

The rationality of the moduli spaces of Coble surfaces and of nodal Enriques surfaces

I. Dolgachev and S. Kondō

Abstract. We prove the rationality of the coarse moduli spaces of Coble surfaces and of nodal Enriques surfaces over the field of complex numbers.

Keywords: Enriques surfaces, Coble surfaces, moduli spaces, rationality problem.

To Igor Rostislavovich Shafarevich on the occasion of his 90th birthday

§ 1. Introduction

The purpose of this note is to prove the rationality of the moduli spaces of Coble surfaces and of nodal Enriques surfaces over the field of complex numbers. A Coble surface is a rational surface obtained by blowing up 10 nodes of a rational plane curve of degree 6, and an Enriques surface is said to be nodal if it contains a smooth rational curve. The moduli space of nodal Enriques surfaces is a codimension-one subvariety in the 10-dimensional moduli space of Enriques surfaces. When the K3-cover of an Enriques surface degenerates, admitting an ordinary double point fixed under an involution, the quotient by the involution is a rational surface obtained from a Coble surface by blowing down the proper transform of the plane sextic. In this way, the moduli space of Coble surfaces can be identified with a codimension-one component of the boundary of the moduli space of Enriques surfaces.

The idea behind the proof is similar to the one used by the second author for the proof of rationality of the moduli space of Enriques surfaces [1]. The K3 surface birationally isomorphic to the double cover of the projective plane branched along the union of a cuspidal plane quintic and its cuspidal tangent contains the lattice $D_8 \oplus U$ in its Picard group. It is shown that the moduli space of K3-covers of Enriques surfaces and the moduli space of K3 surfaces admitting this lattice in its Picard group are birationally isomorphic quotients of a bounded symmetric domain of type IV. A similar idea is used here. We prove that the moduli space of K3-covers of nodal Enriques surfaces (resp. K3-covers of Coble surfaces) is birationally isomorphic to the moduli space of K3 surfaces birationally isomorphic to the double cover of the projective plane branched along the union of a cuspidal plane quintic and its cuspidal tangent line, where the quintic has an additional double point

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(resp. the cuspidal tangent line touches the curve at a non-singular point). It is easily proved, using [2], [3], that the corresponding moduli spaces of plane quintics are rational varieties.

Note that general Enriques and Coble surface are examples of the quotients of a K3 surface by a non-symplectic involution which acts identically on the Picard group. Ma [4] has recently shown the rationality of the moduli spaces of such K3 surfaces in many cases. The case of the K3-covers of Coble surfaces is one of eight exceptional cases where his methods did not work.

The idea that the moduli spaces of Enriques surfaces (resp. nodal Enriques surfaces, Coble surfaces) should be related to the moduli space of cuspidal quintics (resp. their special codimension-one subvarieties) originates from some (still hypothetical) purely geometric constructions of the first author which may relate the corresponding moduli spaces. We discuss these constructions in the last two sections of the paper.

For both of the authors the mathematical work of I. R. Shafarevich has been always a great source of inspiration and admiration. We are happy to have an occasion to dedicate our contribution to him.

§ 2. Preliminaries

A *lattice* is a free abelian group L of finite rank equipped with a non-degenerate symmetric integral bilinear form $L \times L \rightarrow \mathbb{Z}$ whose value on a pair (x, y) will be denoted by $x \cdot y$. For $x \in L \otimes \mathbb{Q}$, we call $x^2 = x \cdot x$ the *norm* of x . For a lattice L and a rational number m , we denote by $L(m)$ the free \mathbb{Z} -module L with the \mathbb{Q} -valued bilinear form obtained from the bilinear form of L by multiplication by m . The signature of a lattice is the signature of the real vector space $L \otimes \mathbb{R}$ equipped with the symmetric bilinear form extended from the one on L by linearity. A lattice is said to be *even* if $x \cdot x \in 2\mathbb{Z}$ for all $x \in L$.

We denote by U the even unimodular lattice of signature $(1, 1)$, and by A_m , D_n or E_k the even *negative*-definite lattice defined by the Cartan matrix of type A_m , D_n or E_k respectively. For an integer m , we denote by $\langle m \rangle$ the lattice of rank 1 generated by a vector with norm m . We denote by $L \oplus M$ the orthogonal direct sum of lattices L and M , and by $L^{\oplus m}$ the orthogonal direct sum of m -copies of L . For any integer k we denote by M_k the set of $x \in M$ with norm k .

We denote by L_{K3} the lattice $E_8^{\oplus 2} \oplus U^{\oplus 3}$. It is isomorphic to the 2-cohomology group $H^2(X, \mathbb{Z})$ of a K3 surface equipped with the structure of a lattice defined by the cup-product. We will refer to L_{K3} as the *K3-lattice*. The lattice $\mathbb{E} = E_8 \oplus U$ is called the *Enriques lattice*. It is isomorphic to the lattice $\text{Num}(S) = \text{Pic}(S)/(K_S)$ of numerical equivalence divisor classes on an Enriques surface S .

Let L be an even lattice and let $L^* = \text{Hom}(L, \mathbb{Z})$, identified with a subgroup of $L \otimes \mathbb{Q}$ with the extended symmetric bilinear form. We denote by A_L the quotient L^*/L and define maps

$$q_L: A_L \rightarrow \mathbb{Q}/2\mathbb{Z}, \quad b_L: A_L \times A_L \rightarrow \mathbb{Q}/\mathbb{Z}$$

by $q_L(x + L) = x \cdot x \pmod{2\mathbb{Z}}$ and $b_L(x + L, y + L) = x \cdot y \pmod{\mathbb{Z}}$. We call q_L the *discriminant quadratic form* of L and b_L the *discriminant bilinear form*. A lattice is said to be *2-elementary* if its discriminant group is a 2-elementary abelian group.

Let $O(L)$ be the orthogonal group of L , that is, the group of isomorphisms of L preserving the bilinear form. Similarly, $O(A_L)$ denotes the group of isomorphisms of A_L preserving q_L . There is a natural map

$$\phi: O(L) \rightarrow O(A_L) \tag{2.1}$$

whose kernel is denoted by $O(L)^*$.

