Inventiones math. 8, 34–54 (1969)

On the Purity of the Degeneration Loci of Families of Curves

I. V. DOLGAČEV (Moscow)

0. Introduction

Let $f\colon X \to Y$ be a morphism of preschemes and let **P** be a certain property of f in a point $x \in X$ (e.g. smoothness, equidimensionality, flatness and so on; see [EGA 4]). Consider the problem of describing the set of points $x \in X$ in which f does not satisfy the property **P** and also the projection of this set on Y. Many results concerning this problem are known. There exists, for example, the theorems (due to Grothendieck, see [EGA 4]) which assert the closedness of these sets. The question of the codimension of above sets is investigated in corresponding theorems of the purity, such as one of Zariski-Nagata ([13, 7]) for the property to be étale morphism and the theorem of the purity of Van der Waerden ([EGA 4], 21.12.12) for the property to be isomorphism.

Here we suggest the theorems of the purity when **P** is the property to be smooth and $f\colon X \to Y$ morphism of smooth schemes over a field with a smooth curve as a general fibre. Our first result is the following.

Theorem of the purity upstairs. Let $\mathrm{Sing}^X(f)$ be the set of points $x \in X$ in which f is not smooth. Assume that $\mathrm{Sing}^X(f) \neq \emptyset$. Then for any irreducible component S_i of $\mathrm{Sing}^X(f)$ we have $\operatorname{codim}(S_i, X) \leqq 2$.

As concerns of the projection of $\mathrm{Sing}^X(f)$ on Y, it seems natural the following

Conjecture. Let X and Y be regular integral schemes, $f\colon X \to Y$ a proper flat morphism of the relative dimension 1 with smooth general fibre. If $\mathrm{Sing}^Y(f) = f(\mathrm{Sing}^X(f)) \neq \emptyset$, then for any irreducible component S_i of $\mathrm{Sing}^Y(f)$ we have $\operatorname{codim}(S_i, Y) = 1$. In other words, $\mathrm{Sing}^Y(f)$ defines a divisor on Y.

Simple arguments based on the Stein's factorization of f and the theorem of the purity of Zariski-Nagata shows that we may assume that all fibres of f are geometrically connected.

We prove here the following result in the direction of this conjecture.

Theorem of the purity downstairs. Let g denote the genus of the general fibre of f. The previons conjecture is true under the following additional assumptions:

i) X and Y are smooth schemes over a field k.

ii) In case $g>1$ f is cohomologically flat (for example, $f_*(\mathcal{O}_X)=\mathcal{O}_Y$ universally, see other conditions in 2.7).

iii) In case $g \geqq 1$, $\operatorname{char}(k)=0$[1].

Note that whereas the first theorem almost immediately follows from the standart arguments using the differential criterion of smoothness, the proof of the second one has considerably more specific character.

It is interesting to note that at the proof of this theorem we used (explicitly or not) all known theorems of the purity, besides mentioned above these are one for étale cohomology ([SGAA]), for the group of Brauer ([GB 3]) and the Grothendieck's theorem of the purity for abelian schemes ([5]). The problem of the elimination of the restrictions imposed on f, X and Y is discussed in the last section of § 3.

In conclusion I wish to thank M. Raynaud, whose valuable remarks helped me to correct the first version of this paper. I also express my thanks to V.I. Danilov and Yu.I. Manin who very attentively looked through the manuscript of this paper and made many useful remarks.

1. Theorem of the Purity Upstairs

1.1. Recall one of equivalent definitions of smoothness ([EGA 4], 17.5.2):

Let $f\colon X \to Y$ be a morphism of locally finite type. One says that f is smooth in a point $x \in X$ if the following conditions are satisfied:

a) f is flat in x,

b) denoting by X_y the fibre of f in the point $y=f(x)$, we have $\mathcal{O}_{X_y,x}$ is a geometrically regular local ring (if the residue field of y is perfect, it is sufficient to require just the regularity of $\mathcal{O}_{X_y,x}$).

We shall use the following differential criterion of smoothness (cf. [EGA 4], 17.11.1): Let S be a prescheme, $f\colon X \to Y$ be a S-morphism of locally finite type. Let

$$f^* \Omega^1_{Y/S} \to \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0 \tag{1.1.1}$$

be the exact sequence of sheaves of relative differentials (loc. cit., 16.4.19). Assume that X and Y are smooth S-preschemes. Then f is smooth in

1. Raynaud in his letter to the author dated 13.1.69 outlined me another proof of the Theorem 3.2 which is based on results of [8] and is valid without any assumption of cohomological flatness and characteristic. Since 3.2 implies the theorem of the purity downstairs we get that restriction (ii) and (iii) cited in the introduction are unnecessary.

the point $x \in X$ if and only if the sheaf $\Omega^1_{X/Y}$ is free in x and the sequence (1.1.1) is exact in x.

In any case if $f: X \to Y$ is smooth in x, then $\Omega^1_{X/Y}$ is free in x ([EGA 4], 17.12.4).

1.2. Let f be a dominant morphism of locally finite type of integral preschemes. Recall that f is called equidimensional in the point $x \in X$ if the dimension of the irreducible component of the fibre $f^{-1}(f(x))$, passing through x is equal to the dimension of the general fibre of f.

If f is smooth in a point x, then it is equidimensional in x ([EGA 4], 17.5.6); furthermore the rank of the free module $\Omega^1_{X/Y,x}$ is equal to the dimension of the general fibre.

We shall say that a morphism f is of the relative dimension n if it is equidimensional in any point of X and the dimension of its general fibre is equal to n.

1.3. Lemma. *Let f: $X \to Y$ be a morphism of locally finite type of integral S-preschemes and let η be the general point of X. Suppose that f is smooth in η and Y is smooth over S. Then we have the exact sequence of sheaves on X:*

$$0 \to f^* \Omega^1_{Y/S} \to \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0. \tag{1.3.1}$$

Proof. Let $\gamma_X = \mathrm{Ker}(f^* \Omega^1_{Y/S} \to \Omega^1_{X/S})$. Since f is smooth in η, $\gamma_{X,\eta} = 0$; in other words, γ_X is a torsion sheaf on X. However, since Y is smooth over S, the sheaf $f^* \Omega^1_{Y/S}$ is locally free on X and can not contain a non-trivial torsion subsheaf. Hence $\gamma_X = 0$. q.e.d.

1.4. We shall need also the following assertion from the theory of local cohomology of coherent algebraic sheaves (cf. [SGA]).

Let X be a locally noetherian preschema, F a coherent $\mathcal{O}_X$-Module, j: $Y \hookrightarrow X$ an open immersion and $Z = X - Y$. We have the canonical homomorphism of sheaves on X:

$$u\colon\ F \to j_* j^* F.$$

We are interesting when u is an isomorphism. We have $\mathrm{Ker}(u) = \mathscr{H}^0_Z(F)$, $\mathrm{Coker}(u) = \mathscr{H}^1_Z(F)$, where $\mathscr{H}^i_Z(F)$ denote sheaves of local cohomology with respect to Z. Furthermore, denoting $\mathrm{depth}_Z(F) = \inf_{x \in Z} \mathrm{depth}\, F_x$ (where $\mathrm{depth}\, F_x = \mathrm{codh}\, F_x$ in notations of [11]), we have $\mathrm{depth}_Z(F) \geqq n \Leftrightarrow \mathscr{H}^i_Z(F) = 0$, $i < n$ ([SGA], Exp. 3). Hence, u is an isomorphism if and only if $\mathrm{depth}_Z(F) \geqq 2$.

