psi(x),$$

where $\Phi_t(x)$ is the $t$th cyclotomic polynomial over $\mathbb{Q}$. We have $\mathbb{Z}[\Gamma] \cong \mathbb{Z}[x]/(x^t - 1)$. The multiplication by $\Psi(x)$ in $\mathbb{Z}[x]$ defines an inclusion of $\Gamma$-modules $\mathbb{Z}[x]/(\Phi_t(x)) \hookrightarrow \mathbb{Z}[x]/(x^t - 1)$ and hence a surjective homomorphism of $k$-tori $R_{E/k}(\mathbb{G}_m) \to T$. Here $T$ is a $\phi(t)$-dimensional $k$-torus defined by the character module $M = \mathbb{Z}[x]/(\Phi_t(x))$. One can view $T$ as the image of $R_{E/k}(\mathbb{G}_m)$ under the endomorphism $\Psi(\gamma)$, where $\gamma$ is a generator of $\Gamma$. One can show that $\Psi(\gamma)$ acts on the subgroup $\langle \zeta_{\ell^{m_\ell}} \rangle$ of $E^*$ by an automorphism. This implies that $T$ contains $\zeta_{\ell^{m_\ell}}$. Thus

$$m_\ell \left[ \frac{\dim T}{\phi(t)} \right] = m_\ell \left[ \frac{\phi(t)}{\phi(t)} \right] = m_\ell = \nu_\ell(\langle \zeta_{\ell^{m_\ell}} \rangle).$$

Now, for any positive integer $n$ let $s = [\frac{n}{\phi(t)}]$. Take $T$ equal to $T_1^s \times \mathbb{G}_m^{n-s\phi(t)}$, where $T_1$ is constructed as above. It is easy to see that $T$ satisfies the last assertion of Theorem 10.

**Corollary 11.** A two-dimensional $k$-torus $T$ with $T(k)$ containing an element of prime order $\ell > 2$ exists if and only if $t_\ell$ takes values in the set $\{1, 2, 3, 4, 6\}$.    $\square$

**Proof.** In fact, the set $\{1, 2, 3, 4, 6\}$ is the set of positive integers $t_\ell$ such that $\phi(t_\ell) \leq 2$. If $\phi(t_\ell) > 2$, Serre's bound implies that no such torus exists. If $\phi(t_\ell) = 2$, Serre's construction from above exhibits such a torus. If $\phi(t_\ell) = 1$, i.e. $t_\ell = 1$ or $2$, we can take $T = \mathbb{G}_{m,k}^2$ in the first case and $T = R_{k(\zeta_\ell)/k}(\mathbb{G}_m)$ in the second case.    ∎

## 4  Del Pezzo Surfaces of Degree 6

Here we recall some well-known facts about toric Del Pezzo surfaces of degree 6. Let $\bar{k}$ be an algebraic closure of $k$. We continue to assume that $k$ is perfect. Recall that a Del Pezzo surface $S$ of degree 6 over $\bar{k}$ is isomorphic to the blow-up of three noncollinear points $p_1, p_2, p_3$ in $\mathbb{P}_{\bar{k}}^2$. The set of $(-1)$-curves on $S$ consists of six curves, the exceptional curves of the blow-up morphism, and the proper transforms of the lines $\overline{p_i, p_j}$. In the anticanonical embedding $S \hookrightarrow \mathbb{P}_{\bar{k}}^6$, they are six lines forming a hexagon.

The surface $S$ is isomorphic to $\mathcal{D}_{\bar{k}} = \mathcal{D} \otimes \bar{k}$, where $\mathcal{D}$ is a unique smooth projective toric surface (defined over $\mathbb{Z}$) with the Picard group of rank 4. It is defined by a complete $\mathbb{Z}^2$-fan $\Sigma$ whose 1-skeleton $\mathrm{Sk}^1(\Sigma)$ consists of six rays, the primitive vectors in $\mathbb{Z}^2$ spanning these rays are $\pm e_1, \pm e_2, \pm(e_1 + e_2)$, where $e_1, e_2$ is a basis in $\mathbb{Z}^2$. The closures of the orbits corresponding to these rays is the set of six $(-1)$-curves.

The group of automorphisms of the surface $\mathcal{D}$ is a group scheme over $\mathbb{Z}$ isomorphic to $\mathbb{G}_m^2 \rtimes D_{12}$, where $D_{12}$ is the dihedral group of order 12, realized as the subgroup of $\mathrm{GL}_2(\mathbb{Z})$ leaving the fan $\Sigma$ invariant.

Let $X$ be a rational Del Pezzo surface of degree 6 over $k$. Then $\bar{X} = X \otimes_k \bar{k}$ is isomorphic to $\overline{\mathcal{D}} = \mathcal{D} \otimes \bar{k}$. Since the set of all $(-1)$-curves on $\bar{X}$ is defined over $k$, its complement $U$ in $X$ becomes isomorphic to a torus over $\bar{k}$. This implies that $U$ is a torsor (= principally homogeneous space) over a two-dimensional $k$-torus $T$ (see [10, Chapter IV, Theorem 8.6]). Since $X$ is rational, $X(k) \neq \emptyset$ and hence $U(k) \neq \emptyset$ ([8, Proposition 4]). This shows that $U$ is an algebraic $k$-torus.