§ 3. The moduli spaces of Enriques, nodal Enriques and Coble surfaces

First we recall the moduli space of lattice polarized K3 surfaces. For any even lattice M of signature $(1, r - 1)$ primitively embeddable in the K3-lattice L_{K3} , one can construct the coarse moduli space $\mathcal{M}_{K3,M}$ (resp. $\mathcal{M}_{K3,M}^a$) of isomorphism classes of M polarized (resp. amply polarized) K3 surfaces X , that is, isomorphism classes of pairs (X, j) , where $j: M \hookrightarrow \text{Pic}(X)$ is a primitive lattice embedding such that the image contains a nef and big (resp. ample) divisor class (see ¹ [5]). Let $N = M_{L_{K3}}^\perp$. Then the period domain is given by

$$\mathcal{D}(N) = \{[\omega] \in \mathbb{P}(N \otimes \mathbb{C}) : \omega \cdot \omega = 0, \omega \cdot \bar{\omega} > 0\}, \tag{3.1}$$

which is the disjoint union of two copies of the $(20 - r)$ -dimensional bounded symmetric domain of type IV. The moduli space is constructed as a quotient,

$$\mathcal{M}_{K3,M} = \mathcal{D}(N)/O(N)^*, \quad \mathcal{M}_{K3,M}^a = (\mathcal{D}(N) \setminus \mathcal{H}_{-2})/O(N)^*,$$

where $O(N)^* = \text{Ker}(O(N) \rightarrow O(A_N))$ and, for any $d \in \mathbb{Z}$,

$$\mathcal{H}_{-2d} = \bigcup_{\delta \in N_{-2d}} \{[\omega] \in \mathcal{D}(N) : \omega \cdot \delta = 0\}. \tag{3.2}$$

We call \mathcal{H}_{-2d} the $(-2d)$ -Heegner divisor. Suppose that N and M satisfy the condition that

(*) the natural maps $O(N) \rightarrow O(A_N)$, $O(M) \rightarrow O(A_M)$ are surjective.

Then

$$\mathcal{M}_{K3,M}/O(A_M) \cong \mathcal{D}(N)/O(N), \quad \mathcal{M}_{K3,M}^a/O(A_M) = (\mathcal{D}(N) \setminus \mathcal{H}_{-2})/O(N)$$

are coarse moduli spaces of K3 surfaces which admit a primitive embedding of M in $\text{Pic}(X)$.

The period point of a marked K3 surface belongs to \mathcal{H}_{-2} if and only if there exists a primitive embedding of $M \oplus \langle -2 \rangle$ in $\text{Pic}(X)$. The image of a generator of $\langle -2 \rangle$ will be an effective divisor class R with self-intersection -2 such that $R \cdot h = 0$ for every divisor class in the image of M . This shows that X does not admit any ample

¹There is an additional technical requirement for the embedding which we refer to loc. cit.

polarization contained in the image of M in $\text{Pic}(X)$. In other words, any nef and ample polarization of X originating in M will blow down R to a double rational point.

We now consider an Enriques surface S . Let $\pi: X \rightarrow S$ be its K3-cover and let σ be the fixed-point-free involution of X . Then $\pi^*(\text{Pic}(S)) = \pi^*(\text{Num}(S)) \cong \mathbb{E}(2)$. We take $\mathbb{E}(2)$ as M and denote by N the orthogonal complement of M in L_{K3} . Then

$$N \cong U \oplus \mathbb{E}(2). \tag{3.3}$$

Note that $\sigma^*|M = 1_M$ and $\sigma^*N = -1_N$. It is known that any period point $[\omega]$ of the K3-cover X is not contained in \mathcal{H}_{-2} (see, for example, [6]). The quotient $\mathcal{D}(N)/\text{O}(N)$ is a normal quasi-projective variety of dimension 10, and $(\mathcal{D}(N) \setminus \mathcal{H}_{-2})/\text{O}(N)$ is the moduli space \mathcal{M}_{En} of Enriques surfaces.

Next we consider *nodal* Enriques surfaces, that is, Enriques surfaces containing a smooth rational curve ((-2) -curve, for short). Let C be a (-2) -curve on an Enriques surface S . Then $\pi^{-1}(C)$ splits into the disjoint sum $C_1 \cup C_2$ of two (-2) -curves. The divisor class $\delta = [C_1 - C_2]$ with $\delta^2 = -4$ belongs to $\pi^*(\text{Pic}(S))^\perp$. If we consider all (-2) -curves on S , the corresponding (-4) -vectors δ generate a negative-definite lattice $R(2)$ in $U \oplus \mathbb{E}(2)$, where R is a root lattice. The root lattice R is a part of the notion of the *root invariant* for Enriques surfaces (see [7]). Since any period point $[\omega]$ of the K3-cover X is orthogonal to an algebraic cycle, we obtain that the period $[\omega]$ belongs to \mathcal{H}_{-4} . Thus we define the moduli space $\mathcal{M}_{\text{En}}^{\text{nod}}$ of nodal Enriques surfaces by

$$\mathcal{M}_{\text{En}}^{\text{nod}} = (\mathcal{H}_{-4} \setminus \mathcal{H}_{-2})/\text{O}(U \oplus \mathbb{E}(2)), \tag{3.4}$$

where $\mathcal{H}_{-4}, \mathcal{H}_{-2}$ are Heegner divisors in the period domain of Enriques surfaces. It is known that such a (-4) -vector δ in $U \oplus \mathbb{E}(2)$ is unique up to the orthogonal group $\text{O}(U \oplus \mathbb{E}(2))$, and the orthogonal complement δ^\perp in $U \oplus \mathbb{E}(2)$ is isomorphic to

$$N = U \oplus \langle 4 \rangle \oplus E_8(2) \tag{3.5}$$

(see [6]). The orthogonal complement $M = N^\perp$ of N in L_{K3} contains $\mathbb{E}(2) \oplus \langle -4 \rangle$ as a sublattice of index 2, where $\mathbb{E}(2) = \pi^*(\text{Pic}(S))$ and $\langle -4 \rangle$ is generated by δ . Then the quotient $\mathcal{D}(N)/\text{O}(N)$ is a 9-dimensional quasi-projective variety. We call a nodal Enriques surface S *general* if $R \cong \langle -2 \rangle$, that is, for any two (-2) -curves C, C' on S , $[C_1 - C_2] = [C'_1 - C'_2]$. Note that, for a general nodal Enriques surface, the decomposition $\mathbb{E}(2) \oplus \langle -4 \rangle$ is unique, that is, it is independent of the choice of (-2) -curves. Hence we have the following.