1.5. Theorem (Purity upstairs). *Let f: $X \to Y$ be an S-morphism of locally finite type, where X, Y are integral smooth S-preschemes and X is regular. Assume that the general fibre of f is one-dimensional and f*

is smooth in the general point of X. *Let* $\mathrm{Sing}^X(f)$ *denote the set of points* $x \in X$, *in which* f *is not smooth, with induced structure of topological space. Suppose that* $\mathrm{Sing}^X(f) \neq \emptyset$. *Then for any maximal point* x_0 *of* $\mathrm{Sing}^X(f)$ *we have*

$$\operatorname{codim}(x_0, X) \stackrel{\text{def}}{=} \dim \mathcal{O}_{X, x_0} \leqq 2.$$

Proof. Let x_0 be a maximal point of $\mathrm{Sing}^X(f)$ (recall that it means that there is no such point $x_1 \in \mathrm{Sing}^X(f)$ for which $x_1 \neq x_0$ and $x_0 \in \overline{\{x_1\}}$). Suppose now that $\operatorname{codim}(x_0, X) > 2$. Let $A = \mathcal{O}_{X, x_0}$, $U = \operatorname{Spec} A$, $U' = U - x_0$, $j\colon U \hookrightarrow U'$ be a canonical immersion and $M = \Omega^1_{X/Y, x_0}$. The ring A is regular and hence due to Auslander-Buchsbaum is factorial. This easily implies that $\operatorname{Pic} U' = 0$ (cf. [EGA 4], 21.6.13). Since x_0 is maximal in $\mathrm{Sing}^X(f)$, for any $x \in U'$ the morphism f is smooth in x and hence by 1.2 $j^*(M^\sim)$ defines an invertible sheaf on U', but $\operatorname{Pic} U' = 0$, hence $j^*(M^\sim) \simeq \mathcal{O}_{U'}$. Since X and Y are smooth, the exact sequence 1.3.1 defines a locally free resolution of the sheaf $\Omega^1_{X/Y}$. Therefore, we have $\dim.\mathrm{proj.}(M) \leqq 1$, and hence, since A is regular,

$$\operatorname{depth} M = \dim A - \dim.\mathrm{proj.}(M) \geqq 2$$

(cf. [11]). Since $\operatorname{depth} A = \dim A > 2$, we have by 1.4 $j_* \mathcal{O}_{U'} = \mathcal{O}_U$, and hence $M^\sim = j_* j^* M^\sim = j_* \mathcal{O}_{U'} = \mathcal{O}_U = A^\sim$. So, $M = A = \Omega^1_{X/Y, x_0}$ is free, and therefore by 1.3 and 1.1 f is smooth in x_0.

1.6. Corollary. *Let* $f\colon X \to Y$ *be a morphism of locally finite type of smooth preschemes over a field with a curve as the general fibre. Suppose that* f *is smooth on an open set* $U \subset X$ *containing points of codimension* $\leqq 2$, *then* f *is smooth on* X.

1.7. *Remark.* Let $f\colon X \to Y$ be a morphism of preschemes satisfying the conditions of 1.5. Let $\vartheta_{X/Y}$ denote the different of f. (See [4].) This a sheaf of ideals on X and from 1.1 and 1.3 it follows that the underlying topological space of the corresponding subscheme of X coincides with $\mathrm{Sing}^X(f)$. Let $\mathcal{D}_{X/Y}$ be the structure sheaf of this subscheme. (Discriminant of f in terms of [4].) The Theorem 1.5 can be deduced also by considering the generalized Koszul complex of $\mathcal{D}_{X/Y}$ (cf. [2], Th. 4) which can be easily constructed taking the sequence dual to 1.3.1.

2. Betti Numbers of Curves Varying in a Family

2.1. Let X be an algebraic curve over a separably closed field k, i.e. a k-scheme of finite type without immersed components with $\dim X = 1$. Assume that X is reduced and let $X = \bigcup_{i=1}^{h} X_i$ its decomposition into irreducible components. Let $\bar{X}_i$ be the normalisation of X_i, $\bar{X} = \amalg \bar{X}_i$, $p\colon \bar{X} \to X$ be the canonical projection. Denoting by P a closed point of X, define the numbers

$$\delta_{X,P} = \dim_k p_*(\mathcal{O}_{\bar{X}})_P / \mathcal{O}_{X,P}; \qquad \delta_X = \sum_{P \in X} \delta_{X,P}.$$

Since the morphism p is finite, the sheaf $p_*(\mathcal{O}_{\bar{X}})/\mathcal{O}_X$ is coherent and concentrated in the finite set of points and hence our definitions are correct. Define also the numbers

$$\delta'_{X,P} = \operatorname{Card}\{p^{-1}(P)\} - 1; \qquad \delta'_X = \sum_{P \in X} \delta'_{X,P}.$$

2.2. If X is a complete algebraic curve, then its arithmetic genus is defined as

$$\pi(X) = \dim_k H^1(X, \mathcal{O}_X).$$

Assume that X is reduced and let g_i be the genus of X_i. The Leray spectral sequence for the morphism f and the sheaf $\mathcal{O}_{\bar{X}}$ yields

$$\pi(X) = \sum_{i=1}^{h} g_i + \delta_X - h + s \tag{2.2.1}$$

where s is the number of connected components of X.

2.3. Define the Betti numbers of X as

$$\beta_i(X) = \begin{cases} s, & i=0 \\ 2\sum_{i=1}^{h} g_i + \delta'_X - h + 1, & i=1 \\ h, & i=2 \\ 0, & i>2. \end{cases}$$

If X is not reduced then we let

$$\beta_i(X) = \beta_i(X_{\mathrm{red}}).$$

2.4. Proposition. *Let X be a complete algebraic curve over a separably closed field k. The étale cohomology of X with coefficients in the constant sheaf $\mu_{n,X} \simeq (\mathbf{Z}/n\,\mathbf{Z})_X$ $((n, \operatorname{char} k) = 1)$ are as follows:*

$$H^i(X_{\mathrm{et}}, \mu_{n,X}) = (\mathbf{Z}/n\,\mathbf{Z})^{\beta_i(X)}.$$

If furthermore X is reduced and $k = \mathbf{C}$, the field of complex numbers, then

$$H^i(X_{\mathrm{cl}}, \mathbf{Z}) = \mathbf{Z}^{\beta_i(X)}.$$

Proof. Since the étale cohomology are topologically invariant ([SGAA], Exp. 10,1.2) we have $H^i(X_{\mathrm{et}}, \mu_{n,X}) = H^i(X_{\mathrm{red,et}}, \mu_{n,X_{\mathrm{red}}})$. After that in both cases the assertion is easily deduced from the Leray spectral sequence for the morphism $p\colon \bar{X}_{\mathrm{red}} \to X_{\mathrm{red}}$ (see for details in [3], Ch. 3).

2.5. Corollary. *Let X be a complete algebraic curve, defined over a separably closed field k. Then the topological Euler-Poincaré characteristic of X which is defined as*

$$EP(X) = \sum_i (-1)^i \beta_i(X)$$

coincides with the 1*-adic Euler-Poincaré characteristic of X (where $1 \neq \operatorname{char} k$). If $k = \mathbf{C}$ and X is reduced, then $EP(X)$ is equal to the usual topological Euler characteristic of X.*

2.6. Definition. *A flat morphism $f\colon X \to Y$ is called cohomologically flat over $y \in Y$ if the canonical map $(f_* \mathcal{O}_X)_y \otimes k(y) \to H^0(X_y, \mathcal{O}_{X_y})$ is bijective.*

2.7. Proposition. *A proper flat morphism $f\colon X \to Y$ with $f_*(\mathcal{O}_X) = \mathcal{O}_Y$ is cohomologically flat over $y \in Y$ in any of the following cases:*

1. *$\mathcal{O}_X$ is cohomologically flat in dimension* 0 *over y (i.e. $f_*(\mathcal{O}_X) = \mathcal{O}_Y$ universally) (cf.* [EGA 3], 7.8.1*).*

2. *f is separable over y, i.e. the geometric fibre $X_{\bar{y}} = X_y \otimes_{k(y)} \overline{k(y)}$ is reduced.*

