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ON SPECIAL ALGEBRAIC K3 SURFACES. I

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To I. R. Šafarevič on his fiftieth birthday

Abstract. In this paper we describe algebraic K3 surfaces on which lie hyperelliptic curves. We prove a direct and an inverse theorem on the representation of such surfaces as a double plane. We explain the connection between surfaces of this type and elliptic surfaces.

Introduction

Let k be an algebraically closed field of characteristic $p \neq 2$. A smooth projective algebraic surface X over k is called a K3 surface if X is regular and the canonical class of X is trivial, i.e. if $H^1(X, \mathcal{O}_X) = 0$ and $\omega_X \simeq \mathcal{O}_X$. We shall call a K3 surface special if there exists a hyperelliptic curve C of genus $g \geq 2$ on it. The goal of this paper is to give an explicit description of all such surfaces. We will show that the special K3 surfaces are precisely those which admit a representation as a double plane (i.e. are birationally isomorphic to an affine surface in $\mathbf{A}_k^3$ with the equation $z^2 = F(x, y)$). Let π denote the class of a K3 surface X, i.e. the smallest of the dimensions of complete linear systems on X which are greater than one. We shall show that for special K3 surfaces π can only take the values 2, 3, 4, or 5. This answers the question of [2]. Moreover, for $\pi > 2$ the condition of speciality is equivalent to the existence of a pencil of elliptic curves of index ≤ 2 on X. In §4 we will show that the universal cover of an elliptic Enriques surface (or any Enriques surface if $\operatorname{char}(k) = 0$) is a special K3 surface.

The main result is the:

Enriques-Campedelli Theorem. *A double plane which is birationally isomorphic to a* K3 *surface is equivalent to one of the following double planes*:

a) $z^2 = F_6(x, y)$, *where* $F_6(x, y) = 0$ *is a curve of degree* 6;

b) $z^2 = F_8(x, y)$, *where* $F_8(x, y) = 0$ *is a curve of degree* 8 *having two ordinary (singular) quadruple points (these can be infinitely near)*;

c) $z^2 = F_{10}(x, y)$, *where* $F_{10}(x, y) = 0$ *is a curve of degree* 10 *having a singular*

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point of multiplicity seven and two ordinary triple points which are infinitely near to first order;

d) $z^2 = F_{12}(x, y)$ *where* $F_{12}(x, y) = 0$ *is a curve of degree* 12 *having a singular point of multiplicity nine and three ordinary points of multiplicity three which are infinitely near to first order.*

This theorem was first proved by Enriques in 1896 [8] and later was reproved by Campedelli [6], [7]. In this paper we give a modernization of Campedelli's proof.

The continuation of this paper will be devoted to moduli and automorphisms of special K3 surfaces.

§1. Basic definitions and auxiliary lemmas

Definition 1.1. A K3 *surface* is a smooth projective algebraic surface X with $H^1(X, \mathcal{O}_X) = 0$ and $\omega_X = \Omega_X^2 \simeq \mathcal{O}_X$. A K3 surface is said to be *special* if there exists a smooth hyperelliptic curve of genus $g \geq 2$ on it.

Lemma 1.2. *Let* $D = \Sigma_i n_i D_i$ *be a connected effective divisor on a* K3 *surface* X. *Assume that* $n_i = 1$ *for at least one value of* i. *Then*

$$\dim |D| \overset{\text{def}}{=} \dim_k H^0(X, \mathcal{O}_X(D)) - 1 = \frac{(D^2)}{2} + 1,$$

$$p_a(D) \overset{\text{def}}{=} \dim_k H^1(D, \mathcal{O}_D) = \dim |D|.$$

The exact sequence of sheaves

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

gives an exact sequence of cohomology groups

$$0 \to H^0(X, \mathcal{O}_X) \to H^0(D, \mathcal{O}_D) \to H^1(X, \mathcal{O}_X(-D))$$
$$\to H^1(X, \mathcal{O}_X) \to H^1(D, \mathcal{O}_D) \to H^2(X, \mathcal{O}_X(-D)) \to H^2(X, \mathcal{O}_X) \to 0.$$

Since X is a K3 surface,

$$H^1(X, \mathcal{O}_X) = 0, \quad H^2(X, \mathcal{O}_X) \simeq H^0(X, \omega_X) \simeq k.$$

On the other hand, because of the conditions on D we have $H^0(D, \mathcal{O}_D) = k$. Hence

$$H^1(X, \mathcal{O}_X(-D)) = 0, \quad \dim_k H^1(D, \mathcal{O}_D) = \dim_k H^2(X, \mathcal{O}_X(-D)) - 1.$$

It remains to use Riemann-Roch for the sheaf $\mathcal{O}_X(-D)$ and Serre duality.

Definition 1.3. The number

$$\pi(X) = \min_{D \subset X} \{\dim |D| \mid \dim |D| > 1\}$$

is called the *class* of the K3 surface X.

Remarks. 1. In the case $k = \mathbf{C}$ the class $\pi(X)$ can take any value ≥ 2 (see [1], Chapter IX). This is apparently also true in general (see [9], p. 256).

2. In [1] K3 surfaces are called Kummer surfaces. At the present time this term is used for a special case of K3's: nonsingular minimal models of the quotient of a

two-dimensional abelian variety by the involution $x \to -x$. Arbitrary K3 surfaces are frequently called generalized Kummer surfaces (cf. [9]). In [2] a "special Kummer surface" was a K3 surface for which the class π is realized by a system of hyperelliptic curves. Our definition is obviously somewhat broader. We do not know an example showing that these definitions are actually different.