Proposition 3.1. *Let $N = U \oplus \langle 4 \rangle \oplus E_8(2)$. Then the moduli space $\mathcal{M}_{\text{En}}^{\text{nod}}$ of nodal Enriques surfaces is birationally isomorphic to $\mathcal{D}(N)/\text{O}(N)$.*

Finally we consider Coble surfaces. A Coble surface is a smooth rational projective surface S such that $|-K_S| = \emptyset$ but $|-2K_S| \neq \emptyset$ (see [8]). A classical example of such a surface is the blow-up of the projective plane at the ten nodes of an irreducible plane curve C of degree 6. The sets of 10 points in the plane realized as the nodes of a rational sextic are examples of *special sets* of points in the sense

of Coble [9] (they were called Cremona special in [10]). They were first studied in [11]. In this note we will restrict ourselves to these classical examples.

Denote by \mathcal{M}_{C_o} the moduli space of Coble surfaces constructed as a locally closed subvariety of the GIT-quotient of the variety of 10-tuples of points in \mathbb{P}^2 modulo the group $\mathrm{PGL}(3)$. By taking the double cover of \mathbb{P}^2 branched along the plane sextic with 10 nodes, the moduli space \mathcal{M}_{C_o} can be described as an open set of an arithmetic quotient of a 9-dimensional bounded symmetric domain of type IV. We briefly recall this.

Denote by X the double cover of the Coble surface S branched along the proper transform of the plane sextic C . Then X is a K3 surface containing the divisors E_0, E_1, \dots, E_{10} , where E_0 is the pullback of a line on \mathbb{P}^2 and E_1, \dots, E_{10} are the inverse images of the exceptional curves over the nodes p_1, \dots, p_{10} of C . It is easily seen that the corresponding divisor classes e_0, e_1, \dots, e_{10} generate the sublattice M_X of $\mathrm{Pic}(X)$ isomorphic to $M = \langle 2 \rangle \oplus \langle -2 \rangle^{\oplus 10}$. Note that M is a 2-elementary lattice of signature $(1, 10)$ with $A_M \cong (\mathbb{Z}/2\mathbb{Z})^{11}$. The orthogonal complement of M_X in $H^2(X, \mathbb{Z})$, denoted by N_X , is a 2-elementary lattice of signature $(2, 9)$ with $q_{N_X} = -q_M$ (see [12], Corollary 1.6.2). The isomorphism class of such lattice is uniquely determined by $-q_M$. Thus N_X is isomorphic to

$$N = \langle 2 \rangle \oplus \mathbb{E}(2) \tag{3.6}$$

(see loc. cit., Theorem 3.6.2). We remark that $M \cong \mathbb{E}(2) \oplus \langle -2 \rangle$.

Let $\mathcal{D}(N)$ be as in (3.1), where N is the lattice (3.6). The quotient $\mathcal{D}(N)/\mathrm{O}(N)$ is a normal quasi-projective variety of dimension 9. The Torelli-type theorem for algebraic K3 surfaces, due to Pyatetskii-Shapiro and Shafarevich [13], implies the following (for more details, see [14]).

Proposition 3.2. *Let $N = \langle 2 \rangle \oplus \mathbb{E}(2)$. Then the moduli space \mathcal{M}_{C_o} of Coble surfaces is isomorphic to an open subset of $\mathcal{D}(N)/\mathrm{O}(N)$.*

Note that $N = \langle 2 \rangle \oplus \mathbb{E}(2)$ is isomorphic to the orthogonal complement of a (-2) -vector in $U \oplus \mathbb{E}(2)$. This implies that the quotient of the (-2) -Heegner divisor \mathcal{H}_{-2} in the period domain of Enriques surfaces by the arithmetic subgroup $\mathrm{O}(U \oplus \mathbb{E}(2))$ is birationally isomorphic to the moduli space \mathcal{M}_{C_o} of Coble surfaces.

§ 4. Plane quintics with a cusp

Let C be a plane quintic curve with a cusp p . Let L be the tangent line to C at the cusp. We consider the following two cases:

- 1) the line L is tangent to C at a smooth point q of C ;
- 2) C has an ordinary node q .

Let $\mathcal{M}_{\mathrm{cusp}}$ be the moduli space of cuspidal quintics, that is, the GIT-quotient of the projective space of plane cuspidal curves of degree 5 by the group $\mathrm{PGL}(3)$. The second author proved earlier that $\mathcal{M}_{\mathrm{cusp}}$ is a rational variety birationally isomorphic to the moduli space $\mathcal{M}_{\mathrm{En}}$ (see [1]). The proof establishes a birational isomorphism between $\mathcal{M}_{\mathrm{En}}$ and the moduli space of K3 surfaces birationally isomorphic to the double covers of \mathbb{P}^2 branched along a cuspidal quintic and its cuspidal tangent. Here

we will follow the same strategy, replacing $\mathcal{M}_{\text{cusp}}$ with its codimension 1 subvarieties $\mathcal{M}'_{\text{cusp}}$ (resp. $\mathcal{M}''_{\text{cusp}}$) corresponding to quintics in case 1) (resp. case 2)).

Theorem 4.1. $\mathcal{M}'_{5,\text{cusp}}$ and $\mathcal{M}''_{5,\text{cusp}}$ are rational varieties of dimension 9.

Proof. We start with case 1). Let C be a quintic curve in this case. By a linear transformation, we may choose coordinates $(x_0 : x_1 : x_2)$ in such a way that $p = (1 : 0 : 0)$ is the cusp, and $V(t_1)$ is the cuspidal tangent line which touches C at the point $q = (0 : 0 : 1)$. Since p is a cusp of C with cuspidal tangent $V(x_1)$, the curve C is given by an equation of the form

$$ax_0^3x_1^2 + x_0^2A_1(x_1, x_2) + x_0A_2(x_1, x_2) + A_3(x_1, x_2) = 0, \quad a \neq 0,$$

where A_1, A_2 and A_3 are homogeneous polynomials of degrees 3, 4 and 5, respectively. Plugging in $x_1 = 0$, we obtain the binary form $x_0^2A_1(0, x_2) + x_0A_2(0, x_2) + A_3(0, x_2)$ in variables x_0, x_2 . It must have a zero at $(0 : 1)$ of multiplicity 2. This implies that $A_2 = x_1A'_2$ and $A_3 = x_1A'_3$ for some polynomials A'_2, A'_3 of degrees 3 and 4, respectively. Thus the equation of C can be rewritten in the form

$$F := ax_0^3x_1^2 + x_0^2A_1(x_1, x_2) + x_0x_1A'_2(x_1, x_2) + x_1A'_3(x_1, x_2) = 0, \quad a \neq 0.$$

Let V be the linear subspace of $S^5(\mathbf{C}^3)^*$ consisting of quintic ternary forms F as above (with a possibly equal to zero). The subgroup G of $\text{GL}(4)$ which leaves V invariant consists of linear transformations

$$x_0 \rightarrow ax_0 + bx_2, \quad x_1 \rightarrow cx_1, \quad x_2 \rightarrow dx_1 + ex_2.$$

Then $\mathcal{M}'_{5,\text{cusp}}$ is birational to the quotient $\mathbb{P}(V)/G$. It follows that the dimension of $\mathcal{M}'_{5,\text{cusp}}$ is equal to 9. Note that G is a soluble algebraic group of dimension 5 acting linearly on the linear space V of dimension 14. The assertion of rationality now follows from a result of Miyata [2] and Vinberg [3].