3. *Y is regular, there exist $n-1$ local parametres $t_1, \ldots, t_{n-1}$ of the ring $\mathcal{O}_{Y,y}$ ($n = \dim Y$) such that the prescheme*

$$X_t = X \otimes_Y \mathcal{O}_{Y,y}/(t_1, \ldots, t_{n-1}) \mathcal{O}_{Y,y}$$

is normal, the general fibre of the projection $X_t \to \operatorname{Spec} \mathcal{O}_{Y,y}/(t_1, \ldots, t_{n-1}) \mathcal{O}_{Y,y}$ is separable and one of the following condition is satisfied:

i) $\operatorname{char} k(y) = 0$,

ii) *if $\operatorname{char} k(y) = p > 0$, then $H^2(X_y, \mathcal{O}_{X_y}) = 0$ and either g.c.d. of the multiplicites of geometrical components of the fibre X_y is prime to p or there exist an étale quasi-section of X_t over $\operatorname{Spec} \mathcal{O}_{Y,y}/(t_1, \ldots, t_{n-1}) \mathcal{O}_{Y,y}$.*

4. *The sheaves $R^i f_* \mathcal{O}_X$ are locally free for $i \geqq 0$.*

Proof. The assertion 1. is obvious; to prove 2. apply the Küneth formula, we have

$$H^0(X_{\bar{y}}, \mathcal{O}_{X_{\bar{y}}}) = H^0(X_y, \mathcal{O}_{X_y}) \otimes_{k(y)} \overline{k(y)}.$$

Since $X_{\bar{y}}$ is connected and has a rational point, we have $H^0(X_{\bar{y}}, \mathcal{O}_{X_{\bar{y}}}) = \overline{k(y)}$, this gives $H^0(X_y, \mathcal{O}_{X,y}) = k(y)$; as it follows from [EGA 3], 7.8 in case 4. f is cohomologically flat in dimension 0. The assertion 3. is an immediate corollary of the criterion of cohomological flatness in dimension 0 for one-dimensional base, due to Raynaud ([8], Th. 4), which can be applied in virtue of the following:

2.8. Lemma. *Let A be a discrete valuation ring, $Y = \operatorname{Spec} A$, $f: X \to Y$ proper flat morphism with geometrically integral general fibre. Then* $f_*(\mathcal{O}_X) = \mathcal{O}_Y$.

Proof. Since f is proper $f_*(\mathcal{O}_X) = M$, where M is A-module of finite type. Since f is flat, M is without torsion. However, finite modules without torsion over a discrete valuation ring are free. Hence M is free. Let y be the general point of Y, $K = \mathcal{O}_{Y,y}$ be the fraction field of A. Since X_y is geometrically integral, we have $M \underset{A}{\otimes} K = (f_* \mathcal{O}_X)_y = H^0(X_y, \mathcal{O}_{X_y}) = K$ (cf. the proof of 2.7,2). Hence $M = A$ and $f_*(\mathcal{O}_X) = \mathcal{O}_Y$.

2.9. Corollary. *Let V be a regular curve over a discrete valuation ring A, i.e. a proper morphism of a regular two-dimensional scheme on* Spec A *(see [6]). Suppose that the residue field of A has zero characteristic. Then V is cohomologically flat over the closed point of* Spec A.

Proof. Since V is regular it easy to see that the conditions of 2.8 are satisfied and we can apply 2.7, 3.

In order to make more clear the role of characteristic we shall give the following example due to Grothendieck (cf. [8]).

2.10. *Example.* Let k_0 be an algebraically closed field of characteristic $p > 0$, $k = k_0((t))$ be the field of formal power series over k_0. Consider over k an elliptic curve A whose reduction is an elliptic curve A_0 with zero invariant of Hasse. Let $f: V \to Y = \operatorname{Spec} k_0[[t]]$ be the minimal model of A (see [10]). Recall it means that V is regular, f is proper with general fibre isomorphic to A and the closed one to A_0. Since the group $H^1(k, A)_p \neq 0$ (see [12]), there exist a principal homogenious space X over A which has order p in $H^1(k, A)$. Let $\varphi: W \to Y$ be the corresponding minimal model of X, let F be the special fibre of φ. It easy to see that $F_0 = F_{\mathrm{red}}$ is an elliptic curve with zero invariant of Hasse and the multiplicity of F is equal to p. We shall show that W is not cohomologically flat over the closed point of Y. Let $\mathcal{J}$ be the Ideal of definition of F_0. Since the selfintersection index of F_0 on W is equal to 0 (see [6]) we obtain that the conormal sheaf $\mathcal{J}/\mathcal{J}^2$ of F_0 is an invertible sheaf on F_0 of degree 0. Since F is a principal divisor on W we have $F \cdot F_0$ is a principal on F_0, hence $\mathcal{O}_{F_0} \otimes \mathcal{J}^p = \mathcal{J}^p/\mathcal{J}^{p+1} \simeq \mathcal{O}_{F_0}$.

Therefore $\mathcal{J}/\mathcal{J}^2$ defines on F_0 a point of the finite order equal to p. Since F_0 has zero invariant of Hasse such point is necessary the unit point of F_0, and hence $\mathcal{J}/\mathcal{J}^2 \simeq \mathcal{O}_{F_0}$. Denote $\mathcal{O}_{F_n} = \mathcal{O}_W/\mathcal{J}^{n+1}$, we have for any $n \geq 1$ the exact sequence

$$0 \to \mathcal{J}^n/\mathcal{J}^{n+1} \to \mathcal{O}_{F_n} \to \mathcal{O}_{F_{n-1}} \to 0.$$

Since $\mathscr{I}$ is invertible on W, we have $\mathscr{I}^n/\mathscr{I}^{n+1}=(\mathscr{I}/\mathscr{I}^2)^n=\mathcal{O}_{F_0}$ for each $n\geqq 1$. This gives

$$H^i(W,\mathscr{I}^n/\mathscr{I}^{n+1})=H^i(F_0,\mathcal{O}_{F_0})=k_0 \qquad (i=0,1)$$

write the exact cohomology sequence

$$0\to H^0(W,\mathcal{O}_{F_0})\to H^0(W,\mathcal{O}_{F_n})\to H^0(W,\mathcal{O}_{F_{n-1}})\to H^1(W,\mathcal{O}_{F_0}).$$

Taking $n=1$ and noticing that the map $H^0(W,\mathcal{O}_{F_1})\to H^0(W,\mathcal{O}_{F_0})$ is surjective, we obtain that

$$H^0(W,\mathcal{O}_{F_1})=k_0^2.$$

Hence the map $H^0(W,\mathcal{O}_{F_1})\to H^1(W,\mathcal{O}_{F_0})=k_0$ has nontrivial kernel. This gives that $H^0(W,\mathcal{O}_{F_2})\to H^0(W,\mathcal{O}_{F_1})$ is not zero. Therefore we obtain that $\dim_{k_0} H^0(W,\mathcal{O}_{F_2})\geqq 2$. Proceeding by induction, we have

$$\dim_{k_0} H^0(W,\mathcal{O}_{F_p})=\dim_{k_0} H^0(F,\mathcal{O}_F)\geqq 2.$$

Since by 2.8 $f_*(\mathcal{O}_X)=\mathcal{O}_Y$ this shows that $f\colon W\to Y$ is not cohomologically flat

2.11. Theorem. *Let $f\colon X\to Y$ be a cohomologically flat proper morphism of the relative dimension one. Assume that Y is integral and the general fibre X_η of f is a smooth geometrically connected curve of the genus g. Then for any geometric point $\bar{y}$ with the centre in $y\in Y$, we have*

$$\beta_0(X_{\bar{y}})=\beta_0(X_{\bar{\eta}})=1,$$
$$\beta_1(X_{\bar{y}})\leqq\beta_1(X_{\bar{\eta}})=2g,$$
$$\beta_2(X_{\bar{y}})\geqq\beta_2(X_{\bar{\eta}})=1.$$