Fix a structure of $k$-torus $T$ on $U$ and consider its action on itself by translation. Then we can extend the action to the action of $T$ on $X$ (see [13, p. 22]). Thus $X$ is a (not necessarily split) toric variety over $k$. Let $E$ be a splitting field of the torus $T$ and $X_E = X \otimes_k E$. Then the split torus $T_E$ acts on $X_E$ making it into a split toric variety over $E$. It is obviously isomorphic to $\mathcal{D}_E := \mathcal{D} \otimes_k E$. Thus $E$ is a splitting field of $X$, i.e. $X_E \cong \mathcal{D}_E$. Conversely, if $E$ is a splitting field of $X$, the torus $T_E$ is split. The Galois group $\Gamma$ of $E/k$ acts on $X_E$ via automorphisms of the toric variety, i.e. it acts on the lattice $N \cong \mathbb{Z}^2$, the dual lattice of the lattice $M$ of characters of $T$, leaving invariant the fan $\Sigma$ defining the toric variety $\mathcal{D}$. The variety $X$ is isomorphic to the descent of $\mathcal{D}$ defined by this action, defined uniquely up to $k$-isomorphism (see [1, 16]).

Choose a splitting field $E$ such that the map $\Gamma \to \mathrm{Aut}(\Sigma)$ is injective. It follows from the classification of subgroups of $D_{12}$ that one of the following cases occurs.

(i)   $\Gamma = \langle \gamma \rangle, |\Gamma| = 2$, the action of $\Gamma$ on $N$ is given by $\gamma : (m, n) \mapsto (n, m)$;

(ii)  $\Gamma = \langle \gamma \rangle, |\Gamma| = 2, \gamma : (m, n) \mapsto (-m, -n)$;

(iii) $\Gamma = \langle \gamma \rangle, |\Gamma| = 3, \gamma : (m, n) \mapsto (-n, m - n)$;

(iv)  $\Gamma = \langle \gamma_1, \gamma_2 \rangle, \Gamma \cong (\mathbb{Z}/2)^2, \gamma_1 : (m, n) \mapsto (n, m), \gamma_2 : (m, n) \mapsto (-m, -n)$;

(v)   $\Gamma = \langle \gamma_1, \gamma_2 \rangle, \Gamma \cong S_3, \gamma_1 : (m, n) \mapsto (-n, m - n), \gamma_2 : (m, n) \mapsto (n, m)$;

(vi)  $\Gamma = \langle \gamma \rangle, \Gamma \cong \mathbb{Z}/6, \gamma : (m, n) \mapsto (m - n, m)$;

(vii) $\Gamma = \langle \gamma_1, \gamma_2 \rangle, \Gamma = D_{12}, \gamma_1 : (m, n) \mapsto (m - n, m), \gamma_2 : (m, n) \mapsto (n, m)$.

We easily get (see [1]) that

$$\mathrm{rank}\,\mathrm{Pic}(X) = \begin{cases} 3 & \text{in cases (i), (ii),} \\ 2 & \text{in cases (iii), (iv), (v),} \\ 1 & \text{in cases (vi), (vii).} \end{cases}$$

Note that in all cases except case (i), the torus $T$ is anisotropic (i.e. $M^\Gamma = \{0\}$).

**Proposition 1.** Assume that a cyclic group $G = \langle \sigma \rangle$ of prime order $\ell \geq 5$ acts minimally on a $k$-rational Del Pezzo surface $X$ of degree 6. Let $T$ be the complement of the union of $(-1)$-curves on $X$ that acts on $X$ via its structure of a toric surface over $k$. Then $\sigma$ is defined via the action by an element $\tilde{\sigma} \in T(k)$. The torus $T$ splits over $k(\zeta_\ell)$ with cyclic Galois group $\langle \gamma \rangle$ of order 6 and $t_\ell = 6$. The $G$-surface $(X, G)$ is unique up to $k$-isomorphism. $\qquad \square$

**Proof.** By assumption $\ell \geq 5$, the group $G$ cannot be isomorphic to a subgroup of $D_{12}$, hence can be identified with a subgroup of $T(k)$. The assumption that $(X, G)$ is minimal implies that $X$ is $k$-minimal because the set of $(-1)$-curves does not contain orbits of length $\ell$. Let $E$ be a minimal splitting field of $X$ (and hence for $T$) with Galois group $\Gamma$. Then $\Gamma$ is isomorphic to a subgroup of $D_{12}$ and only cases (vi) and (vii) from the above list are possible. Indeed, in the remaining cases rank $\mathrm{Pic}(X) > 1$ and hence $X$ is not $k$-minimal.