Definition 1.4. A morphism $f: X' \to X$ of complete integral algebraic surfaces is called a *two-sheeted cover* (or *double cover*) if one of the following equivalent conditions holds:

1) f induces a separable quadratic extension of function fields $k(X')/k(X)$.

2) There exists an open subset $U \subset X'$ such that $f|U$ is a finite morphism of degree 2.

3) The morphism f splits into a composition $X' \xrightarrow{f_1} X_1' \xrightarrow{f_2} X$, where f_1 is a birational morphism and f_2 is a finite morphism of degree 2.

A proof of the equivalence of assertions 1)–3) uses a standard technique of the theory of schemes ([10], Chapter 3) and presents no difficulty.

Definition 1.5. A *double plane* is an algebraic surface which is birationally isomorphic to the affine surface $\mathrm{Spec}(k[x, y, z]/(z^2 - F(x, y)))$.

Proposition 1.6. *An algebraic surface X is a double plane if and only if it is birationally equivalent to a surface X which is a double cover of the projective plane $\mathbf{P}_k^2$.*

The proof is obvious.

Lemma 1.7. *Let $g: X' \to X$ be a double cover of smooth surfaces. Then for any divisors D_1 and D_2 on X we have*

$$(f^*(D_1) \cdot f^*(D_2))_{X'} = 2(D_1 \cdot D_2)_X.$$

The proof follows trivially from general properties of Chow rings.

Lemma 1.8. *If D is an integral divisor on a K3 surface, then the linear system $|D|$ has no base points.*

For a proof see [1], Chapter VIII, Lemma 2.

Proposition 1.9. *Let C be a hyperelliptic curve on a K3 surface X with $p_a(C) = m$. The linear system $|C|$ gives a double cover $f: X \to V$, where V is a surface of degree $m - 1$ in $\mathbf{P}_k^m$.*

For a proof see [1], Chapter VIII, Lemma 3.

Proposition 1.10. *Let $f: X' \to X$ be a double cover of smooth surfaces. Let $X' \xrightarrow{f_1} X_1' \xrightarrow{f_2} X$ be the Stein factorization of f (see condition 3) of Definition 1.4). Then the singular points of the surface X_1' are precisely the inverse images of the singular points of the branch curve W of the finite cover f_2.*

In fact, the surface X_1' is normal and we can therefore use local properties of finite Galois coverings of algebraic varieties (see [12]).

Definition 1.11. The curve W of the hypothesis of Proposition 1.10 is called the *branch curve* of the double cover.

Corollary 1.12. *Assume that the branch curve of the double cover* $f\colon X' \to X$ *is nonsingular and that the surface* X' *is a minimal model. Then* f *is a finite morphism.*

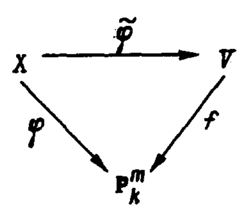
Lemma 1.13. *Let* $f\colon X' \to X$ *be a finite morphism of degree two of a normal surface* X' *onto a smooth surface* X. *Assume that the branch curve* W *of the morphism* f *has only ordinary double points* $P_1, \cdots, P_n$. *Let* $\overline{X} \to X$ *be the blowing-up of these points. Then the normalization* $\widetilde{X}'$ *of the surface* $X' \times_X \overline{X}$ *is a smooth surface, and the projection* $\widetilde{X}' \to \overline{X}$ *induces a finite cover whose branch curve is the proper transform of the curve* W.

This assertion is a special case of "Jung's method" for resolving the singularities of surfaces. Its verifications reduces to a trivial local computation, which we omit.

§2. The Enriques-Campedelli Theorem

For the duration of this section F_n will denote a relative minimal model of a rational surface possessing a rational curve S_n with $(S_n^2) = -n$ (n is a nonnegative integer different from 1) (see [1], Chapter 4). L will denote any one of the fibers of the canonical morphism $F_n \to \mathbf{P}_k^1$ a section of which is the curve S_n.

Proposition 2.1. *Let* $\phi\colon X \to \mathbf{P}_k^m$ *be a morphism of a normal projective surface whose image* $\phi(X)$ *is a normally imbedded surface of degree* $m-1$. *Then there exists a factorization*



where f *is a birational morphism, and* V *is one of the following surfaces*: $\mathbf{P}_k^2$, F_n *or the surface* $\overline{F}_n$ *obtained by blowing down the curve* S_n *on* F_n *at the normal point.*

A hyperplane section of the surface $\phi(X)$ is a curve of degree $m-1$ which is normally imbedded in $(m-1)$-dimensional projective space. It follows from the Riemann-Roch Theorem that such a curve is rational. By Proposition 2 of [4] it follows from this that the normalization V of the surface $\phi(X)$ is one of the surfaces enumerated above. Since X is normal, we obtain the desired assertion.

Lemma 2.2. *Let* A *be a regular geometric two-dimensional local ring with residue field* k *of characteristic different from* 2. *Let* g *be an automorphism of* A *of order* 2. *Then either the ring of invariants* A^g *is regular, or its completion* $\hat{A}^g$ *is isomorphic to the completion of the local ring of the vertex of a quadratic cone.*

We can obviously assume the ring A to be complete, and hence to be isomorphic to the ring $k[[x, y]]$. In this case the action of g is equivalent to a linear map, and therefore we may assume that either $g(x) = x$ and $g(y) = -y$, or $g(x) = -x$ and $g(y) = -y$. In the former case $A^g \simeq k[[x, y^2]]$ is regular, and in the second $A^g \simeq k[[x^2, y^2, xy]] \simeq k[[u, v, w]]/(uv - w^2)$. This proves the lemma.