Proof. The assertion about β_0 is an immediate corollary of the Connectedness Theorem of Zariski (see [EGA 3], 4, 3.10), and one concerning β_2 is trivial. We prove the assertion about β_1. Let $A=\mathcal{O}_{Y,y}$; obviously we may assume that $Y=\operatorname{Spec} A$, that the residue field of A is separably closed and that $F=X_y$ is the special fibre of the morphism $f\colon X\to Y$. By the invariance of the Euler-Poincaré characteristic (with coefficients in the structure sheaf) of fibres of a flat proper morphism ([EGA 3], 7.9.11), we have, since $\dim_{k_0} H^0(X_y,\mathcal{O}_{X_y})=1$

$$\pi(F)=\pi(X_\eta)=g.$$

Let $F_0=F_{\mathrm{red}}$ and $\pi'=\dim_k H^1(F_0,\mathcal{O}_{F_0})$, the ariphmetic genus of F_0. The natural immersion $F_0\hookrightarrow F$ induces the epimorphism $H^1(F,\mathcal{O}_F)\to H^1(F_0,\mathcal{O}_{F_0})$ (since $\dim F=1$). This yields

$$\pi'\leqq\pi(F)=g. \tag{1}$$

Hence it is sufficient to show that

$$\beta_1(F_0) \leqq 2\pi'. \tag{2}$$

By (2.2.1) and 2.3 this inequality is equivalent to the following one:

$$2\delta_{F_0} - \delta'_{F_0} - h + 1 \geqq 0. \tag{3}$$

Let $F_0 = \bigcup_{i=1}^{h} F_i$, where F_i are irreducible components, $\tilde{F} = \amalg F_i$, let $\bar{F} = \amalg \bar{F}_i$ be the normalisation of F_0 and let F' be the curve, the underlying topological space of which coincides with one of F and for any open $U \subset F$, $\Gamma(U, \mathcal{O}_{F'}) = \{(f_1, \ldots, f_n) | f_i \in \Gamma(U \cap F_i, \mathcal{O}_F), f_i(P) = f_j(P)$ for all pairs i, j such that $P \in F_i \cap F_j\}$. We have natural morphisms

$$\bar{F} \xrightarrow{p_3} \tilde{F} \xrightarrow{p_2} F' \xrightarrow{p_1} F_0.$$

Denote by $p = p_1 \circ p_2 \circ p_3$; $\bar{F} \to F_0$, $p' = p_1 \circ p_2$: $\tilde{F} \to F_0$ the corresponding compositions. Let $P \in F_0$, F_i $(j = 1, \ldots, n_p)$ be irreducible components of F_0, which contain P. We have

$$p_*(\mathcal{O}_{\bar{F}})_P \supseteq p'_*(\mathcal{O}_{\tilde{F}})_P = \bigoplus_{J=1}^{n_p} \mathcal{O}_{F_{i_j}, P} \supseteq p_{1,*}(\mathcal{O}_{F'})_P = \mathcal{O}_{F', P} \supseteq \mathcal{O}_{F_0, P}.$$

Furthermore

$$\dim_k p_*(\mathcal{O}_{\bar{F}})_P / p'_*(\mathcal{O}_{\tilde{F}})_P = \sum_{j=1}^{n_p} \delta_{F_{i_j}, P},$$

where $P_j \in \tilde{F}$ and $p'_*(P_j) = P$,

$$\dim_k p'_*(\mathcal{O}_{\tilde{F}})_P / \mathcal{O}_{F', P} = n_p - 1.$$

We also define the number $\gamma_P = \dim_k \mathcal{O}_{F', P} / \mathcal{O}_{F, P}$.

Hence, we have

$$\delta_{F_0, P} = n_p - 1 + \sum_{j=1}^{n_p} \delta_{F_{i_j}, P_j} + \gamma_P.$$

Similar arguments show that

$$\delta'_{F_0, P} = n_p - 1 + \sum_{j=1}^{n_p} \delta'_{F_{i_j}, P_j}.$$

Now (3) can be rewritten in the form:

$$\sum_{P \in F_0} (n_p - 1) + 2 \sum_{P \in F_0} \sum_{j=1}^{n_p} \delta_{F_{i_j}, P_j} + 2 \sum_{P \in F_0} \gamma_P - \sum_{P \in F_0} \sum_{j=1}^{n_p} \delta'_{F_{i_j}, P_j} - h + 1 \geqq 0.$$

Let $\mathfrak{c}_{F_{i_j}}$ be a local component of the conductor of the curve F_{i_j} in its normalisation $\bar{F}_{i_j}$ considered as a divisor on $\bar{F}_{i_j}$. We have $2\delta_{F_{i_j}, P_j} \geqq$

$\deg(\mathfrak{c}_{F_{i_j},P_j})$ ([9], Th. 10). This implies

$$2\delta_{F_{i_j},P_j} \geqq \deg \mathfrak{c}_{F_{i_j},P_j} \geqq \delta'_{F_{i_j},P_j}. \tag{4}$$

The obvious inductive argument shows that

$$\sum_{P \in F_0} (n_P - 1) - h + 1 \geqq 0.$$

Collecting all these inequalities we get (2).

2.12. Suppose that for the fibre F of the morphism $f: X \to Y$ we have an equality $\beta_1(F) = 2g$. In this case all inequalities in the proof of 2.6 must be equalities. We have

(1) and (2) $\Rightarrow \pi(F_0) = \pi(F)$;

(4) $\Rightarrow$ all curves F_i are nonsingular;

(5) $\Rightarrow$ the irreducible components of F are connected as a tree;

(6) $\Rightarrow \forall P \in F_0$, $\gamma_P = 0$, this shows that in each point P components of F_0 are intersecting transversally.

2.13. Let $f: V \to \operatorname{Spec} A$ be a regular curve over a discrete valuation ring A with the algebraically closed residue field k (cf. 2.9). Suppose that the general fibre of f is a geometrically connected smooth curve of the genus g. By the Connectedness theorem of Zariski (see [EGA 3], 4.3.10) this implies that the special fibre F of f is connected. Furthermore it is easy to see, applying [EGA 4], 15.4.2 that F has no immersed points.

We shall use the notations and results of the intersection theory on V (see [6, 10]).

Lemma. *Assume additionally that V has no exceptional curves of the* 1st *kind and that $g > 0$. Considering F as a divisor on V, suppose that $F = nC + D$, where C is an integral curve of the arithmetic genus $g' \geqq g$ not contained in the support of D. Then we have $g' = g$, $D = \emptyset$ and either $n = 1$ or $n > 1$ and $g' = g = 1$.*

Proof. Recall that we have for any effective divisor Z with support on F ([6], Th. 3.2):

$$\chi(Z, \mathcal{O}_Z) \overset{\text{def}}{=} p_a(Z) = -\frac{(Z \cdot K) + (Z^2)}{2} \tag{1}$$

where K is the relative canonical divisor on V determined (not uniquely) by an isomorphism $\mathcal{O}_V(K) = \omega_V = \Omega^1_{V/A}$. We have also ([6], no. 1):

$$(F \cdot Z) = 0. \tag{2}$$

The invariance of the Euler characteristic ([EGA 3], 7.9.11) implies that

$$p_a(F)=1-g. \tag{3}$$

Since C is reduced we have $H^0(C, \mathcal{O}_C)=k$ and hence

$$p_a(C)=\dim_k H^0(C, \mathcal{O}_C)-\pi(C)=1-g'. \tag{4}$$

From (2) it follows that

$$0=(C\cdot F)=n(C^2)+(C\cdot D).$$

Since F is connected and C is not the component of D we have therefore

$$(C^2)\leqq 0 \quad \text{and} \quad (C^2)=0 \Leftrightarrow D=\emptyset. \tag{5}$$

Let $D=n' C'+D'$, where C' is reduced and is not the component of D', we have from (2):

$$0=(F\cdot C')=n(C'\cdot C)+n'(C'^2)+(D'\cdot C').$$

Hence $(C'^2)<0$ and since $p_a(C')\leqq 1$ (C' is reduced!) we have from (1) $(C'\cdot K)\geqq -1$. The case $(C'\cdot K)=-1$ is impossible, since it would be imply that C' is an exceptional curve of the 1st kind (the Castelnoovo's criterion, [6], 3.9). Therefore we have $(C'\cdot K)\geqq 0$ and since C' is an arbitrary component, we get

$$(D\cdot K)\geqq 0. \tag{6}$$

Now we have from (1) and (2):

$$2g-2=(K\cdot F)+(F^2)=(K\cdot F)=n(K\cdot C)+(D\cdot K).$$

Applying (4) we obtain

$$2g-2=n(2g'-2)-n(C^2)+(D\cdot K).$$

Denote $\Delta=g'-g$, we have from (5)

$$n(C^2)=(n-1)(2g'-2)+2\Delta+(D\cdot K)\leqq 0.$$

However from the conditions of the lemma and (6) we have that all summands in the right side are nonnegative. Hence we get

$$\Delta=0\Rightarrow g'=g,$$
$$(C^2)=0\Rightarrow D=\emptyset \qquad \text{(in virtue of (5))}$$

and $(n-1)(g'-1)=0$.