Let us consider case (vi). In this case $\Gamma = \langle \gamma \rangle$ is a cyclic group of order 6 and $\gamma$ acts on the lattice $N = \mathbb{Z}^2$ by a matrix $\left( \begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix} \right)$. This matrix satisfies the cyclotomic equation $x^2 - x + 1 = 0$. The lattice $M$ of characters of $T$ is dual to the lattice $N$, and hence it is isomorphic as a $\Gamma$-module to $\mathbb{Z}[x]/(x^2 - x + 1)$. Thus the torus $T$ and the surface $X$ are determined uniquely by the extension $E/k$. Let us show that, under our assumptions, $E = k(\zeta_\ell)$ and hence $t_\ell = 6$.

Since $G \subset T(k)$ extends the translation action on $T$ by an element of order $\ell$, the group $T(E) \cong (E^*)^2$ contains $G$. The image of $G$ under the projection homomorphism $T(E) \to E^*$ contains the group $\mu_\ell$ of $\ell$th roots of 1. Therefore, $E' = k(\zeta_\ell)$ is contained in $E$. Let $T(E)[\ell] \cong (M/\ell M) \otimes \mu_\ell$ be the $l$-torsion subgroup of $T(E)$. The group $\mathrm{Gal}(E/E')$ acts trivially on $T(E)[\ell]$, or, equivalently, the actions of $\Gamma$ and $\mathrm{Gal}(E'/k)$ on $T(E)[\ell]$ coincide. By Minkowski's lemma [15, Lemma 1], the natural homomorphism $\mathrm{GL}(M) \to \mathrm{GL}(M/\ell M)$ is a bijection on the set of elements of finite order. This shows that the actions of the Galois groups coincide on the whole character lattice $M$ of $T$. In view of minimality of the splitting field, we get $E = E'$.

The previous argument also shows that case (vii) does not occur. In fact, if $E$ is a minimal splitting field with Galois group isomorphic to $D_{12}$, then it follows from above that $E' = k(\zeta_\ell)$ is contained in $E$ and must coincide with $E$. Since the extension $E'/k$ is cyclic this is impossible. $\qquad \blacksquare$

Applying Corollary 11, we obtain that $\ell = 7$ if char $k = 0$ and $k \cap \mathbb{Q}(\zeta_\ell) = \mathbb{Q}$. On the other hand, if for example $k = \mathbb{F}_{17}$ and $\ell = 13$, we get $t_\ell = 6$. This shows that $\ell = 13$ is possible in this case.

**Remark 2.**   Another example of a nonsplit toric Del Pezzo surface is a Del Pezzo surface $X$ of degree 8 isomorphic to a nonsingular quadric $\mathbf{F}_0$ over $\bar{k}$. Recall that $\mathbf{F}_0$ can be given by a fan $\Sigma$ with 1-skeleton spanned by vectors $\pm e_1, \pm e_2$. The automorphism group of the toric surface (i.e. the subgroup of $\mathrm{Aut}(\mathbf{F}_0)$ preserving the toric structure) is equal to $\mathbb{G}_m^2 \rtimes D_8$, where $D_8$ is the dihedral group of order 8. Fix a toric structure on $\mathbb{F}_0$ by fixing a quadrangle of rulings and a structure of a torus on its complement. Let $E/k$ be a Galois extension with Galois group $\Gamma$. Chose an injective homomorphism $\Gamma \to \mathrm{GL}(2, \mathbb{Z})$ that leaves invariant the fan $\Sigma$. As we have mentioned above, there is a unique descent of the toric variety $\mathbb{F}_0$ to a toric $k$-variety $X$. This is a toric Del Pezzo surface of degree 8. Since all two-dimensional $k$-tori are rational, the surface $X$ is a $k$-rational.

Following the proof of the previous proposition, we see that an automorphism of order $\ell \geq 5$ can act minimally on $X$ only if $T$ is isotropic with $\Gamma$ isomorphic to a cyclic group of order 2 acting by $(m, n) \mapsto (n, m)$ or anisotropic with $\Gamma$ isomorphic to a cyclic group of order 4 or $D_8$. In the first case $T = R_{E/k}(\mathbb{G}_m)$ and $T(k) = E^*$ does not contain an element of order $\ell$, unless $t_\ell = 2$. In the second case $T_E$ is the quotient of the torus of $R_{E/k}(\mathbb{G}_m)$ corresponding to the submodule $\mathbb{Z}[x]/(x^2 + 1)$ of $\mathbb{Z}[\Gamma] \cong \mathbb{Z}[x]/(x^4 - 1)$. The group $T(k)$ contains an element of order $\ell \geq 5$ only if $t_\ell = 4$. If $\Gamma \cong D_8$, we obtain that $E$ contains $k(\zeta_\ell)$. Since $k(\zeta_\ell) = E$, this case is impossible.    $\square$

## 5   The Main Theorem

Let us start proving our main result, which is Theorem 1 from Section 1. Let $\sigma$ be an element of prime order $\ell > 2$ in $\mathrm{Cr}_2(k)$. As before, we assume that $\ell \neq p = \mathrm{char}\, k$.