Theorem 2.3. *Let X be a special* K3 *surface. Then there exists a double cover $f\colon X \to V$, where V is one of the following surfaces*: $\mathbf{P}_k^2$, F_0, F_n $(n = 2, 3, 4)$, *or* $\overline{F}_2$.

Let C be a hyperelliptic curve on X and m its genus. Applying Propositions 1.9 and 2.1, we obtain a double cover $f\colon X \to V$, where V is $\mathbf{P}_k^2$, F_n or $\overline{F}_n$. We shall show that the case $n > 4$ is impossible, and for $n = 3$ and $n = 4$ the morphism $f\colon X \to \overline{F}_n$ passes through the canonical morphism $F_n \to \overline{F}_n$, which blows down the section S_n at the singular point.

Assume that the first case holds, i.e. $V = F_n$ with $n > 4$. Let $f\colon X \xrightarrow{\phi_1} X' \xrightarrow{\phi_2} F_n$ be the Stein factorization of the morphism f (i.e. ϕ_1 is birational and ϕ_2 is a finite morphism of degree 2). We assume that the morphism ϕ_1 is an isomorphism in some neighborhood of $\phi_2^{-1}(S_n)$. Then either $f^{-1}(S_n)$ is $R_1 + R_2$, where the R_i are rational curves on X, or $f^{-1}(S_n)$ is an irreducible curve. In any case $(f^{-1}(S_n)^2) \geq -8$. Since, on the other hand, $(f^{-1}(S_n)^2) = -2n$, we obtain from this that $n \leq 4$.

Now suppose $f^{-1}(S_n) = D_1 + \cdots + D_r + R$, where D_i $(1 \leq i \leq r)$ are rational curves (perhaps reducible) blown down by the morphism ϕ_1 into distinct singular points of X' lying on $\phi_2^{-1}(S_n)$, and R is the proper transform of $\phi_2^{-1}(S_n)$. We have

$$(f^{-1}(S_n)^2) = -2n = \sum_{i=1}^{r} (D_i^2) + 2\sum_{i=1}^{r} (D_i R) + R^2.$$

Again it is obvious that either R is an irreducible curve, or $R = R_1 + R_2$, where the R_i are rational curves. In either case $(R^2) \geq -8$. Since $(D_i^2) = -2$ (cf. Lemma 1.2) and $(D_i R) \geq 1$, we obtain

$$n = r - \sum_{i=1}^{r} (D_i R) - \frac{(R^2)}{2} \leqslant 4.$$

We note that $n > 0$ can happen here only in the case $r = 1$, $n = 3$, $R_1 = R_2$.

Now we assume that the second case holds: $V = \overline{F}_n$ $(n > 2)$. Let P be a singular point of $\overline{F}_n$. If the morphism $f\colon X \to \overline{F}_n$ were finite over a neighborhood of P, the local ring $\mathcal{O}_P$ would be the quotient of a regular ring by the action of the group of order two. By Lemma 2.2 it would follow from this that P is either a nonsingular point or an ordinary double point. The latter obviously contradicts the fact that $n > 2$. Thus $f^{-1}(P)$ is a divisor on X. By the universality of blowing-up, we will obtain from this that f splits into a composition $X \to F_n \to \overline{F}_n$. This proves the theorem.

Proposition 2.4. *Let $f\colon X \to F_0$ be a double cover with X a* K3 *surface. Then the branch curve W of the morphism f has degree* (4.4). *Moreover, on X there exist elliptic curves E_1 and E_2 with $(E_1 \cdot E_2) = 2$, for which the linear system $|E_1 + E_2|$*

is irreducible, and any nonsingular curve $D \in |E_1 + E_2|$ *is a hyperelliptic curve of genus* 3.

Let $L_1 = L$ and $L_2 = S_0$ be the effective generators of $\mathrm{Pic}(F_0)$. Then $(f^{-1}(L_i)^2) = 0$, and hence $|f^{-1}(L_i)|$ is a pencil of elliptic curves. If E_i $(i = 1, 2)$ are nonsingular elliptic curves belonging to this pencil, then obviously $(E_1 \cdot E_2) = 2(L_1 \cdot L_2) = 2$. Since $((E_1 + E_2)^2) = 4$, we have $\dim |E_1 + E_2| = 3$. It is obvious that the system $|E_1 + E_2|$ is irreducible. For any nonsingular curve $D \in |E_1 + E_2|$ the pencil $|E_1|$ cuts out a linear series g_2^1 of dimension 1 of degree 2 on D. Hence D is a hyperelliptic curve. Its genus is equal to $(D^2/2) + 1 = 3$. If W is the branch curve of f, then $(L_i \cdot W) = 4$, since $f^{-1}(L_i)$ is an elliptic curve. Hence $W \sim 4L_1 + 4L_2$, i.e. it has degree (4, 4).

Proposition 2.5. *Let* $f\colon X \to F_2$ *be a double cover with* X *a* K3 *surface. Then the branch curve* W *of the morphism* f *is equivalent to the divisor* $8L + 4S_2$. *Moreover, on* X *there exist an elliptic curve* E *and two nonsingular rational curves* R_1 *and* R_2 *with* $(R_1 \cdot R_2) = 0$ and $(R_1 \cdot E) = (R_2 \cdot E) = 1$. *Here the linear system* $|2E + R_1 + R_2|$ *is irreducible and any nonsingular curve* $D \in |2E + R_1 + R_2|$ *is a hyperelliptic curve of genus* 3.