2.14. Corollary. *In hypothesises of* 2.11, *we have for any* $y\in Y$

$$EP(X_{\bar{y}})\geqq EP(X_{\bar{\eta}})=2-2g.$$

Furthermore, if char $k(y)=0$ *and* X *and* Y *are regular the equality takes place if and only if either* X_y *is smooth or* $g \leqq 1$ *and* $X_{y,\mathrm{red}}$ *is smooth curve of the genus* g.

Proof. As it follows from 2.11 we have

$$EP(X_{\bar{y}})-EP(X_{\bar{\eta}})=(\beta_1(X_{\bar{\eta}})-\beta_1(X_{\bar{y}}))+(\beta_2(X_{\bar{y}})-\beta_2(X_{\bar{\eta}}))\geqq 0$$

and the equality takes place if and only if

$$\beta_1(X_{\bar{y}})=\beta_1(X_{\bar{\eta}})=2g,$$
$$\beta_2(X_{\bar{y}})=\beta_2(X_{\bar{\eta}})=1.$$

In virtue of 2.12 it implies that $X_{y,\mathrm{red}}$ is a smooth curve of the genus equal to g. Suppose that X_y is not geometrically reduced (applying [EGA 4], 15.4.2 it is easy to see that $X_{\bar{y}}$ has no immersed points and hence in classical terms "$X_{\bar{y}}$ is a multiple nonsingular fibre") and $g \geqq 2$. The assertion is local and hence we can assume that $Y=\operatorname{Spec} A$, where A is a local ring with the algebraically closed residue field and y is the closed point of A. ([EGA 4], 17.7.3.) Let $n=\dim A$, since the property smoothness is constructive ([EGA 4], 17.7.11), there exist $n-1$ local parametres $t_1, \ldots, t_{n-1} \in A$ such that the general fibre of the canonical morphism $X \otimes_Y A/(t_1, \ldots, t_{n-1}) A \to \operatorname{Spec} A/(t_1, \ldots, t_{n-1}) A$ is smooth. Denote by B the discrete valuation ring $A/(t_1, \ldots, t_{n-1}) A$ let f': $X_t = X \otimes_Y B \to \operatorname{Spec} B$ be the canonical projection. The special fibre F of f' is isomorphic to $X_y \otimes_{k(y)} \overline{k(y)}$, where $\overline{k(y)}$ is the algebraic closure of $k(y)$. Since flatness is preserved by the change base, f' is flat. This easily implies that X_t is reduced.

Let p: $\bar{X}_t \to X_t$ be the canonical morphism of normalization of X_t. Denote by $\bar{F}$ the special fibre of the projection $\bar{f}$: $\bar{X}_t \to \operatorname{Spec} B$. Each irreducible component of $\bar{F}_{\mathrm{red}}$ is a curve of the genus $\geqq g$. Applying the lemma to the desingularisation of X_t (which exist due to results of Hironaka and obviously can be selected without exceptional curves of the 1st kind) we obtain that $\bar{F}$ is an integral curve of the genus g. Since $g > 1$ this implies (by means, for example, the Hurwitz formula) that the restriction of p on $\bar{F}$ defines the isomorphism p': $\bar{F} \to F_{\mathrm{red}}$ (here we use that char $k(y)=0$). Let $\bar{x}_0 \in \bar{X}_t$ be the general point of $\bar{F}$, $x_0 = p(\bar{x}_0)$. The ring $\mathcal{O}_{X_t, x_0}$ is integral and its normalisation coincides with $\mathcal{O}_{\bar{X}_t, \bar{x}_0}$. Since $p^{-1}(F)=\bar{F}$ it is easy to see that $\mathfrak{m}_{y_0} \mathcal{O}_{\bar{X}, \bar{x}_0} = \mathfrak{m}_{x_0}$ and since p' is an isomorphism we get that the corresponding extension $k(\bar{x}_0)/k(x_0)$ is trivial. Hence we obtain that a) X_t is geometrically unibranched in x_0 ([EGA 4], 6.15.1) and b) the canonical morphism p: $\bar{X}_t \to X_t$ is nonramified in $\bar{x}_0$. Applying [EGA 4], 18.10.1 this implies that p is étale in $\bar{x}_0$ and hence (loc. cit., 17.5.8) X_t is normal in y_0 and p is an isomorphism in $\bar{x}_0$. This yields that F is reduced in its general point and since $F = X_{\bar{y}}$ has no immersed points we get that it is reduced. This contradiction proves the assertion of the corollary.

2.15. *Remarks.* a) In the case of $\dim X = 2$ without the assumption of cohomological flatness the previous statement had been proved by Šafarevič ([1], Ch. 4, Th. 7) who used the intersection theory on X. In this case the equality is valid if and only if either X_y is smooth or $X_{y,\mathrm{red}}$ is a smooth elliptic curve. The general case gives the third possibility namely, $X_{y,\mathrm{red}}$ is a rational curve and such examples can be easily constructed with fibering of a threefold over a surface.

b) The generalizing of 2.12 to the statement about semicontiniousness of the Euler-Poincaré characteristic and Betti numbers is false if we do not assume that f is a separable morphism or $\dim Y = 1$. I conjecture that in case when f is separable it is true.

3. Purity Downstairs

3.1. Let $f: X \to Y$ be a flat proper morphism of the relative dimension 1 with X, Y smooth over a field k. Suppose that the general fibre of f is a smooth geometrically connected curve of the genus g. We want to show that at certain restriction on f the set $\mathrm{Sing}^Y(f) = f(\mathrm{Sing}^X(f))$ (see notations of 1.5), if it is not empty, defines a divisor on Y. For the future we assume that $\mathrm{Sing}^Y(f) \neq \emptyset$.

Lemma. *For any maximal point* y_0 *of* $\mathrm{Sing}^Y(f)$ *we have*

$$\operatorname{codim}(y_0, Y) \stackrel{\mathrm{def}}{=} \dim \mathcal{O}_{Y, y_0} \leqq 2.$$

Proof. Since f is equidimensional, for any maximal point x_0 of the fibre $f^{-1}(y_0)$ we have $\operatorname{codim}(x_0, X) = \operatorname{codim}(y_0, Y)$ ([EGA 4], 13.2.9). It is obvious that $f^{-1}(y_0)$ contains a maximal point of $\mathrm{Sing}^X(f)$. In virtue of 1.5 this implies that $\operatorname{codim}(y_0, Y) \leqq 2$.

Suppose now that there exist a maximal point y_0 of $\mathrm{Sing}^Y(f)$ with $\operatorname{codim}(y_0, Y) = 2$. Let $A = \mathcal{O}_{Y, y_0}$, A^{sh} be its strict henselization ([EGA 4], 18.5.8), y_0 be the image of y_0 in $\operatorname{Spec} A^{\mathrm{sh}}$. By "localizing f" we may assume that $Y = \operatorname{Spec} A^{\mathrm{sh}}$ ([EGA 4], 17.7.3). Let $Y' = Y - y_0$, $X' = X_Y \times Y'$. Thus we have the commutative diagram:

$$\begin{array}{ccc} X' & \xhookrightarrow{i} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xhookrightarrow{j} & Y \end{array}$$

where i and j are canonical immersions and f' is a restriction of f on X'. Since y_0 was maximal on $\mathrm{Sing}^Y(f)$ and the smoothness is preserved at the base change ([EGA 4], 17.7.3) f' is smooth. Thus we must to prove the following.