**Proposition 2.**   Assume $\ell \geq 5$ and $\sigma$ acts minimally as an automorphism of a $k$-rational conic bundle $X$. Then $t_\ell \leq 2$ and $\sigma$ is conjugate in $\mathrm{Cr}_2(k)$ to an element defined by a rational point on a two-dimensional algebraic $k$-torus.    $\square$

**Proof.**   Let $\pi : X \to C$ be a conic bundle structure on $X$. Since $X$ is $k$-rational, the set $X(k)$ is not empty and hence the set $C(k)$ is not empty. Thus $C \cong \mathbb{P}_k^1$.

Assume $t_\ell > 2$, by Corollary 9, $\mathrm{Aut}_k(C) \cong \mathrm{PGL}_2(k)$ does not contain elements of order $\ell$. Thus $G = \langle \sigma \rangle$ acts trivially on the base $C$ and hence acts on the generic fiber $X_\eta$ by $k(\eta)$-automorphisms. Since $k$ is algebraically closed in $k(\eta)$, we have $t_\ell(k(\eta)) = t_\ell(k) > 2$, so we can apply Corollary 9 again and get that the action on $X_\eta$, and hence on $X$, is trivial.

Now we assume that $t_\ell \leq 2$. Let $c$ be the number of reducible fibers in the conic fibration.

Assume first that $c = 0$, i.e. $\bar{X}$ is a minimal ruled surface $\mathbf{F}_e$. Since the base $C$ of the fibration is isomorphic to $\mathbb{P}^1_k$, this easily implies that $X \cong \mathbf{F}_e$ over $k$. Let $\bar{G}$ be the image of $G$ in $\mathrm{Aut}_k(C)$.

Suppose $\bar{G}$ is trivial, $G$ has two fixed points, maybe conjugate on the general fiber isomorphic to $\mathbb{P}^1_{k(C)}$. This defines a two-section $S$ of the fibration fixed by $G$. If $e > 0$, one of the components of $\bar{S}$ is the exceptional section. The other component must be defined over $k$ and $G$-invariant. Since $G$ acts identically on the base, each component is fixed by $G$ pointwise. Since the fixed locus of $G$ is smooth (because $\ell \neq \mathrm{char}\ k$), the two components do not intersect. The complement of $S$ and two fibers is a $G$-invariant $k$-torus $T$. If $e = 0$, then $G$ leaves invariant the other projection to $\mathbb{P}^1_{\bar{k}}$ and hence leaves invariant fibers over two points, maybe conjugate over $k$. The complement of these two fibers and two fibers of the first projection is an algebraic torus defined over $k$.

If $\bar{G}$ is not trivial, then it fixes two points $x_1, x_2$ on $C$, maybe conjugate over $k$. A surface with $e = 1$ is not $G$-minimal. If $e \geq 2$, the surface $X$ has a unique section $S$ with self-intersection $-e$. The fibers $F_1, F_2$ over $x_1, x_2$ are $G$-invariant and each contains a fixed point on $S$. Since $\ell \neq \mathrm{char}\ k$, the group $G$ must have another fixed point $y_i \in F_i, i = 1, 2$. The closed subscheme $\{y_1, y_2\}$ is defined over $k$. After we perform a $G$-invariant elementary transformation at these points, we obtain a surface isomorphic to $\mathbf{F}_{e-2}$. Continuing in this way, we arrive at either $\mathbf{F}_0 \cong \mathbb{P}^1_k \times \mathbb{P}^1_k$, or $\mathbf{F}_1$, the latter being not $G$-minimal.

Now suppose $c > 0$. If $(\bar{X}, G)$ is minimal, then $G$ contains an element that switches irreducible components of some reducible fiber ([6, Lemma 5.6], the proof of this lemma works whenever the ground field is algebraically closed and its characteristic is coprime with the order of the group). Thus, the cyclic group $G$ of order $\ell$ is mapped nontrivially to a group of even order. Since $\ell$ is odd, this is impossible. Since $(X, G)$ is minimal, this implies that $X \to C$ is relatively minimal over $k$. Since $\mathrm{Pic}(X)$ is generated by $K_X$ and components of fibers of the conic fibration, we have $\mathrm{Pic}(X) \cong \mathbb{Z}^2$. Suppose $X$ is not $k$-minimal, then there exists a birational $k$-morphism from $X$ to a rational $k$-minimal surface $X'$ with $\mathrm{Pic}(X') \cong \mathbb{Z}$. Thus $X'$ is a rational $k$-minimal Del Pezzo surface. In terminology of [7] this makes a link of type I. Applying Theorem 2.6 (i) from *loc. cit.*, we see that the possible values of $K_X^2$ are $8, 6, 5$, and $3$.

If $K_X^2 = 8$, we have $X' = \mathbb{P}^2_k$ and $X \cong \mathbf{F}_1$, so $(X, G)$ is not minimal.

If $K_X^2 = 6$, $X'$ is a quadric $Q$ in $\mathbb{P}^3_k$ with $\mathrm{Pic}(Q) \cong \mathbb{Z}$. The morphism $X \to X'$ blows down a pair $E$ of conjugate $(-1)$-curves. We have $E \in |-K_X - 2F|$, where $F$ is the class of a fiber in the conic bundle. Obviously, $G$ leaves $E$ invariant and hence $(X, G)$ is not minimal.