Since $(L^2) = 0$, $|f^{-1}(L)|$ is a pencil of elliptic curves. Let $E \in |f^{-1}(L)|$ be a nonsingular curve belonging to this pencil. Since $(S_2^2) = -2$, we have $(f^{-1}(S_2)^2) = -4$. Arguing as in the proof of Theorem 2.3, we will obtain that $f^{-1}(S_2) = R_1 + R_2$, where the R_i are isomorphic to S_2 and $(R_1 \cdot R_2) = 0$. Since $(E(R_1 + R_2)) = 2(L \cdot S_2) = 2$ and $(ER_i) > 0$, we have $(E \cdot R_1) = (E \cdot R_2) = 1$. Consider the linear system $|2E + R_1 + R_2|$. By Lemma 1.2

$$\dim|2E + R_1 + R_2| = \frac{(2E + R_1 + R_2)^2}{2} + 1 = 3.$$

Since, on the other hand, $\dim|2E + R_i| = 2$, $i = 1, 2$, it follows that $|2E + R_1 + R_2|$ is an irreducible linear system. For any nonsingular curve $D \in |2E + R_1 + R_2|$ the pencil $|E|$ cuts out a linear series of dimension 1 and degree 2 on D. Hence D is a hyperelliptic curve. Its genus is equal to $(D^2)/2 + 1 = 3$.

Let $W \sim aL + bS_2$ be the branch curve of the morphism f. Then $(W \cdot L) = 4$, since $|f^{-1}(L)|$ is a pencil of elliptic curves. On the other hand, $(W \cdot (2L + S_2)) = 8$, since $|f^{-1} \cdot (2L + S_2)| = |2E + R_1 + R_2|$ is a linear system of hyperelliptic curves of genus 3. Elementary computations now show that $a = 8$ and $b = 4$.

Proposition 2.6. *Let* $f\colon X \to F_3$ *be a double cover with* X *a* K3 *surface. Then the branch curve* W *of the morphism* f *is equivalent to the divisor* $10L + 4S_3$. *Moreover, on* X *there exist an elliptic curve* E *and nonsingular rational curves* R_1 *and* R_2 *with* $(R_1 \cdot R_2) = (R_1 \cdot E) = 1$ *and* $(R_2 \cdot E) = 0$. *Here the linear system* $|3E + 2R_1 + R_2|$ *is irreducible and any nonsingular curve* $D \in |3E + 2R_1 + R_2|$ *is a hyperelliptic curve of genus* 4.

Again, as in the preceding propositions, $|f^{-1}(L)|$ is a pencil of elliptic curves. Let E be one of the curves of this pencil. Since $(S_3^2) = -3$, we have $(f^{-1}(S_3)^2) = -6$. Arguing as in the first half of the proof of Theorem 2.3, we have that $f^{-1}(S_3) = 2R_1 + R_2$. Here R_2 is an exceptional curve on X which is mapped by the morphism f into a point, and $2R_1$ is the proper transform of S_3. Since $((2R_1 + R_2)^2) = -8 - 2 + 4(R_1 \cdot R_2) = -6$, we have $(R_1 \cdot R_2) = 1$. Moreover, $(E \cdot (2R_1 + R_2)) = 2(L \cdot S_3) = 2$, and since obviously $(R_2 \cdot E) = 0$, we have $(E \cdot R_1) = 1$. We now consider the linear system $|3E + 2R_1 + R_2|$. It is obvious that

$$\dim|3E + 2R_1 + R_2| = \frac{((3E + 2R_1 + R_2)^2)}{2} + 1 = 4.$$

On the other hand, $\dim|3E + R_1 + R_2| < \dim|3E + 2R_1| = 3$. Hence the system $|3E + 2R_1 + R_2|$ is irreducible. For any nonsingular curve D of this system $(E \cdot D) = (E \cdot (3E + 2R_1 + R_2)) = 2$, and hence the pencil $|E|$ cuts out a linear series of dimension 1 and degree 2 on D. Thus D is a hyperelliptic curve of genus $(D^2)/2 + 1 = 4$.

Let $W \sim aL + bS_3$ be the branch curve of the morphism f. Then $(W \cdot L) = 4$, since $|f^{-1}(L)|$ is a pencil of elliptic curves. On the other hand, $(W \cdot (3L + S_3)) = 10$, since $|f^{-1}(3L + S_3)| = |3E + 2R_1 + R_2|$ is a linear system of hyperelliptic curves of genus 4. Simple computations now show that $a = 10$ and $b = 4$.

Proposition 2.7. *Let $f\colon X \to F_4$ be a double cover with X a* K3 *surface. Then the branch curve of the morphism f is equivalent to the divisor $12L + 4S_4$. Moreover, on X there exist an elliptic curve E and a nonsingular rational curve R with $(R \cdot E) = 1$. Here the linear system $|4E + 2R|$ is irreducible and any nonsingular curve of this system is a hyperelliptic curve of genus* 5.

The proof of this is completely analogous to the proofs of the preceding two propositions, and we omit it.