3.2. *Theorem of the purity downstairs* (local form). Let $Y = \mathrm{Spec}\, A$ be a local strictly henselian regular two-dimensional scheme, $y_0 \in Y$ its closed point, $Y' = Y - y_0$ be the punctured local scheme. Let $f: X \to Y$ be a proper flat morphism of the relative dimension 1 with smooth geometrically connected curve of the genus g as a general fibre. Assume that in case $g > 1$ f is cohomologically flat and in the case $g \geqq 1$ the residue field k of A has zero characteristic. Suppose that the restriction of f on $X' = X \underset{Y}{\times} Y'$ is smooth, then f is smooth.

3.3. Lemma (Grothendieck). *Let A be a regular noetherian ring of the dimension two, $\mathfrak{m}$ its maximal ideal, $X = \mathrm{Spec}\, A$, $X' = X - \{\mathfrak{m}\}$. Then for any locally free sheaf L' on X' there exists its free extension on X, i.e. a free sheaf L on X whose restriction on X' coincides with L'.*

Proof. Let j be the canonical immersion $X' \hookrightarrow X$. Since L' is locally free, $j_* L$ is a coherent sheaf on X (cf. [SGA], Exp. VIII, III-2). Denote by M an A-module of finite type for which $j_* L = M^\sim$. Since $j_* j^* j_* L = j_* L = M^\sim$ we have by 1.4 depth $M \geqq 2$, and hence, since A is regular, we get dim. proj. $M = \dim A - \operatorname{depth} M = 0$ (see [11]). Thus M is projective and, since A is local, it is free.

Corollary 1. *A locally free sheaf on X' is free.*

Corollary 2. *A is pure, i.e. any étale covering of X' is a restriction of some étale covering of X (cf.* [SGA], *Exp.* 10, 3.4*).*

3.4. We shall need some notations and results from the theory of the Brauer's group of a prescheme, developed by Grothendieck (see [GB i], $i = 1, 2, 3$).

Let X be a prescheme, the second cohomology group $H^2(X, G_{m,X})$ in étale topology of X is called the cohomological group of Brauer of X and is denoted by $\mathrm{Br}'(X)$ ([GB 2], 2.7). This group coincides with the group of Brauer of X, defined as the group of classes of Azumaya's algebres on X (see [GB 1], 1.2) in the following cases ([GB 2], 2.2 and 2.5):

i) X is noetherian and $\dim X \leqq 1$.

ii) X is regular and $\dim X \leqq 2$.

iii) X is local henselian.

3.5. Lemma. *Let A be a two-dimensional local regular strictly henselian ring, $Y = \mathrm{Spec}\, A$, x its closed point, $Y' = Y - x$. Then* $\mathrm{Br}'(Y') = \mathrm{Br}\, Y' = 0$.

Proof. Since $\dim Y' = 1$, we have $\mathrm{Br}'(Y') = \mathrm{Br}\, Y'$. Let $\mathscr{A}'$ be an algebra of Azumaya on Y'. Consider the canonical injection $j\colon Y' \hookrightarrow Y$ and let $\mathscr{A} = j_* \mathscr{A}'$. In virtue of the Corollary 1 to the Lemma 3.3, $\mathscr{A}'$ is free as an $\mathcal{O}_{Y'}$-Module. Since $\dim A = 2$, $j_* \mathcal{O}_{Y'} = \mathcal{O}_Y$ (see 1.4) and therefore $\mathscr{A}$ is also free as an $\mathcal{O}_Y$-Module. Hence $\mathscr{A}$ is an algebra of Azumaya on Y ([GB 1], Cor. 5.2). However, since A is strictly henselian, any algebra of Azumaya on Y is trivial (theorem of Azumaya, [GB 1], Cor. 6.2). Therefore $\mathscr{A}' = j^* \mathscr{A}$ is also trivial. q.e.d.

3.6. Lemma. *With notations of* 3.2, *we have for the special fibre* F *of* f:

$$\beta_1(F) = 2g, \qquad \beta_2(F) = 1.$$

Proof. Endow the preschemes X, X', Y' and Y with étale topology and consider on X' the sheaf $\mu_{n,X'} \simeq (\mathbf{Z}/n\mathbf{Z})_{X'}$, $(n, \mathrm{char}\, k) = 1$. We have

$$R^i f'_* \mu_{n,X'} = \begin{cases} \mu_{n,Y'}^{2g} & i=1 \\ \mu_{n,Y'} & i=2 \\ 0 & i>2. \end{cases}$$

Really, since f' is smooth, for any geometric point $\bar{y}$ of Y' the corresponding fibre $X_{\bar{y}}$ is a smooth curve. Since f' is cohomologically flat (see 2.7(2)), by the invariance of the ariphmetic genus of the fibres of a flat proper morphism ([EGA 3], 7.9.11), the genus of X_y is equal to g. Hence from 2.4 it follows that

$$H^i(X_{\bar{y}}, \mu_{n, X_{\bar{y}}}) = \begin{cases} (\mathbf{Z}/n\mathbf{Z})^{2g} & i=1 \\ (\mathbf{Z}/n\mathbf{Z}) & i=2 \\ 0 & i>2. \end{cases} \tag{1}$$

This is also an immediate corollary of the theorem of specialization for cohomology groups ([SGAA], XVI, 2.2). Applying the "Base change theorem" ([SGAA], Exp. 12, 3.1), we have

$$(R^i f'_* \mu_{n,X'})_{\bar{y}} = H^i(X_{\bar{y}}, \mu_{n, X_{\bar{y}}}) \simeq \begin{cases} (\mathbf{Z}/n\mathbf{Z})^{2g} & i=1 \\ \mathbf{Z}/n\mathbf{Z} & i=2 \\ 0 & i>2. \end{cases}$$

By [SGAA], Exp. 9, 2.13 this implies that the sheaves $R^i f'_* \mu_{n,X'}$ $(i = 1, 2)$ are locally constant on Y' and therefore (loc. cit. 2.1) are represented by an étale covering $\bar{Y}'$ of Y'. As it follows from 3.3 (namely, Corollary 2), $\bar{Y}'$ is induced by some étale covering of Y, which is trivial, since Y is strictly henselian ([EGA 4], 18.8.1). Thus $\bar{Y}'$ is trivial, i.e. $\bar{Y}' = \coprod \bar{Y}'_i$, where $\bar{Y}'_i \xrightarrow{\sim} Y'$. This shows that the sheaves $R^i f'_* \mu_{n,X'}$ are constant on Y' and hence we have (1).

Consider now the Kummer exact sequences on Y':

$$0 \to H^0(Y', \mathcal{O}_{Y'})^*/H^0(Y', \mathcal{O}_{Y'})^{*n} \to H^1(Y', \mu_{n,Y'}) \to (\text{Pic } Y')_n \to 0,$$
$$0 \to (\text{Pic } Y')^{(n)} \to H^2(Y', \mu_{n,Y'}) \to H^2(Y', G_{m,Y'})_n \to 0.$$

As it was explained in the proof of 1.5, Pic $Y' = 0$ (that is also immediately follows from the Corollary 1 to the Lemma 3.3). Furthermore, applying 3.5, we have $\text{Br}'(Y') = H^2(Y', G_{m,Y'}) = 0$. Again using 3.2 we have $H^0(Y', \mathcal{O}_{Y'}) = A$ and since A is strictly henselian, $A^* = A^{*n}$ ([EGA 4], 18.5.13). Thus

$$H^i(Y', \mu_{n,Y'}) = 0, \qquad i = 1, 2. \tag{2}$$

Let's consider now the Leray spectral sequence

$$E_2^{i,j} = H^i(Y', R^j f'_* \mu_{n,X'}) \Rightarrow H^n(X', \mu_{n,X'}).$$

We have by (2) $E_2^{1,j} = E_2^{2,j} = 0$, $j \geqq 0$. This implies:

$$H^0(Y', R^i f'_* \mu_{n,X'}) = H^i(X', \mu_{n,X'}) = \begin{cases} (\mathbf{Z}/n\mathbf{Z})^{2g}, & i = 1 \\ \mathbf{Z}/n\mathbf{Z}, & i = 2. \end{cases}$$