If $K_X^2 = 5$, then $X' = \mathbb{P}_k^2$ and $X \to X'$ blows down a fourtuple $E$ of $(-1)$-curves. We have $E \in |-2K_X - 3F|$ and again $(X, G)$ is not minimal.

Finally, if $K_X^2 = 3$, then $X$ is a Del Pezzo surface isomorphic to a cubic surface. In this case a $(-1)$-curve is a line on $X$. Thus $X$ contains a line $L$ defined over $k$ such that the conic bundle is formed by the pencil of planes through the line. The line $L$ is $G$-invariant and the pair $(X, G)$ is not minimal.

Thus we may assume that $X$ is $k$-minimal. Applying Theorem 4, we obtain $K_X^2 \geq 5$. The conic bundle $\bar{X}$ contains $c = 8 - K_X^2 \leq 3$ reducible fibers. Since $G$ in its action on $C$ has no orbits of length 2 or 3, $G$ has at least $c$ fixed points on $C$.

If $c = 3$, $G$ acts identically on $C$. The surface $\bar{X}$ admits a birational morphism to $\mathbb{P}_{\bar{k}}^2$, the blow-up of 4 points. The exceptional curves are sections of the conic bundle. Their union $E$ belongs to the linear system $|-2K_{\bar{X}} - 3F|$, where $F$ is the class of a fiber of the conic bundle on $\bar{X}$. This shows that $G$ leaves $E$ invariant, and, since the order of $G$ is an odd prime number $\ell$, it leaves each component invariant. Thus $G$ has four fixed points on each fiber, hence acts trivially on the general fiber, hence trivially on $X$.

Assume $c = 2$, i.e. $K_X^2 = 6$ and $\bar{X}$ is obtained by blowing up two points on a quadric. They are the base points of a pencil of conic plane sections of the quadric. The exceptional divisor $E$ is a two-section of the conic bundle that belongs to the linear system $|-K_{\bar{X}} - 2F|$. As in the previous case $G$ fixes each component of $E$. Let $T$ be the complement of the union $B$ of $E, F_1, F_2$, where $F_1$ and $F_2$ are the singular fibers. Since $(X, G)$ is minimal, the Galois group switches the components of $E$ and the components of the fibers. This shows that $B(k)$ consists of two points, the singular points of the fibers. If $k$ is infinite, $X(k)$ is dense in $X(\bar{k})$ because $X$ is rational. This shows that $T(k) \neq \emptyset$. If $k = \mathbb{F}_q$ is finite, it follows from the Weil Theorem [10, Chapter IV, Section 5, Corollary 1] that $\#X(k) \equiv 1 \mod q$, hence $\#X(k) > 2$ and $T(k) \neq \emptyset$ again. Thus $T$ is a $G$-invariant two-dimensional torus. Since $\sigma$ is of order $\ell \geq 5$, its image in the automorphism group of $T$ (as an algebraic group) is trivial. Thus $\sigma$ is realized by an element of $T(k)$.

Assume $c = 1$. Then $X$ is obtained by blowing up one point on a minimal ruled surface. This point must be a $k$-point and the exceptional curve is a component of a unique singular fiber. Since $\ell$ is odd it is $G$-invariant. Thus $(X, G)$ is not minimal. ∎

**Proposition 3.** Assume that $\sigma$ of prime order $\ell \geq 7$ acts minimally on a $k$-rational Del Pezzo surface $X$ of degree $d$. Then one of the following cases occurs:

(i)  $d = 6, t_\ell = 6$;

(ii)  $d = 8, t_\ell = 4$ or $t_\ell = 2$;

(iii)  $d = 9, t_\ell \leq 3$.

In all cases $X$ has a structure of a toric surface and $\sigma$ belongs to $T(k)$, where $T$ is an open subset of $X$ isomorphic to a $k$-torus. $\qquad\square$

**Proof.** Assume first that $d \leq 5$. In this case $\mathrm{Aut}(\bar{X})$ is isomorphic to a subgroup of the Weyl group of a root system of type $A_4$, $D_{12}$, $E_6$, $E_7$, or $E_8$ [6, Lemma 5.2]. The classification of elements of finite order in Weyl groups shows that $\ell \leq 7$, and the equality may occur only if $d = 2$ or $d = 1$.