Proposition 2.8. *Let $f\colon X \to \overline{F}_2$ be a double cover with X a* K3 *surface. Then there exists a double cover $f'\colon X \to V$, where either $V = F_0$ or $V = F_2$.*

Let P be the singular point of the surface $\overline{F}_2$. If $f^{-1}(P)$ is a divisor on X, then, by the universality of blowings-up, the morphism f splits into a composition $X \to F_2 \to \overline{F}_2$, where $F_2 \to \overline{F}_2$ is a blowing down of the section S_2 into the singular point. But if the morphism f is finite over some neighborhood U of P, then, by Lemma 2.2, the morphism f does not split over $U \setminus P$. Since $\overline{F}_2$ is isomorphic to a quadratic cone in $\mathbf{P}_k^3$, we may assume that f is a morphism of X onto a surface of order two in $\mathbf{P}_k^3$. The inverse image of the hyperplane section $f(X)$ defines a linear system $|D|$ of curves of genus $\pi = 3$. On the other hand, the inverse image of a hyperplane section passing through the vertex of the cone defines a divisor $E_1 + E_2$ belonging to $|D|$. Since $(D \cdot E_i) = 2$ (the E_i are the inverse images of the generators of the cone), we have $(E_i^2) = 0$ and $(E_1 \cdot E_2) = 2$. Hence the E_i are elliptic curves on X. Each pencil $|E_i|$ defines a morphism $f_i\colon X \to \mathbf{P}_k^1$. Since $(E_1 \cdot E_2) = 2$, the morphism $f_1 \times f_2\colon X \to \mathbf{P}_k^1 \times \mathbf{P}_k^1 = F_0$ is the desired double cover.

Theorem 2.9 (Enriques-Campedelli). *A special* K3 *surface is birationally equivalent to one of the following four types of double planes:*

a) $z^2 = F_6(x, y)$, *where* $F_6(x, y) = 0$ *is a curve of degree* 6;

b) $z^2 = F_8(x, y)$, *where* $F_8(x, y) = 0$ *is a curve of degree* 8 *which has two ordinary singular points of multiplicity four (perhaps infinitely near: case* b'));

c) $z^2 = F_{10}(x, y)$, *where* $F_{10}(x, y) = 0$ *is a curve of degree* 10 *which has a singular point of multiplicity seven and two ordinary triple points which are infinitely near to first order*;

d) $z^2 = F_{12}(x, y)$, *where* $F_{12}(x, y) = 0$ *is a curve of degree* 12 *which has a singular point of multiplicity nine and three ordinary triple points which are infinitely near to first order.*

Proof. By Theorem 2.3 and Proposition 2.8 there exists a double cover $f\colon X \to V$, where V is one of the surfaces $\mathbf{P}_k^2$ or $F_n (n = 0, 2, 3, 4)$.

Case 1. $V = \mathbf{P}_k^2$. In this case for any line L on $\mathbf{P}_k^2$ we have $(f^{-1}(L)^2) = 2(L^2) = 2$. Hence the linear system $|f^{-1}(L)|$ consists of curves of genus 2, and the intersection index of L with the branch curve equals 6. Thus $\pi(X) = 2$, and the branch curve of f has degree 6. From this it follows that the surface X is birationally equivalent to a double plane of type a).

Case 2. $V = F_0$. By Proposition 2.4 the branch curve W of the morphism $f\colon X \to V$ has degree (4, 4). Let P be a point on F_0 which does not belong to W, let $\tilde{F}_0 \to F$ be the blow-up of P, $\tilde{X} = X \times_{F_0} \tilde{F}_0$, and $\tilde{f}\colon \tilde{X} \to \tilde{F}_0$ the projection. After taking the composition of $\tilde{f}$ with the canonical morphism $\tilde{F}_0 \to \mathbf{P}_k^2$ (blowing-down of the proper transforms on $\tilde{F}_0$ of the generators of F_0 passing through P), we obtain a double cover $\tilde{X} \to \mathbf{P}_k^2$. The branch curve of this cover is the proper image of W relative to the birational map $F_0 \to \tilde{F}_0 \to \mathbf{P}_k^2$ and is obviously a curve of degree 8 with two singular points of multiplicity four (the images of curves on F_0 which can be blown down).

Since the surface $\tilde{X}$ is birationally equivalent to the desired surface X, we see that X is birationally equivalent to a double plane of type b).

Case 3. $V = F_2$. By Proposition 2.5 the branch curve W of the morphism $f\colon X \to V$ is equivalent to the divisor $8L + 4S_2$.

Let P_1 and P_2 be distinct closed points on F_2 and not belonging to W. Let $\tilde{F} \to F_2$ be the blow-up of these points. Blowing down the proper images of generators L_1 and L_2 passing through P_1 and P_2, we obtain a double cover of surfaces $\tilde{X} = X \times_{F_2} \tilde{F}_2 \to F_0$ whose branch curve $\tilde{W}$ is a curve of degree (8, 4) with two ordinary quadruple points lying on the section S_0. Let Q be one of these points and L a generator passing through Q. Blowing up the point Q and blowing down the proper images of the sections S_0 and L, we obtain a double cover $\tilde{\tilde{X}} = \tilde{X} \times_{F_0} \tilde{F}_0 \to \mathbf{P}_k^2$ with a branch curve of degree eight with two infinitely near singular points of multiplicity four. Since the surfaces $\tilde{\tilde{X}}$ and X are birationally isomorphic, we will obtain that X is birationally isomorphic to a double plane of type b').

Case 4. $V = F_3$. By Proposition 2.6 the branch curve W of the morphism $f\colon X \to V$ is equivalent to the divisor $10L + 4S_3$. In this case the curve W contains as one of its irreducible components the section S_3. Let $W' = W \setminus S_3$. Then $(W' \cdot S_3) = 1$ and $(W' \cdot L) = 3$. Making elementary transformations at any three points P_1, P_2, P_3 not lying on W, we obtain a double cover $\tilde{f}: \tilde{X} \to F_0$, where $\tilde{X}$ is birationally equivalent to X and the branch curve of $\tilde{f}$ is the section S_0 and some curve W' of degree (10, 3) with three singular points of multiplicity four on S_0. Now making an elementary transformation with center at one of these singular points, we obtain a double cover $X \times_{F_0} \tilde{F}_0 \to \mathbf{P}^2_k$ with a branch curve of degree 10 with a singular point of multiplicity seven and two ordinary triple points which are infinitely near to it. Thus we get case c) of the theorem.