The case 2) can be argued in the same way. We may assume that the node does not lie on the cuspidal tangent line. First transform C to a curve such that $p = (1 : 0 : 0)$ is a cusp with the cuspidal tangent line $V(x_1)$, and $q = (0 : 1 : 0)$ is a node. Arguing as above, we find that C can be given by an equation

$$F := ax_0^3x_1^2 + x_0^2A_1(x_1, x_2) + x_0x_2A_2(x_1, x_2) + x_2^2A_3(x_1, x_2) = 0, \quad a \neq 0.$$

Let V' be a linear subspace of $S^5(\mathbf{C}^3)^*$ consisting of quintic ternary forms F as above (with a possibly equal to zero). Its dimension is equal to 13. The subgroup G' of $\text{GL}(4)$ leaving V' invariant consists of projective transformations

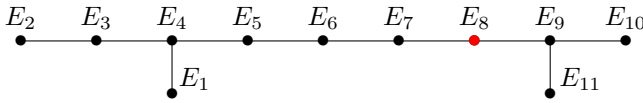
$$x_0 \mapsto ax_0 + bx_2, \quad x_1 \mapsto cx_1, \quad x_2 \rightarrow dx_2.$$

It is a soluble algebraic group of dimension 4 acting linearly on V . The variety $\mathcal{M}''_{5,\text{cusp}}$ is birational to the quotient variety $\mathbb{P}(V')/G'$. It follows that the dimension of $\mathcal{M}''_{5,\text{cusp}}$ is 9. Invoking the same result of Miyata and Vinberg, we obtain that $\mathcal{M}''_{5,\text{cusp}}$ is rational.

§ 5. K3 surfaces associated with a plane quintic with a cusp

In this section, we shall show that $\mathcal{M}'_{5,\text{cusp}}$ (resp. $\mathcal{M}''_{5,\text{cusp}}$) is isomorphic to an open subset of an arithmetic quotient of a 9-dimensional bounded symmetric domain of type IV.

First we consider case 1). Let C be a cuspidal quintic as in this case and let L be the cuspidal tangent. Consider the plane sextic curve $C + L$. Let p be the cusp and q the smooth point of tangency of L with C . Let \overline{X} be the double cover of \mathbb{P}^2 branched along $C + L$. Then \overline{X} has a rational double point of type E_7 over p locally isomorphic to $V(z^2 + y(y^2 + x^3))$, and a rational double point of type A_3 over q locally isomorphic to $V(z^2 + x(x + y^2))$. Denote by X a minimal resolution of \overline{X} and by τ the covering transformation. Then X is a K3 surface containing 11 smooth rational curves E_1, \dots, E_{11} with intersection graph pictured below:



We see that E_1, \dots, E_7 form the intersection graph of type E_7 , E_8 is the inverse image of L and E_9, E_{10}, E_{11} form the intersection graph of type A_3 . The covering transformation τ preserves each of E_1, \dots, E_9 and interchanges E_{10} and E_{11} . Note that the linear system

$$|E_1 + E_3 + 2(E_4 + \dots + E_9) + E_{10} + E_{11}|$$

defines an elliptic fibration with a singular fibre of type \tilde{D}_9 , and E_2 is a section of this fibration. This implies that these 11 curves generate a sublattice M_X of $\text{Pic}(X)$ isomorphic to $M = U \oplus D_9$. Here U is generated by the class of a fibre and the section E_2 , and D_9 is generated by E_1, E_4, \dots, E_{11} . Since the discriminant of M is equal to 4, and there are no even unimodular lattices with signature $(1, 10)$, M_X is primitive in $H^2(X, \mathbb{Z})$ and $M_X = \text{Pic}(X)$ for general X .

Let N_X be the orthogonal complement of M_X in $H^2(X, \mathbb{Z})$. Then N_X has signature $(2, 9)$. It follows from [12], Corollary 1.6.2 that $q_{N_X} \cong -q_M$. Note that $A_M \cong \mathbb{Z}/4\mathbb{Z}$. It also follows from loc. cit., Theorem 1.14.2 that the isomorphism class of N_X is uniquely determined by q_M . Thus N_X is isomorphic to

$$N = \langle 4 \rangle \oplus U \oplus E_8. \tag{5.1}$$

Obviously $O(M_X) \cong \mathbb{Z}/2\mathbb{Z}$. We have the following lemma, which is easy to prove.

Lemma 5.1. *The group $O(A_{S_X})$, and hence $O(A_{T_X})$, is generated by the covering involution τ . In particular, the natural maps (2.1), $O(M_X) \rightarrow O(A_{M_X})$ and $O(N_X) \rightarrow O(A_{N_X})$ are surjective.*

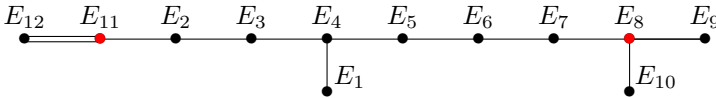
Let $\mathcal{D}(N)$ be as in (3.1) with the lattice N in (5.1). The quotient $\mathcal{D}(N)/O(N)$ is a normal quasi-projective variety of dimension 9.

We fix a primitive embedding of M in the K3-lattice L_{K3} with $N = M^\perp$. We also fix a basis $\{e_i\}$ of M which has the same incidence relation as $\{E_i\}$. It follows from Lemma 5.1 that there exists an isometry from $H^2(X, \mathbb{Z})$ to L_{K3} sending the

classes of E_i to e_i . This defines a M -lattice polarization on the corresponding K3 surface (see [5]) satisfying condition $(*)$. Note that the M -marking determines the action of the involution of τ on $H^2(X, \mathbb{Z})$: τ^* acts trivially on e_1, \dots, e_9 , interchanges e_{10} and e_{11} , and acts on N as -1 . Conversely, let $(X, j), (X', j')$ be two M -amply polarized K3 surfaces whose periods coincide in $\mathcal{D}(N)/O(N)$. We denote by τ (resp. τ') the involution of X (resp. X'). It follows from Lemma 5.1 that there exists a Hodge isometry $\phi: H^2(X, \mathbb{Z}) \rightarrow H^2(X', \mathbb{Z})$ preserving the M -markings. The Torelli type theorem for algebraic K3 surfaces implies that there exists an isomorphism $\varphi: X \rightarrow X'$ with $\varphi^* = \phi$. Moreover $\varphi \circ \tau = \tau' \circ \varphi$. Hence φ induces an isomorphism between the corresponding plane quintics. Thus we have the following theorem.