Assume $d = 2$. Let $\pi : \bar{X} \to \mathbb{P}^2_{\bar{k}}$ be a birational morphism that blows down seven disjoint $(-1)$-curves $C_1, \ldots, C_7$ and $e$ be the divisor class of their sum. Obviously, $e$ has only one effective representative. Let $\mathcal{R} = (\mathbb{Z}K_{\bar{X}})^\perp$ be considered as a quadratic lattice isomorphic to the root lattice of type $E_7$. It is known that the stabilizer subgroup of $e$ in the Weil group $W(E_7)$ is isomorphic to the permutation group $S_7$ (see [5]). The index of this subgroup is equal to 576. Elements of the $W(E_7)$-orbit of $e$ correspond to sets of seven disjoint $(-1)$-curves on $\bar{X}$. They are paired into 288 pairs, two sets in the same pair differ by the Geiser involution on $\bar{X}$. Recall that the linear system $|-K_{\bar{X}}|$ defines a degree 2 finite map $\bar{X} \to \mathbb{P}^2_{\bar{k}}$ ramified over a nonsingular plane quartic. The cover transformation is the Geiser involution. It sends $C_i$ to the curve $C_i'$ such that $C_i + C_i' \in |-K_{\bar{X}}|$. Since $288 \equiv 1 \mod 7$, a group of order 7 has a fixed point on the set of such pairs and acts identically on each element in the fixed pair. Thus we can find a $\sigma$-invariant class $e$ as above. It is known that $\mathcal{R}^{\langle\sigma\rangle} \cong \mathbb{Z}$ (see [6, Table 5]). Since $e \cdot K_{\bar{X}} = -7$, the divisor class $2e + 7K_{\bar{X}}$ spans $\mathcal{R}^{\langle\sigma\rangle}$. Since the Galois group $\Gamma$ commutes with $\langle\sigma\rangle$, it either leaves the class $2e + 7K_{\bar{X}}$ invariant or contains an element that changes $2e + 7K_{\bar{X}}$ to $-2e - 7K_{\bar{X}}$. In the first case, $e$ is $\langle\sigma\rangle \times \Gamma$- invariant contradicting the assumption that $(X, \langle\sigma\rangle)$ is minimal. In the second case, $\Gamma$ contains an element $\gamma$ that sends $e$ to $e' = -e - 7K_{\bar{X}}$. This implies that $\gamma(C_1 + \cdots + C_7) = \gamma(C_1) + \cdots + \gamma(C_7)$, where $C_i + \gamma(C_i) \in |-K_{\bar{X}}|$. Thus $\gamma$ acts as the Geiser involution. It sends the divisor class $C_i - C_j$ to $C_j - C_i$. Together with $2e + 7K_{\bar{X}}$, the divisor classes of $C_i - C_j$ generate $\mathcal{R}$, hence $\gamma$ acts as the minus identity on $\mathcal{R}$ and $\mathcal{R}^\Gamma = \{0\}$. This implies that $X$ is $k$-minimal, and because $d \leq 5$ and $X$ is $k$-rational, this contradicts Theorem 4.

Assume $d = 1$. An element of order 7 in $W(E_8)$ is conjugate to an element of order 7 in $W(E_7)$, where the inclusion $W(E_7) \subset W(E_8)$ corresponds to the inclusion of the Dynkin diagrams. This implies that $\sigma$ acting on $\bar{X}$ is a lift of an automorphism $\sigma'$ of order 7 of a Del Pezzo surface $\bar{X}'$ of degree 2 under the blow-up of a $\sigma'$-invariant point. Let $C_8$ be the corresponding $\sigma$-invariant exceptional curve. Let $e' = C_1 + \cdots + C_7$ be the pre-image of the $\sigma'$-invariant class in $\mathrm{Pic}(\bar{X}')$ from the previous case $d = 2$ and $e = e' + C_8$. Let $\mathcal{R} = (K_{\bar{X}})^\perp$. Then $\mathcal{R}^{\langle\sigma\rangle}$ is generated (over $\mathbb{Q}$) by $v = \frac{1}{3}(e - 3C_8 + 5K_{\bar{X}})$ and $w = C_8 + K_{\bar{X}}$. We have $(v, v) = -4$, $(v, w) = 1$, $(w, w) = -2$. The Galois group leaves $\mathcal{R}^{\langle\sigma\rangle}$

invariant and acts as an isometry of the quadratic form $-4x^2 + 2xy - 2y^2$. It is easy to compute the orthogonal group of this quadratic form to obtain that an element of finite order in this group is either the identity or the minus the identity. In the first case we obtain that $X$ is not $\langle\sigma\rangle$-minimal. In the second case, by using the argument from the previous case, we obtain that $\Gamma$ contains an element $\gamma$ that acts as the Bertini involution. Recall that $|-2K_{\bar{X}}|$ defines a degree 2 map $\bar{X} \to Q$, where $Q$ is quadric cone in $\mathbb{P}^3_{\bar{k}}$. Its cover transformation is the Bertini involution. It sends $C_i$ to $C'_i \in |-2K_{\bar{X}} - C_i|$. It acts as the minus identity on $\mathcal{R}$ and hence $X$ is $k$-minimal. We conclude as in the previous case.