The Kummer exact sequence on X' gives the following exact sequences

$$0 \to H^0(X', \mathcal{O}_{X'})^*/H^0(X', \mathcal{O}_{X'})^{*n} \to H^1(X', \mu_{n,X'}) \to (\text{Pic } X')_n \to 0,$$
$$0 \to (\text{Pic } X')^{(n)} \to H^2(X', \mu_{n,X'}) \to H^2(X', G_{m,X'})_n \to 0.$$

Since the general fibre of f is geometrically integral and Y' is normal, we have $f'_*(\mathcal{O}_{X'}) = \mathcal{O}_{Y'}$ ([EGA 3], 4.3.12), therefore

$$H^0(X', \mathcal{O}_{X'})^* = H^0(Y', \mathcal{O}_{Y'})^* = A^*,$$

and hence by (3) we have

$$\begin{aligned} (\text{Pic } X')_n &= H^1(X', \mu_{n,X'}) = (\mathbf{Z}/n\mathbf{Z})^{2g}, \\ (\text{Pic } X')^{(n)} &\hookrightarrow H^2(X', \mu_{n,X'}) = \mathbf{Z}/n\mathbf{Z}. \end{aligned} \tag{4}$$

Since the relative dimension of the morphism f is equal to 1, we have an epimorphism $\text{Pic } X \to \text{Pic } F$ (see [EGA 4], 21.9.12 or [SGAA], Exp. 13, 3.2). The exact Kummer sequence on X together with the Base change theorem yields the isomorphism

$$(\text{Pic } X)_n \simeq H^1(X, \mu_{n,X}) \simeq H^1(F, \mu_{n,F})$$

and the exact commutative diagram:

$$\begin{array}{ccccccccc}
0 \to & (\operatorname{Pic} X)^{(n)} & \to & H^2(X, \mu_{n,X}) & \to & H^2(X, G_{m,X})_n & \to 0 \\
& \downarrow & & \downarrow \wr & & \downarrow & \\
0 \to & (\operatorname{Pic} F)^{(n)} & \to & H^2(F, \mu_{n,F}) & \to & H^2(F, G_{m,F})_n & \to 0. \\
& \downarrow & & & & & \\
& 0 & & & & &
\end{array}$$

Since the field k is separably closed $H^2(F, G_{m,F})_n = 0$ ([SGAA], Exp. 9, 4.6). Hence we have

$$(\operatorname{Pic} X)^{(n)} = (\operatorname{Pic} F)^{(n)} = H^2(F, \mu_{n,F}) = (\mathbf{Z}/n\,\mathbf{Z})^{\beta_2(F)},$$
$$(\operatorname{Pic} X)_n = H^1(F, \mu_{n,F}) = (\mathbf{Z}/n\,\mathbf{Z})^{\beta_1(F)}.$$

Since codim $F = 2$ and X is regular, we have ([EGA 4], 21.6.12):

$$\operatorname{Pic} X = \operatorname{Pic} X'. \tag{5}$$

By (4) this implies

$$\begin{aligned}
(\operatorname{Pic} X)_n &= (\mathbf{Z}/n\,\mathbf{Z})^{\beta_1(F)} = (\operatorname{Pic} X')_n = (\mathbf{Z}/n\,\mathbf{Z})^{2g},\\
(\operatorname{Pic} X)^{(n)} &= (\mathbf{Z}/n\,\mathbf{Z})^{\beta_2(F)} = (\operatorname{Pic} X')^{(n)} = \mathbf{Z}/n\,\mathbf{Z}
\end{aligned} \tag{6}$$

hence $\beta_1(F) = 2g$, $\beta_2(F) = 1$. q.e.d.

3.7. Corollary. *With conditions of* 3.2 *we have for any n prime to* char k:

$$\operatorname{Br}'(X')_n = \operatorname{Br}(X')_n = 0.$$

Proof. Since dim $X' = 2$ and X' is regular, we have $\operatorname{Br}'(X') = \operatorname{Br}(X')$. Consider the Kummer exact sequence on X', we get the following exact sequence:

$$0 \to (\operatorname{Pic} X')^{(n)} \to H^2(X', \mu_{n,X'}) \to \operatorname{Br}'(X')_n \to 0.$$

As we have seen at the proof of 3.5 (the equalities (3) and (6)):

$$(\operatorname{Pic} X')^{(n)} = \mathbf{Z}/n\,\mathbf{Z}; \qquad H^2(X', \mu_{n,X'}) = \mathbf{Z}/n\,\mathbf{Z}.$$

This yields that $\operatorname{Br}'(X')_n = 0$.

3.8. *Remark.* In case $k = \mathbf{C}$ one can prove the assertion analogous to 3.4 by means of usual topological arguments there Y is changed by a 4-ball, $Y' = Y - y_0$ by a 3-sphere. We have the commutative diagram

$$\begin{array}{ccc}
X' & \hookrightarrow & X \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
Y' & \hookrightarrow & Y
\end{array}$$

Y' has trivial homologies in dimensions 1 and 2 (the analogue of (2)). Thus, since f' is obviously a Serre's fibration, this implies that $H^i(X') = H^i(F')$, where $i = 1, 2$ and F' is a fibre of f' (the analogue of (3)). Contracting U to the point y_0 and using the covering homotopy axiom, we get that X is contractible to F. This gives $H^i(X) = H^i(F)$. However, since codim $F = 4$, we have $H^i(X) = H^i(X')$, $i = 1, 2$ (the analogue of (5)). Finally, we have $H^i(X) = H^i(F) = H^i(X') = H^i(F')$, $i = 1, 2$.

3.9. Theorem (Purity Downstairs). *Let f: $X \to Y$ be a flat proper morphism of smooth schemes over a field k. Assume that f is of the relative dimension* 1 *with a smooth geometrically connected curve of the genus g as a general fibre. Suppose that in case $g > 1$ f is cohomologically flat (see* 2.6*) and in case $g \geqq 1$ char $k = 0$. Let $\mathrm{Sing}^Y(f) = f(\mathrm{Sing}^X(f))$ with the induced structure of a topological space. Then if $\mathrm{Sing}^X(f) \neq \emptyset$, we have for any maximal point $y_0 \in \mathrm{Sing}^Y(f)$ $\mathrm{codim}(y_0, Y) = 1$. In other words, $\mathrm{Sing}^Y(f)$ defines a divisor on Y.*

Proof. Using the arguments of 3.1 it is sufficient to prove 3.2. Applying the Lemma 3.4 and Corollary 2.14 we obtain our assertion in case $g > 1$. Hence, it remains to consider two cases, namely, $g = 1$ and $g = 0$.

Case $g = 1$. Since f is of the relative dimension 1 over a henselian ring, it is projective morphism (really, the special fibre F of f is complete curve and hence projective, taking ample $\xi_0 \in \mathrm{Pic}(F)$ and lifting it to an element $\xi \in \mathrm{Pic}(X)$ we can apply [EGA 3], 4.7.1). Hence F': $X' \to Y'$ is a smooth projective morphism. Under these conditions there exist the relative Picard scheme $\mathbf{Pic}(X'/Y')$ (see [FGA]) and its connected component of the unit section defines an abelian scheme J' over Y' of the relative dimension 1 (i.e. an elliptic scheme in terms of Grothendieck, see [5], 4.7). As it easily follows from the definitions of $\mathbf{Pic}(X'/Y')$ the fibres of $J' \to Y'$ are the jacobians varieties of the corresponding fibres of f'. In virtue of the Grothendieck's theorem of the purity for abelian schemes ([5], 4.7). J' can be prolonged to the abelian scheme J over Y.