Theorem 5.2. *Let $N = \langle 4 \rangle \oplus U \oplus E_8$. Then the moduli space $\mathcal{M}'_{5, \text{cusp}}$ is isomorphic to an open subset of $\mathcal{D}(N)/O(N)$.*

Next we study case 2). Again consider the plane sextic curve $C + L$. Let L intersect C at the cusp p and two distinct points q_1, q_2 . Let \bar{X} be the double cover of \mathbb{P}^2 branched along $C + L$. Then \bar{X} has a rational double point of type E_7 over p and three rational double points of type A_1 over q_1, q_2, q . Denote by X the minimal resolution of \bar{X} and by τ the covering transformation. Then X is a K3 surface containing 12 smooth rational curves E_1, \dots, E_{12} whose intersection graph is pictured below:



Here E_9, E_{10} or E_{12} corresponds to the exceptional curve over q_1, q_2 or q , respectively, and E_1, \dots, E_7 correspond to the exceptional curves over p , E_8 is the inverse image of L and E_{11} is the inverse image of the line passing through p and q . The covering transformation τ preserves each of E_1, \dots, E_{12} . Note that the linear system

$$|2(E_4 + \dots + E_8) + E_1 + E_3 + E_9 + E_{10}|$$

defines an elliptic fibration with singular fibres of type \tilde{D}_8 and of type \tilde{A}_1 , and E_2 is a section of this fibration. This implies that these 12 curves generate the sublattice M_X of $\text{Pic}(X)$ isomorphic to $M = U \oplus D_8 \oplus A_1$. Here U is generated by the class of the fibre and the section E_2 , the sublattice D_8 is generated by E_1, E_4, \dots, E_{10} , and the sublattice A_1 is generated by E_{12} . Since the fixed locus of τ consists of a smooth curve of genus 4 and four smooth rational curves E_2, E_4, E_6, E_8 , the invariant sublattice of $H^2(X, \mathbb{Z})$ under the action of τ^* coincides with M_X [15], Theorem 4.2.2. In particular, M_X is primitive in $H^2(X, \mathbb{Z})$ and $M_X = \text{Pic}(X)$ for generic X .

Let N_X be the orthogonal complement of M_X in $H^2(X, \mathbb{Z})$. Then M_X has signature $(2, 9)$, and N_X is isomorphic to

$$N = \langle 2 \rangle \oplus U(2) \oplus E_8. \tag{5.2}$$

It is easy to see that $O(A_{M_X}) \cong \mathbb{Z}/2\mathbb{Z}$. This implies the following lemma.

Lemma 5.3. *The group $O(A_{M_X})$ is generated by the isometry of S_X acting trivially on E_3, \dots, E_{12} and switching E_1 and E_2 . The group $O(A_{N_X})$ is generated by the covering involution τ . In particular, the natural maps $O(M_X) \rightarrow O(A_{M_X})$ and $O(N_X) \rightarrow O(A_{N_X})$ are surjective.*

Let $\mathcal{D}(N)$ be as in (3.1) with N defined in (5.2). Using the same argument as in case 1), we prove the following theorem.

Theorem 5.4. *Let $N = \langle 2 \rangle \oplus U(2) \oplus E_8$. Then the moduli space $\mathcal{M}''_{5,\text{cusp}}$ is isomorphic to an open subset of $\mathcal{D}(N)/O(N)$.*

Remark 5.5. Ma [4] proved the rationality of the moduli space of K3 surfaces in the case 2) by using another model of the quotient $X/(\tau)$.

§ 6. Proof of the rationality

In this section, we prove the rationality of \mathcal{M}_{Co} and $\mathcal{M}_{\text{En}}^{\text{nod}}$.

Theorem 6.1. *Let N (resp. N') be the lattice in (3.6) (resp. (5.1)). Then*

$$\mathcal{D}(N)/O(N) \cong \mathcal{D}(N')/O(N').$$

Proof. We have $N(1/2) = \langle 1 \rangle \oplus U \oplus E_8$. The odd unimodular lattice $N(1/2)$ contains the unique even sublattice of index 2 isomorphic to N' . Thus we can regard N' as a sublattice of $N(1/2)$. Then any isometry of $N(1/2)$ preserves N' and hence $O(N(1/2)) \subset O(N')$. Conversely, consider $(N')^* = \langle \frac{1}{4} \rangle \oplus U \oplus E_8$. The discriminant group $A_{N'}$ is a finite cyclic group of order 4 and contains the unique subgroup $N(1/2)/N'$ of order 2. This implies that any isometry of N' can be extended to an isometry of $N(1/2)$, and hence $O(N') \subset O(N(1/2))$. Thus we have $O(N(1/2)) = O(N')$. Now consider the bounded symmetric domain $\mathcal{D}(N(1/2)) = \mathcal{D}(N')$. Then

$$\mathcal{D}(N(1/2))/O(N(1/2)) = \mathcal{D}(N')/O(N').$$

Obviously

$$\mathcal{D}(N(1/2))/O(N(1/2)) \cong \mathcal{D}(N)/O(N).$$

Therefore we have proved the assertion.

Theorem 6.2. *Let N (resp. N') be the lattice in (3.5) (resp. (5.2)). Then*

$$\mathcal{D}(N)/O(N) \cong \mathcal{D}(N')/O(N').$$

Proof. Consider

$$N'(1/2) = \langle 1 \rangle \oplus U \oplus E_8(1/2), \quad (N'(1/2))^* = \langle 1 \rangle \oplus U \oplus E_8(2).$$

Then N is the even sublattice of $(N'(1/2))^*$. Hence $O(N'(1/2)) \subset O(N)$. Conversely, consider $N^* = \langle 1/4 \rangle \oplus U \oplus E_8(1/2)$ and the discriminant quadratic form

$$q_N: N^*/N \rightarrow \mathbb{Q}/2\mathbb{Z}.$$

We remark that $N'(1/2)$ is characterized as being the maximal submodule K of N^* such that $q_N(K/N) \subset \mathbb{Z}/2\mathbb{Z}$. Any isometry of N can be extended to one, denoted by ϕ , of N^* . Then the above remark implies that ϕ preserves $N'(1/2)$. Hence $O(N) \subset O(N'(1/2))$. Therefore

$$\mathcal{D}(N)/O(N) = \mathcal{D}(N'(1/2))/O(N'(1/2)) \cong \mathcal{D}(N')/O(N').$$

Combining Propositions 3.2, 3.1 and Theorems 5.2, 5.4, 6.1, 6.2, we have the following theorem.