Case 5. $V = F_4$. By Proposition 2.7 the branch curve W of the morphism $f: X \to F_4$ is equivalent to the divisor $12L + 4S_4$. In this case the curve S_4 is contained in W, and the curve $W' = W - S_4$ does not intersect S_4.

Carrying out a construction analogous to case 4, we obtain case d) of the theorem.

Corollary 2.10. *For any special* K3 *surface the class* π *can only take the values* 2, 3, 4 *or* 5.

In fact, the inverse image of a line on a double plane of type a)–d) is a hyperelliptic curve of genus 2, 3, 4 or 5.

Corollary 2.11. *Each special* K3 *surface with class* $\pi = 2$ *(respectively* 3, 4, 5) *is a double cover of* $\mathbf{P}^2_k$ *(respectively* F_0 *or* F_2, F_3, F_4*).*

As was mentioned at the beginning of the proof of Theorem 2.9, each special K3 surface X is a double cover of one of the surfaces $\mathbf{P}^2_k$, F_0, F_2, F_3 or F_4. If $\pi = 2$, then the linear system $|C|$ of curves of genus 2 defines a double cover $X \to \mathbf{P}^2_k$. If $\pi = 3$, the linear system $|C|$ of curves of genus 3 defines a morphism $f: X \to \mathbf{P}^3_k$ whose image V is a surface of degree two. Each such surface is either the quadric F_0 or the cone $\overline{F}_2$. Now apply Proposition 2.8. If $\pi = 5$, then X can be mapped two-to-one only onto F_4, since otherwise by Propositions 2.4–2.6 there would exist a curve of smaller genus on X. For $\pi = 4$ the morphism $X \to \mathbf{P}^4_k$, defined by a system of curves of genus 4, can pass through only one of the surfaces F_n $(n = 2, 3, 4)$(see the proof of Theorem 2.3). The case $n = 2$ is impossible by Proposition 2.5. The case $n = 4$ is impossible since there does not exist an embedding of F_4 into $\mathbf{P}^4_k$.

§3. Converse of the Enriques-Campedelli Theorem

Lemma 3.1. *Let X be a normal algebraic surface and D an integral curve on X whose image in the group* Pic(X) *is divisible by two. Then there exists an irreducible unramified cover $F: X' \to X \setminus D$ of degree* 2.

Using the étale topology, we consider on $U = X \setminus D$ the Kummer sequence [3]:

$$0 \to \Gamma(U, \mathcal{O}_U^*)/\Gamma(U, \mathcal{O}_U^*)^2 \to H^1(U, \mu_2) \to \mathrm{Pic}(U_2) \to 0$$

(recall that char $k \neq 2$). Since X is normal, the canonical restriction morphism $r: \mathrm{Pic}(X) \to \mathrm{Pic}(U)$ is surjective, and its kernel is generated by the divisor D. Since the class of D is divisible in $\operatorname{Pic}(X)$ by 2, the quotient group $\operatorname{Pic}(X)/\operatorname{Ker}(r) = \operatorname{Pic}(U)$ contains an element of order two. Hence

$$H^1(U, \mu_2) \neq 0.$$

It remains to use the fact that the group $H^1(U, \mu_2)$ classifies the principal bundles over U with structure group $\mathbf{Z}/2\mathbf{Z}$. A nontrivial such bundle also defines the desired unramified cover $f: X' \to U$.

Lemma 3.2. *In the notation of Lemma 1.13 we have*

$$c_2(\widetilde{X}') = 2(c_2(X) + n) + 2\sum_{i=1}^{h} (g(W_i) - 1),$$

where $c_2(Z)$ is the second Chern number of the tangent sheaf to the surface Z (= the topological Euler characteristic of Z in case $k = \mathbf{C}$ and the l-adic Euler characteristic in the general case), and $g(W_i)$, $1 \leq i \leq h$, is the geometric genus of an irreducible component of the branch curve W.

If $\overline{X}$ is the blow-up of the double points of W on X, then $c_2(\overline{X}) = c_2(X) + n$. Now use the "Severi correspondence formula" [11].

Theorem 3.3. *Let $z^2 = F_{2n}(x, y)$ $(n = 3, 4, 5, 6)$ be a double plane. Assume that the branch curve $F_{2n}(x, y) = 0$ is irreducible and, except for the singularities mentioned in Theorem 2.9, has only ordinary double points. Then the minimal projective nonsingular model of this double plane is a special* K3 *surface with class $\pi \leq n - 1$.*

Proof. Suppose $n = 3$, and $W \subset \mathbf{P}^2_k$ is a projective curve of degree 6 which is irreducible and has only ordinary double points (or is smooth). By Lemma 3.1 there exists an unramified cover $U' \to U = \mathbf{P}^2_k \backslash W$ of degree two. This cover can obviously be extended to a finite morphism $f: X \to \mathbf{P}^2_k$, ramified only over W (cf. [12], p. 4). Let V be the blow-up of $\mathbf{P}^2_k$ with center at the singularities of W. By Lemma 1.13 the normalization X' of the surface $X \times_{\mathbf{P}^2_k} V$ is a nonsingular surface, and the projection $f': \overline{X}' \to V$ defines a finite double cover, ramified over the proper transform $\overline{W}$ of the curve W. Let $L_1, \cdots, L_n$ be exceptional curves on V which are inverse images of singular points of W relative to the projection $V \to \mathbf{P}^2_k$. We have $K_V = -3\overline{H} + L_1 + \cdots + L_n$, where $\overline{H}$ is the proper transform of the line H in $\mathbf{P}^2_k$.