Finally, it remains to consider the case $d \geq 7$. If $d = 7$, the surface $\bar{X}$ is obtained by blowing up two points in $\mathbb{P}^2_{\bar{k}}$ and is obviously not $G$-minimal (the proper transform of the line joining the two points is $G$-invariant and is defined over $k$). If $d = 8$, the surface is either $\mathbf{F}_0$ or $\mathbf{F}_1$. The surface $\mathbf{F}_1$ is obviously not $G$-minimal. Let $\bar{X} = \mathbf{F}_0$. Since $\ell$ is odd, $G$ preserves the rulings on $\bar{X} \cong (\mathbb{P}^1_{\bar{k}})^2$. Since $X$ is a $G$-minimal Del Pezzo surface (the case of a minimal conic bundle with $\mathrm{Pic}(X)^G \cong \mathbb{Z}^2$ was already considered in Proposition 2), $\mathrm{Pic}(X) \cong \mathbb{Z}$ and therefore the Galois group $\Gamma_k$ switches the two rulings. Since $G$ commutes with the Galois group, it cannot act identically on the base of any of the two rulings (otherwise it acts identically on both rulings, hence acts identically on the surface). This implies that $\sigma$ has four fixed points on $\bar{X}$, the vertices of a quadrangle of lines. Since $\sigma$ is a $k$-automorphism, the set of these points defines a closed subset of $X$ and the quadrangle descends to a divisor $D$ on $X$ whose complement is a $k$-torus $T$. So we obtain that $X$ is a toric Del Pezzo surface of degree 8. As we saw in Remark 2, this case is realized only if $t_\ell = 4$ or $t_\ell = 2$.

If $d = 9$, the surface $\bar{X} \cong \mathbb{P}^2_{\bar{k}}$. By definition, $X$ is a Severi–Brauer variety of dimension two. Since $X(k) \neq \emptyset$, it must be trivial, i.e. $X \cong \mathbb{P}^2_k$. Thus $\mathrm{Aut}(X) \cong \mathrm{PGL}_3(k)$. Applying Corollary 9, we obtain that this can be realized only if $t_\ell \leq 3$. The fixed locus of $\sigma$ in $\mathbb{P}^2_{\bar{k}}$ is either the union of a line and a point, or the set of three distinct points. In the first case the line and the point must be defined over $k$. By joining the point with two $k$-points on the line, we obtain a triangle of $\sigma$-invariant lines. Its complement is a $k$-torus on which $\sigma$ acts as a translation. In the second case, the three fixed points are fixed under $\sigma$ because $\ell \geq 5$. They define a $\sigma$-invariant triangle on $\mathbb{P}^2_k$ whose complement is a $k$-torus.

The remaining case $d = 6$ was considered in Proposition 1 from the previous section. An automorphism of $\sigma$ of order $\ell \geq 5$ acts on a unique Del Pezzo surface $X$ and its image $\mathrm{Aut}(X) \cong T(k) \rtimes D_{12}$ is contained in $T(k)$. This proves Theorem 1. ∎

**Remark 3.** It is shown in [6, Table 8], that a Del Pezzo surface $X$ of degree 1 over $\mathbb{C}$ does not admit an automorphism of order 7, minimal or not. One can show that this is true over an arbitrary field $k$ (even of characteristic 7).

One can also prove the nonexistence of a minimal action of a cyclic group of order 7 on a Del Pezzo surface of degrees $d = 1$ or 2 by using the ideas of Mori theory. We give the proof only in the case $d = 2$ leaving the case $d = 1$ as an exercise to the reader. Let $G = \langle \sigma \rangle$. The group $G \times \Gamma$ acts on the Mori cone $\mathrm{NE}(\bar{X})$. It is a polyhedral cone generated by the divisor classes of fifty-six $(-1)$-curves on $\bar{X}$. The intersection of $\mathrm{NE}(\bar{X})$ with $\mathrm{Pic}(\bar{X})^G \cong \mathbb{Z}^2$ has two extremal rays, both $G$-invariant. They are switched by the Geiser involution. The surface $\bar{X}$ does not admit a $G$-invariant structure of a conic bundle. In fact, since $K_X^2 = 2$, the number of singular fibers of such a conic bundle must be equal to six. So $G$ fixes them and switches the components of some of these fibers. Since $G$ is of odd order, this is impossible. Now, it follows from the proof of Theorem 5 that each $G$-invariant extremal ray is generated by a $\Gamma$-orbit of a $(-1)$-curve that consists of disjoint curves. Since the number of $(-1)$-curves is equal to fifty-six and is divisible by 7, such an orbit must consist of seven curves (if the orbit is a singleton, there are at least six more invariant $(-1)$-curves contradicting to the fact that rank $\mathrm{Pic}(\bar{X})^G = 2$). Thus the $G$-invariant Mori cone is generated by two extremal rays corresponding to two seven-tuples of disjoint $(-1)$-curves. Since $X$ is $\Gamma - G$-minimal, the group $\Gamma$ switches the two sets sending each curve $C_i$ from the first set to the curve $C_i' \in |-K_X - C_i|$ from the second set. Therefore, it acts on $\mathrm{Pic}(X)$ as the Geiser involution. Thus $X$ is $k$-minimal contradicting its rationality. $\qquad\square$

## 6 The Case of Characteristic 0

Assume char $k = 0$. We assume also that

$$(*) \ k \cap \mathbb{Q}(\zeta_\ell) = \mathbb{Q}.$$

Thus

$$t_\ell = \ell - 1.$$

Assume $\ell \geq 7$. By Proposition 2, $\sigma$ cannot act minimally on a conic bundle. By Proposition 3, $\sigma$ can only act minimally on a Del Pezzo surface $X$ of degree 6 in which case $\ell = 7$. By Proposition 1, $X$ is a unique (up to isomorphism) toric surface $T$ over $k$ split over $E = k(\zeta_7)$. The Galois group $\Gamma$ acts on $T_E$ via the subgroup $H \subset \mathrm{Aut}(\Sigma)$ isomorphic to the cyclic group of order 6. The action of $H$ on $T_E$ is $\Gamma$-equivariant, and hence admits a

descent to an action of $H$ on $X$. The seven-torsion subgroup $T(k)[7]$ of $T(k)$ is $H$-invariant. Hence $H$ acts on the cyclic group $\langle \sigma \rangle$ of order 7 by automorphisms. This shows that all nontrivial powers of $\sigma$ are conjugate in $\mathrm{Cr}_2(k)$.