Theorem 6.3. *There are birational isomorphisms*

$$\Upsilon : \mathcal{M}_{\text{En}} \cong \mathcal{M}_{5, \text{cusp}}, \tag{6.1}$$

$$\Upsilon' : \mathcal{M}_{\text{Co}} \cong \mathcal{M}'_{5, \text{cusp}}, \tag{6.2}$$

$$\Upsilon'' : \mathcal{M}_{\text{En}}^{\text{nod}} \cong \mathcal{M}''_{5, \text{cusp}}. \tag{6.3}$$

By Theorem 4.1, we obtain the following main theorem.

Theorem 6.4. \mathcal{M}_{Co} and $\mathcal{M}_{\text{En}}^{\text{nod}}$ are rational varieties.

Remark 6.5. The K3-cover X of a general nodal Enriques surface is isomorphic to a minimal non-singular model of the *Cayley quartic symmetroid*, the locus Y of singular quadrics in a general 3-dimensional linear system L of quadrics in \mathbb{P}^3 (see [16]). The surface Y has 10 nodes corresponding to reducible quadrics in L . The set of 10 points in \mathbb{P}^3 realized as the ten nodes of a Cayley quartic symmetroid is one of the special sets of points in \mathbb{P}^3 in the sense of Coble (see [9]). There is a beautiful relationship between Cayley quartic symmetroids and rational sextics (see loc. cit.). The variety of such sets modulo projective equivalence is birationally isomorphic to the GIT-quotient of the Grassmannian G of webs of quadrics in \mathbb{P}^3 modulo $\text{PGL}(4)$. It is birationally isomorphic to some finite cover of $\mathcal{M}_{\text{En}}^{\text{nod}}$. The rationality of $G/\text{PGL}(4)$ is a difficult problem.

§ 7. A geometric construction: Enriques surfaces

Let $\mathcal{M}_{\text{En}}(2)$ be the moduli space of degree 2 polarized Enriques surfaces, that is, the coarse moduli space of pairs (X, h) , where h is a nef divisor class with $h^2 = 2$. It is known that $h = F_1 + F_2$, where F_1, F_2 are nef divisors with $F_i^2 = 0$ and $F_1 \cdot F_2 = 1$, or $h = 2F_1 + R$, where F_1 is as above and R is (-2) -curve with $F_1 \cdot R = 0$ (see [17], Corollary 4.5.1). We call h *non-degenerate* if h is as in the first case, and *degenerate* otherwise.

For any h as above, the linear system $|2h|$ defines a degree 2 map $\phi_h : X \rightarrow \mathbb{P}^4$ whose image is a quartic del Pezzo surface D . If h is non-degenerate, D has 4 ordinary double points; otherwise, it has 2 ordinary double points and one rational double point of type A_3 . We call D a 4-nodal quartic del Pezzo surface in the first case and a degenerate 4-nodal quartic del Pezzo surface in the second (see [17], Ch.0, § 4). The set of fixed points of the deck transformation σ of the double cover ϕ_h consists of a smooth curve \overline{W} and 4 isolated points. The image W of \overline{W}

on D is a curve of arithmetic genus 5 in the linear system $|\mathcal{O}_D(2)|$. It does not pass through singular points of D . The map $(\phi_h)|_W: \overline{W} \rightarrow W$ is the normalization map.

Let $\mathcal{M}_{\text{En}}(2)^{\text{ndeg}}$ be the GIT-quotient $|\mathcal{O}_D(2)|//\text{Aut}(D)$, where D is a non-degenerate quartic del Pezzo surface, and let $\mathcal{M}_{\text{En}}(2)^{\text{deg}}$ be the same when D is a degenerate quartic del Pezzo surface. The first variety (resp. the second) is a projective variety of dimension 10 (resp. 9). The disjoint union

$$\mathcal{M}_{\text{En}}(2) = \mathcal{M}_{\text{En}}(2)^{\text{ndeg}} \cup \mathcal{M}_{\text{En}}(2)^{\text{deg}}$$

can be viewed as a compactification of the moduli space of degree 2 polarized Enriques surfaces. Consider the forgetful rational maps

$$\Phi_1: \mathcal{M}_{\text{En}}(2)^{\text{ndeg}} \dashrightarrow \mathcal{M}_{\text{En}}, \quad \Phi_2: \mathcal{M}_{\text{En}}(2)^{\text{deg}} \dashrightarrow \mathcal{M}_{\text{En}}^{\text{nod}}.$$

It is known that Φ_1 is of degree $2^7 \cdot 17 \cdot 31$ (see [18]) and Φ_2 is of degree $2^3 \cdot 17$ (see [19]).

Any quartic del Pezzo surface is equal to the base locus of a pencil of quadrics in \mathbb{P}^4 . If D is a 4-nodal quartic del Pezzo surface, the pencil contains three singular quadrics, two of corank 2 and one of corank 1. If D is a degenerate 4-nodal quartic del Pezzo surface, it contains only two singular quadrics, both of corank 2 (see [17], Ch. 0, § 4). The locus C of singular quadrics in the net $\mathcal{N} = |I_W(2)|$ of quadrics containing W is a curve of degree 5. It has two singular double points corresponding to the singular quadrics of corank 2 containing D . The line ℓ joining the two singular points is the pencil $|I_D(2)|$ of quadrics containing D . In the case of a non-degenerate polarization, the line ℓ intersects C at some other non-singular point q . In the case of a degenerate polarization, ℓ is tangent to a branch of one of the singular points. Let \overline{C} be the normalization of C . We assume that W is non-singular; this happens for any unnodal Enriques surface and for a general nodal surface. In this case the curve \overline{C} is a non-singular curve of genus 4. Its plane quintic model is given by the linear system $|K_{\overline{C}} - \overline{q}|$, where \overline{q} is the pre-image of q under the normalization. Recall that the canonical model of a non-singular curve of genus 4 is the complete intersection of a cubic surface and a quadric surface. The rulings of the quadric define two g_3^1 's on the curve (they coincide if the quadric is a cone). In our case, the plane quintic model has two singular points, and the pencils of lines through these points define two different g_3^1 's. Thus the quadric containing the canonical model of \overline{C} is non-singular. Each of the two lines passing through the point \overline{q} intersects \overline{C} at two points corresponding to the branches of the singular points of C . Thus, in the degenerate case, the point \overline{q} coincides with one of the points of ramification of the corresponding g_3^1 . For a general canonical curve of genus 4 we have 12 points of ramification in each g_3^1 .