On the other hand, it is obvious that $\overline{W} \sim 6\overline{H} - 2L_1 + \cdots + 2L_n$. From this it follows (cf. [11]) that $K_{X'} \sim f'^*(K_V) + \frac{1}{2} f'^*(\overline{W}) \sim 0$. By Lemma 3.2, $c_2(X') = 2(3 + n) + 2(9 - n) = 24$. By Noether's formula

$$1 - q + p_g = \frac{c_2(X') + (K^2_{X'})}{12},$$

from which it follows that $q = \dim_k H^1(X, \mathcal{O}_{X'}) = 0$. Hence the surface X' is regular and $K_{X'} \sim 0$, and this is the definition of a K3 surface.

Now suppose $n = 4$ and $W \subset \mathbf{P}^2_k$ is an irreducible projective curve of degree 8, having perhaps, besides the two ordinary quadruple points A_1 and A_2, only ordinary double points $P_1, \cdots, P_n$. Blowing up the points A_1 and A_2 on $\mathbf{P}_k^2$ and blowing down the proper transform of the line A_1A_2, we get that the proper transform $\overline{W}$ of W on the quadric F_0 is an irreducible curve of type (4,4), having perhaps ordinary double points $P'_1, \cdots, P'_n$. The remaining part of the argument is carried out in this case analogously to that of the previous case (replacing $\mathbf{P}_k^2$ by F_0).

The proofs of the remaining cases follow along the model of the preceding ones, and we omit them. We only note that for $n = 4$ (case of infinitely near points) it is necessary to replace $\mathbf{P}_k^2$ by F_2, for $n = 5$, by F_3, for $n = 6$ by F_4 (cf. Propositions 2.5, 2.6 and 2.7) and to use Lemmas 3.1 and 3.2.

Remark 3.4. In case $k = \mathbf{C}$ it follows easily from the results of G. N. Tjurina (see [1], Chapter IX) that for a "generic" double plane of type a), b) or b'), c), d)(cf. Theorem 2.9) $\pi = 2, 3, 4, 5$ respectively. This is apparently also true in general.

§4. Elliptic surfaces and special K3 surfaces

We recall some definitions.

Definition 4.1. A *pencil of elliptic curves* on a projective algebraic surface X is a morphism $f\colon X \to B$, where B is a smooth curve, and the general fiber of f is a smooth elliptic curve. The index of a pencil is the index of the generic fiber X_η of the morphism f, i.e. the greatest common divisor of the degrees of effective divisors on X_η defined over the field $k(B)$.

Let $r\colon \operatorname{Pic}(X) \to \operatorname{Pic}(X_\eta)$ be the canonical restriction morphism. Passing to Néron-Severi groups we have a homomorphism $\overline{r}\colon \operatorname{NS}(X) \to \operatorname{NS}(X_\eta) \to \mathbf{Z}$. It is easy to see that the index of the pencil f on X equals the order of the cokernel of the homomorphism $\overline{r}$. In other words, it is equal to $\min\{(F \cdot C) | C$ a transversal curve over B, F an arbitrary fiber of $f\}$.

Definition 4.2. A smooth projective algebraic surface is said to be *elliptic* if there exists a pencil of elliptic curves on it. The *index* I of an elliptic surface X is the minimal index of all the possible pencils of elliptic curves on X.

Lemma 4.3. *For an algebraic* K3 *surface X to be an elliptic surface with index I it is necessary and sufficient that on X there exist a connected curve C with $p_a(C) = 1$ and an irreducible curve S with $(C \cdot S) = I$.*

Necessity is obvious. To prove the sufficiency we need to use Lemmas 1.2 and 1.8 and consider the morphism $f\colon X \to \mathbf{P}_k^1$ defined by the linear system $|C|$.

Theorem 4.4. *Every special* K3 *surface X of class $\pi > 2$ is an elliptic surface. Moreover, for $\pi = 4$ or 5 the index of this surface is equal to one* (1), *and for $\pi = 3$ it is ≤ 2.*

Proof. By 2.11 for $\pi = 3$ (respectively 4 or 5) there exists a double cover X over F_0 or F_2 (respectively F_3 or F_4). It remains to apply Lemma 4.3 and Propositions 2.4 and 2.5 (respectively 2.6 or 2.7).

Theorem 4.5. *Each elliptic* K3 *surface with index $I \leq 2$ is a special* K3 *surface.*

Proof. Assume $I = 1$ and that C and S are curves on X chosen via Lemma 4.3. Since $(S^2) \geq -2$, we have

$$((4C + 2S)^2) = 16(C \cdot S) + 4(S^2) = 16 + 4(S^2) \geqslant 8.$$

Moreover,

$$((4C + 2S)^2) > ((4C + S)^2) = 8 + (S^2),$$
$$((4C + 2S)^2) > ((3C + 2S)^2) = 12 + 4(S^2).$$

Thus the linear system $|4C + 2S|$ has no fixed components and consists of curves of genus ≥ 5. Since $(C^2) = 0$, we have $(C \cdot (4C + 2S)) = 2$.

Hence the pencil $|C|$ cuts out a linear series g_2^1 on a nonsingular curve $D \in |4C + 2S|$. Hence D is hyperelliptic and X is special.