This proves the following.

**Theorem 12.**   Assume (\*) is satisfied. Then $\mathrm{Cr}_2(k)$ does not contain elements of prime order $> 7$ and all elements of order 7 are conjugate.                    □

It follows from the proof of Proposition 3 that an element of order $\ell = 5$ can be realized as an automorphism of a Del Pezzo surface of degree 8 defined over $\mathbb{Q}$. One can also show that it can be realized as a minimal automorphism of a Del Pezzo surface of degree 5 over $\mathbb{Q}$ that arises from the Cremona transformation $(x, y, z) \mapsto (xz, x(z - y), z(x - y))$ with four fundamental points $(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)$. The blow-up of these points is a Del Pezzo surface of degree 5 (see [3, p. 20]). Note that $\mathrm{PGL}_3(\mathbb{Q})$ does not contain elements of order $\geq 5$, so these two transformations are not conjugate to a projective transformation. Explicit examples of Cremona transformations of orders 5 and 7 over $\mathbb{Q}$ were given by N. Elkies in his letter to J. P. Serre on November 10, 2005 (see http://math.harvard.edu/~elkies/serre.pdf).

**References**

[1]   Batyrev, V., and Yu. Tschinkel. "Rational points of bounded height on compactifications of anisotropic tori." *International Mathematics Research Notices* 12 (1995): 591–635.

[2]   Colliot-Thlne, J.-L., and J.-J. Sansuc. "La *R*-quivalence sur les tores." *Annales Scientifiques de l'Ecole Normale Suprieure* 10, no. 4 (1977): 175–229.

[3]   de Fernex, T. "On planar Cremona maps of prime order." *Nagoya Mathematical Journal* 174 (2004): 1–28.

[4]   Demazure, M. "Surfaces de Del Pezzo, 1–5." In *Séminaire sur les Singularités des Surfaces*, edited by M. Demazure, H. Pinkham, and B. Teissier, 21–69. Lecture Notes in Mathematics 777. Berlin: Springer, 1980.

[5]   Dolgachev, I. "Weyl Groups and Cremona Transformations." In *Singularities*, Part 1, 283–94. Proceedings of Symposia in Pure Mathematics 40. Providence, RI: American Mathematical Society, 1983.

[6]   Dolgachev, I., and V. Iskovskikh, "Finite subgroups of the plane Cremona group." In Algebra, Arithmetic and Geometry: In honor of Y. I. Manin, 443–549, vol. 1, Progress in Math. vol. 269, Springer-Verlag, 2009.

[7]   Iskovskikh, V. A. "Factorization of birational mappings of rational surfaces from the point of view of Mori theory." *Uspekhi Matematicheskikh Nauk* 51 (1996): 3–72.

[8]   Klyachko, A., and V. Voskresenskiĭ, "Toric Fano varieties and systems of roots." *Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya* 48, no. 2 (1984): 237–63.

[9]   Lipman, J. "Desingularization of two-dimensional schemes." *Annals of Mathematics* 107, no. 2 (1978): 151–207.

[10]  Manin, Yu. *Cubic Forms. Algebra, Geometry, Arithmetic*, 2nd ed. North-Holland Mathematical Library 4. Amsterdam: North-Holland, 1986.

[11]  Minkowski, H. "Zur Theorie der positiven quadratischen Formen." *Journal für die reine und angewandte Mathematik* 101 (1887): 192–202.

[12]  Mori, S. "Threefolds whose canonical bundles are not numerically effective." *Annals of Mathematics* (2) 116 (1982): 133–76.

[13]  Robbiani, M. "On the arithmetic of isotropic del Pezzo surfaces of degree six." *Journal für die reine und angewandte Mathematik* 503 (1998): 1–45.

[14]  Serre, J. -P. *Cohomologie Galoisienne*, 5th ed. Lecture Notes in Mathematics 5. Berlin: Springer, 1994.

[15]  Serre, J.-P. "Bounds for the Orders of Finite Subgroups of $G(k)$." In *Group Representations Theory*, edited by M. Geck, D. Testerman, and J. Thévenaz. Lausanne, Switzerland: EPFL Press, 2006.

[16]  Voskresenskiĭ, V. *Algebraic Tori*. Moscow: Nauka, 1977.

[17]  Voskresenskiĭ, V. "Projective invariant Demazure models." *Izvestiya Akademii Nauk SSSR: Seriya Matematicheskaya* 46, no. 2 (1982): 195–210.