Let \mathcal{X}_4 be the coarse moduli space of pairs (T, t) which consist of a non-singular curve T of genus 4 and a point t on it. Let \mathcal{X}'_4 be a hypersurface in \mathcal{X}_4 of pairs (T, t) such that $t + 2t'$ belongs to a g_3^1 on T . The forgetful map $(T, t) \mapsto t$ defines a map $\mathcal{X}_4 \rightarrow \mathcal{M}_4$, where \mathcal{M}_4 is the coarse moduli space of curves of genus 4. In the non-degenerate case, it (birationally) identifies \mathcal{X}_4 with the universal curve over \mathcal{M}_4 . In the degenerate case, it is a finite cover of degree 24. Projecting T from

the point t , we obtain a plane quintic curve with a node and a cusp. Conversely, the normalization of such a curve defines a pair (T, t) as above, where t' corresponds to the branch of the cusp and t is the pre-image of the residual point of the line joining the two singular points. In this way we obtain a birational isomorphism

$$\mathcal{X}'_4 \cong \mathcal{M}''_{5,\text{cusp}}.$$

By assigning to $(X, h) \in \mathcal{M}_{\text{En}}(2)^{\text{ndeg}}$ (resp. $(X, h) \in \mathcal{M}_{\text{En}}(2)^{\text{deg}}$) the quintic plane curve C parameterizing singular quadrics of the branch curve of the degree 2 map $\phi_h: X \rightarrow \mathbb{D}$, we obtain a rational map²

$$\begin{aligned} \Psi_1: \mathcal{M}_{\text{En}}(2)^{\text{ndeg}} &\dashrightarrow \mathcal{X}_4 \\ \text{(resp. } \Psi_2: \mathcal{M}_{\text{En}}(2)^{\text{deg}} &\dashrightarrow \mathcal{X}'_4). \end{aligned}$$

The degree of this map is equal to the number of projective equivalence classes of nets of quadrics with the fixed curve of singular quadrics. It is known that the number of such equivalence classes is equal to the number of non-effective even theta characteristics on the normalization of the curve [21], Ch. 6. Since the canonical model of \bar{C} lies on a non-singular quadric, all even characteristics are non-effective, and their number is equal to $2^3(2^4 + 1) = 2^3 \cdot 17$. Since the degrees of the maps Ψ_2 and Φ_2 coincide, it is natural to make the following conjecture.

Conjecture 7.1. Let Υ'' be the birational isomorphism in Theorem 6.3. Then

$$\Psi_2 = \Upsilon'' \circ \Phi_2.$$

Next we assume that h is a non-degenerate degree 2 polarization. Let $(T, t) \in \mathcal{X}_4 \setminus \mathcal{X}'_4$. Assume that \bar{q} is not a point of ramification of a g_3^1 on T . Consider the rational map

$$f: \mathbb{P}^3 \dashrightarrow \mathbb{P}^4$$

given by the linear system of cubic surfaces containing T . We choose a basis of the linear system in the form $(V(F), V(x_0Q), \dots, V(x_3Q))$, where $T = V(F) \cap V(Q)$ is the intersection of a cubic and a quadric, and x_0, \dots, x_3 are projective coordinates in \mathbb{P}^3 . The image of the map is a singular cubic hypersurface K with equation

$$y_0Q(y_1, \dots, y_4) + F(y_1, \dots, y_4) = 0.$$

The singular point $\mathfrak{o} = [1, 0, \dots, 0]$ of K is the image of the quadric $V(Q)$. Let L be the tangent to T at the point t . Its image on K is a line ℓ not containing the point \mathfrak{o} . Consider the projection of K from the line ℓ to the plane. Let $K' \rightarrow K$ be the blowing up of K along ℓ . The projection defines the structure of a conic bundle on K' . It is known that the discriminant curve is of degree 5 (see [21]). We claim that in our case it is a cuspidal quintic. Of course, the cusp is the projection of the point \mathfrak{o} . It is clear that the cusp is distinguished from an ordinary node by the property that the pencil of lines through the point contains only one

²The varieties $\mathcal{X}_4, \mathcal{X}'_4$ are known to be rational [20], and so is the variety $\mathcal{M}_{\text{En}}(2)^{\text{ndeg}}$.

line intersecting the curve at the point with multiplicity ≥ 3 . A line through the singular point of the discriminant curve corresponds to a hyperplane H containing the line ℓ and the singular point \mathfrak{o} . Let $y = f(t)$ be the image of the point t on K . Its coordinates are $(0 : a_0 : \dots : a_3)$, where $t = (a_0 : \dots : a_3)$. Let $\ell' = \overline{\mathfrak{o}y}$ be the line on K joining y to \mathfrak{o} .³ It is easy to see that the plane in H spanned by ℓ and ℓ' intersects K along the line ℓ' taken with multiplicity 2. This plane corresponds to the projection of \mathfrak{o} , that is, the cusp of the discriminant curve. The cubic surface $S = H \cap K$ is the image of a plane Π in \mathbb{P}^3 which contains the tangent line L . Assume that L is not tangent to the quadric $V(Q)$. The restriction of the map f to Π is given by the net of cubic curves which are tangent to each other at the point t , and the base points of the net lie on the conic $V(Q) \cap \Pi$. Since Π contains the tangent line to $V(Q)$, the conic $V(Q) \cap \Pi$ is tangent to L at t , and hence equals the union of two lines l_1, l_2 intersecting at t . The cubic $V(F)$ intersects the conic at t and four additional points $p_1, p_2 \in l_1$ and $p_3, p_4 \in l_2$. The conic bundle on S contains four singular conics: $2\ell'$, and the images of the reducible conics $\overline{p_1p_2 + p_3p_4}$, $\overline{p_1p_3 + p_2p_4}$, $\overline{p_1p_4 + p_2p_3}$. The surface S has two singular points of type A_1 : one is the node of K and the other lies on ℓ' . We see that any plane Π not tangent to $V(Q)$ has four singular conics. Thus, there is only one line intersecting the discriminant curve at less than 4 points. It corresponds to the plane Π tangent to the quadric $V(Q)$ at the point t . This proves the assertion.