Sublemma. *The schemes X' and J' are Y'-isomorphic.*

Proof. As it easily can be seen this assertion is equivalent to the fact that the general fibre X'_η of f' has a rational point over the field of fractions of the ring A, in other words X'_η is a trivial principal homogenious space over elliptic curve J'_η. Let $i: \eta \hookrightarrow Y'$ be a canonical injection of the general point, J_η in obvious way represents on η an abelian sheaf in étale topology. Considering the Leray spectral sequence for i we have the exact sequence:

$$0 \to H^1(Y', i_* J'_\eta) \to H^1(\eta, J'_\eta) \xrightarrow{\varphi} \coprod_{y \in Y'} H^1(\tilde{K}(y), J'_\eta)$$

where $\tilde{K}(y)$ denotes the field of fractions of the strict henselisation of the ring $\mathcal{O}_{Y',y}$. As it well-known the elements of the group $H^1(\eta, J'_\eta)$ are interpreted as principal homogenious spaces over J_η, considering X'_η as an element of this group, we have that its image at the map φ is zero (since f' is smooth and hence X' has a rational point over $\tilde{K}(y)$ for each point y). Therefore X' belongs to the Tate-Šafarevič group $H^1(Y', i_* J'_\eta)$. To prove the sublemma it is sufficient to show that this group is trivial. Since $\dim J' = 2$ and both J' and Y' are regular, we can apply the results of Artin-Tate-Grothendieck concerning the connection between the Brauer group and the Tate-Šafarevič group (see [GB 3], no. 4). Since $J' \to Y'$ is smooth and has a section these results give the isomorphism $H^1(Y', i_* J'_\eta) \simeq \operatorname{Br} J'$. Since $\operatorname{Br} J'$ is a torsion group ([GB 2], 1.4) and char $k=0$, we can apply Corollary 3.7 in order to get $\operatorname{Br} J' = 0$.

Hence, we get that schemes X and J are birationally isomorphic over Y, namely, they coincides on the open set $X' \simeq J'$. Since $\operatorname{codim}(X - X', X) = 2$ and both X and J are regular, the theorem of the purity of Van der Waerden ([EGA 4], 21.12.12), applied to the graph of the birational correspondence between X and J implies that X is Y-isomorphic to J. However the second scheme is smooth over Y, hence we are through.

Case $g=0$. Again as in the proof of 3.3 we may assume that $Y = \operatorname{Spec} A$ is strictly henselian. Since in the case $g=0$ the geometric fibres of the morphism $f'\colon X' \to Y'$ are projective lines, X' is a Severi-Brauer preschema over Y' (see [GB 1]). Let $\mathcal{A}_{Y'}$ be the corresponding Algebra Azumaya. Applying the Lemma 3.3, we get that $\mathcal{A}_{Y'}$, is matricial. This implies that the Severi-Brauer preschema X' is trivial, i.e. $X' = \mathbf{P}(L') = \operatorname{Proj}(\mathbf{S}(L'))$, where $\mathbf{S}(L')$ denotes a symmetric Algebra of a locally free sheaf L' on Y'. Again applying 3.2, we obtain $L' = j^*(L)$ where L is a free sheaf on Y. Hence, $X' = \mathbf{P}(L') = \mathbf{P}(L) \times_Y Y'$ ([EGA 2], 4.1.3.1).

So, we get that X is birationally isomorphic to the scheme $\mathbf{P}(L)$, namely, they coincide on the open subset $X' \simeq \mathbf{P}(L) \times_Y Y'$. The same argument as in the previous case shows that X is isomorphic to $\mathbf{P}(L)$. However, the latter scheme is smooth over Y ([EGA 4], 17.3.9). Hence X is smooth over Y. q.e.d.

3.10. Corollary. *In hypotheses of* 3.5 *we have if f is smooth over an open set $V \subset Y$ containing points of codimension* $\leqq 1$, *then f is smooth.*

3.11. *Remarks.* Here we discuss the problem of the elimination of restrictions imposed on f, X and Y, namely the assumptions (i)–(iii) cited in the introduction.

1. The restriction (i) arises in the Theorem 1.5 which was used for the reduction to the local form 3.2 of the Theorem 3.9 (see 3.1). This

shows that there are two ways to eliminate (i): the first consists on the elimination (i) in 1.5 and in the second we have to prove 3.9 not using 1.5. Note that in the latter case the assertion 1.5 would be an immediate corollary of 3.9.

2. In order to eliminate (ii) we need the proof of 2.14 not using the assumption of cohomological flatness. I guess that 2.14 just as in the case of dim $Y=1$ is valid without such assumption.

3. The restriction (iii) arises from the application of a result from the theory of Ogg-Šafarevič for two-dimensional bases. To a pity all known results (not numerous) in this theory are up to p-torsion.

4. It is obvious that without the assumption of flatness of f the Theorem 3.9 is false. In order to get a counterexample it is sufficient to blow up a subschema of a suitable closed fibre of f. There exist less trivial counterexamples (due to Manin in case $g=0$ and due to Kawai in complex case and $g=1$).

Added in Proof. The Theorem 1.5 have been generalized by the author to the following result. Let $f: X \to Y$ be a morphism of locally finite type of integral schemes. Suppose that either X and Y are smooth S-schemes or X is locally complete intersection over Y. Then $\operatorname{codim}(\operatorname{Sing}^X(f), X) \leqq n+1$, where n is the dimension of the general fibre of f.

Using this result one can reduce the Theorem 3.9, in which the condition on X and Y to be smooth is changed by the condition of regularity, to its local form 3.2. Thus the restriction i) cited in the introduction can be eliminated.

Bibliography

[EGA i] ($i=2,3,4$) Grothendieck, A.: Elements de Géométrie Algébrique. Chapitre i. Publ. Math., 8 ($i=2$), 11, 17 ($i=3$), 20, 24, 28, 32 ($i=4$). Paris 1961.

[FGA] – Fondements de la Géométrie Algébrique (Collection d'exposes au séminaire Bourbaki), Secrétariat Mathematique, Paris 1957–1962.

[SGA] – Séminaire de Géométrie Algébrique, mimeo. notes, IHES, Paris 1962.

[GB i] – Le groupe de Brauer i ($i=1,2,3$), Sem. Bourbaki, May 1965, n° 290 ($i=1$); Nov. 1965, n° 297 ($i=2$); miméo. notes, IHES, Paris, ($i=3$).

[SGAA] Artin, M., et A. Grothendieck: Cohomologie étale des schémas, Séminaire de l'IHES, miméo. notes, Paris 1963/64.

1. Algebraic surfaces (edited by I. R. Šafarevič). Trudy Mat. Inst. Steklov, No. 75, Izdat. Acad. Nauk SSSR, Moscow 1965.
2. Buchsbaum, D., and D. S. Rim: A generalized Koszul complex. Bull. Amer. Math. Soc. **69**, 382–385 (1963).
3. Danilov, V., and I. Dolgačev: Etale topology of schemes (to be published, in russian).
4. Dolgačev, I., and A. Paršin: The different and discriminant of the regular maps. Matem. Zametki **4**, No. 5, 519–523 (1968).
5. Grothendieck, A.: Un théoreme sur les homomorphismes de schémas abeliens. Inventiones math. **2**, 59–78 (1966).
6. Lichtenbaum, S.: Curves over discrete valuation rings. Amer. Journ. **90**, 380–406 (1968).
7. Nagata, M.: On the purity of the branch loci in regular local rings. Illin. Journal of Math. **3**, 319–333 (1959).

8. Raynaud, M.: Spécialisation du foncteur de Picard. Critère numérique de représentabilité. C. R. Acad. Sc. Paris **264**, 1001 – 1004 (1967).
9. Rosenlicht, M.: Equivalence relations on algebraic curves. Ann. of Math. **56**, 169 – 191 (1952).
10. Šafarevič, I.: Lectures on minimal models and birational transformations of two dimensional schemes. Tata Inst. of Fundamental Research. Bombay 1966.
11. Serre, J. P.: Algèbre locale, Multiplicités. Lecture notes in Math., No. 11. Berlin-Heidelberg-New York: Springer 1965.
12. Vvedenski, O. N.: Duality in elliptic curves over local fields. II. Izv. Acad. Nauk. SSSR **30**, 891 – 922 (1966) [in russian].
13. Zariski, O.: On the purity of the branch locus of algebraic functions. Proc. Nat. Acad. Sci. USA **44**, 791 – 796 (1958).

I. V. Dolgačev
Moscow State University
Department of Mathematics
Moscow, V-234, USSR

(Received 1. 9. 1968; in revised form 27. 12. 1968)