Let \mathcal{C} be the coarse moduli space of pairs (K, ℓ) , where K is a cubic threefold with one ordinary double point and ℓ is a line not containing the singular point. It is known that the map $\mathcal{C} \rightarrow \mathcal{M}_{5, \text{cusp}}$ which assigns to (K, ℓ) the discriminant curve of the conic bundle defined by ℓ is of degree equal to the number of odd theta characteristics on the normalization of the discriminant curve [21], Remark 6.27. The latter number is equal to $2^4 \cdot (2^5 - 1) = 2^4 \cdot 31$. Thus our construction defines a rational map of degree $2^4 \cdot 31$,

$$\Psi_2 : \mathcal{X}_4 \dashrightarrow \mathcal{M}_{5, \text{cusp}}.$$

Composing it with the rational map $\Psi_1 : \mathcal{M}_{\text{En}}(2)^{\text{ndeg}} \dashrightarrow \mathcal{X}_4$ we get a rational map

$$\Psi_2 \circ \Psi_1 : \mathcal{M}_{\text{En}}(2)^{\text{ndeg}} \dashrightarrow \mathcal{M}_{5, \text{cusp}}$$

of degree $2^7 \cdot 17 \cdot 31$. Comparing this with the degree of the map

$$\Phi_1 : \mathcal{M}_{\text{En}}(2)^{\text{ndeg}} \dashrightarrow \mathcal{M}_{\text{En}},$$

we propose the following conjecture (see [22]).

Conjecture 7.2. Let Υ be the birational isomorphism in (6.1). Then

$$\Upsilon \circ \Phi_1 = \Psi_2 \circ \Psi_1.$$

³In [22], the first author mistakenly chose the line ℓ' on K instead of ℓ for the centre of the projection map.

§ 8. A geometric construction: Coble surfaces

We assume that S is an unnodal Coble surface in the sense that it has no (-2) -curves. We know that the orthogonal complement of K_S in $\text{Pic}(S)$ isomorphic to the Enriques lattice \mathbb{E} . As in the case of unnodal Enriques surfaces, we consider a polarization $h = [F_1 + F_2]$ of degree 2. Such polarizations correspond to lattice embeddings $U \hookrightarrow \mathbb{E}$. One can show that the linear system $|2h|$ defines a regular map $\phi_h: S \rightarrow \mathbb{P}^4$ whose image is a 4-nodal quartic del Pezzo surface D (see [10]). The set of fixed points of the deck transformation σ of S consists of a smooth curve \overline{W} of genus 4 and three isolated points. The image of the anti-canonical curve $\mathfrak{C} \in |-2K_S|$ is a singular point q of D (the other three singular points are the images of the isolated fixed points). The curve \overline{W} intersects \mathfrak{C} at two points. Its image W on D is a curve of arithmetic genus 5 with a double point at q . It is equal to the complete intersection of a net of quadrics. Thus a Coble surface is obtained as a degeneration of an Enriques surface when the branch curve W of the map ϕ_h passes through a singular point of D .

Let us look at the discriminant curve of the net \mathcal{N} of quadrics with base locus equal to W . It is a plane quintic with two double points corresponding to corank 2 quadrics in the pencil $|\mathcal{O}_D(2)|$. What is different here is that one of the double points is a cusp. In fact, it is known that the tangent cone of a double point of the discriminant curve corresponding to a corank 2 quadric in the net consists of quadric tangent to the singular line ℓ of the corank 2 quadric. Since the point q is a base point of the net \mathcal{N} , the restriction of \mathcal{N} to the singular line ℓ has q as a base point. Hence there is only one pencil of quadrics touching ℓ at one point (different from q). This proves the assertion. Let T be the normalization of the discriminant curve D of \mathcal{N} . As in the case of Enriques surfaces, it is isomorphic to the intersection of a non-singular quadric $V(Q)$ and a cubic surface $V(F)$. The quintic curve D is obtained by projecting T from a point t on T . Since the projection has a cusp, the point t is a residual point of a point of ramification of one of the g_3^1 on T . Thus, as in the case of nodal Enriques surfaces, we obtain a rational map

$$\Psi'_1: \mathcal{M}_{\text{Co}}(2) \dashrightarrow \mathcal{X}'_4.$$

The degree of this map is the same as the degree of the map Ψ_1 , that is, equal to $2^3 \cdot 17$.

Next we consider the restriction of the map $\Psi_2: \mathcal{X}_4 \dashrightarrow \mathcal{M}_{5,\text{cusp}}$ to \mathcal{X}'_4 . In the notation of the previous section, the cubic $V(F)$ intersects the tangent line L at the point t and is tangent at the point $p_2 = p_3 = t'$. The cubic surface S is the image of the plane Π under the rational map given by the net of cubics tangent at the point t , tangent to the line $\overline{tt'}$ at t' and passing through the points p_4, p_5 . Thus, we have only two singular conics on S in the pencil of conics defined by the hyperplane H . This shows that the line defined by H intersects the discriminant curve at two points, with multiplicity 3 at the cusp and multiplicity 2 at the other point. This gives a rational map

$$\Psi'_2: \mathcal{X}'_4 \dashrightarrow \mathcal{M}'_{5,\text{cusp}}.$$

The degree of this map is the same as in the Enriques case, that is, equal to $2^4 \cdot 31$. The composite

$$\Psi'_2 \circ \Psi'_1: \mathcal{M}_{\text{Co}}(2) \dashrightarrow \mathcal{M}'_{5, \text{cusp}}$$

is of degree $2^7 \cdot 17 \cdot 31$. On the other hand, since the automorphism group of a general Coble surface is isomorphic to the automorphism group of a general Enriques surface (see [10]), the same count as in [18] shows that the degree of the rational map

$$\Phi': \mathcal{M}_{\text{Co}}(2) \dashrightarrow \mathcal{M}_{\text{Co}}$$

is equal to $2^7 \cdot 17 \cdot 31$. This suggests the following conjecture.

Conjecture 8.1. Let Υ' be the birational isomorphism in Theorem 6.3. Then

$$\Upsilon' \circ \Phi' = \Psi'_2 \circ \Psi'_1.$$

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Igor Dolgachev

Department of Mathematics,
University of Michigan, Ann Arbor, USA
E-mail: idolga@umich.edu

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Shigeyuki Kondō

Graduate School of Mathematics,
Nagoya University, Japan
E-mail: kondo@math.nagoya-u.ac.jp