Suppose $I = 2$. The linear system $|C + S|$ has no fixed components and consists of curves of genus

$$g = \frac{((C+S)^2)}{2} + 1 = \frac{4 + (S^2)}{2} + 1 \geqslant 2.$$

Since $(C(C + S)) = (C \cdot S) = 2$, the pencil $|C|$ cuts out a linear series g_2^1 on any nonsingular curve $D \in |C + S|$. Hence D is a hyperelliptic curve and X is a special K3 surface.

Definition 4.6. An *Enriques surface* is a smooth algebraic surface X with $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$ and $\omega_X^{\otimes 2} \simeq \mathcal{O}_X$.

Proposition 4.7. *Let X be an Enriques surface. Then there exists a finite unramified cover of degree two $f: X' \to X$, where X' is a* K3 *surface.*

Since $\omega_X \not\simeq \mathcal{O}_X$, the sheaf ω_X defines an element of order two in the group $\mathrm{Pic}(X)$. By the Kummer sequence

$$0 \to H^0(X, \mathcal{O}_X^*)/H^0(X, \mathcal{O}_X^*)^2 \to H^1(X, \mu_2) \to \mathrm{Pic}(X_1) \to 0$$

we will obtain that $H^1(X, \mu_2) \neq 0$. Since the group $H^1(X, \mu_2)$ classifies principal homogeneous spaces with structure group $\mathbf{Z}/2\mathbf{Z}$, a nontrivial p.h.s. defines a finite unramified double cover $f: X' \to X$. Elementary computations (see [11]) show that $\omega_{X'} \simeq f^*(\omega_X) \simeq \mathcal{O}_{X'}$ and $H^1(X', \mathcal{O}_{X'}) = 0$. Hence X' is a K3 surface.

Lemma 4.8 (L. Godeaux). *Let X be an elliptic surface with $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$, let $f: X \to \mathbf{P}_k^1$ be the corresponding pencil of elliptic curves, and F an arbitrary fiber over a closed point. Let $\Gamma_1 \sim (1/m_1)F, \cdots, \Gamma_n \sim (1/m_n)F$ be the "supports" of multiples of fibers of f. Then*

$$\omega_X \simeq \mathcal{O}_X((n-1)F - \Gamma_1 - \ldots - \Gamma_n).$$

Let $\omega_X = \mathcal{O}_X(K_X)$. Since $H^1(X, \mathcal{O}_X) = 0$, for any complete (integral) divisor D with $p_a(D) = 1$ we have $\dim|D + K| = 0$. Let D' denote the uniquely defined effective divisor of the linear system $|D+K|$. In particular, $F' = \lambda_1\Gamma_1 + \cdots + \lambda_n\Gamma_n$, where $\lambda_i < m_i$ (since $H^2(X, \mathcal{O}_X) = H^0(X, \omega_X) = 0$).

On the other hand, since $F' = (m_1 - 1)\Gamma_1 + \Gamma_1'$, we have $\Gamma_1 = (\lambda_1 - m_1 + 1)\Gamma_1 + \cdots + \lambda_n\Gamma_n$. From this we obtain $\lambda_1 - m_1 + 1 \geq 0$, which gives $m_1 - 1 \leq \lambda_1$, and since $\lambda_1 < m_1$, then $\lambda_1 = m_1 - 1$. The equalities $\lambda_i = m_i - 1$ are proved analogously.

Finally we obtain

$$K_X \sim F' - F \sim (m_1 - 1)\Gamma_1 + \ldots + (m_n - 1)\Gamma_n - F$$
$$\sim (n-1)F - \Gamma_1 - \ldots - \Gamma_n.$$

Corollary 4.9. *In the notation of the preceding lemma, if $\omega_X^{\otimes 2} \simeq \mathcal{O}_X$, then $n = 2$, $m_1 = m_2 = 2$ and $\omega_X \simeq \mathcal{O}_X(\Gamma_1 - \Gamma_2)$.*

We have

$$\mathcal{O}_X \simeq \omega_X^{\otimes 2} \simeq \mathcal{O}_X((2n-2)F - 2\Gamma_1 - \ldots - 2\Gamma_n)$$
$$= \mathcal{O}_X((n-2)F + (m_1 - 2)\Gamma_1 + \ldots + (m_n - 2)\Gamma_n).$$

Since all the $m_i \geq 2$, we have $n = 2$ and $m_i = 2$. Moreover,

$$\omega_X \simeq \mathcal{O}_X(F - \Gamma_1 - \Gamma_2) \simeq \mathcal{O}_X(\Gamma_1 - \Gamma_2).$$

Theorem 4.10. *Let X be an elliptic Enriques surface. Then there exists a finite unramified double cover $f\colon X' \to X$, where X' is a special* K3 *surface.*

Proof. Let f be the cover constructed in Lemma 4.1. We shall show that X' is a special K3 surface. Let J be the Jacobian surface of F ([1], Chapter VI). It is easy to see that J is a rational surface (cf. [5]). Hence the Šafarevič group $Ш(J_\eta)$ of the generic fiber of J is trivial. Using the theory of Ogg and Šafarevič, we obtain that the group of principal homogeneous spaces $WC(J_\eta)$ is the direct sum of the groups of "local invariants" $WC(J_\eta) = \bigoplus_{x \in \mathbf{P}_k^1(k)} H^1(J_x, \mathbf{Q}/\mathbf{Z})$.

By Corollary 4.9 the surface X defines an element X_η in $WC(J_\eta)$ of the form (α, β), where $2\alpha = 2\beta = 0$. Hence the order of X_η in this group equals two. Since the index of X_η is equal to its order [13], X_η has a rational point over the quadratic extension $k(\eta)$. The composition $X' \to X \to \mathbf{P}_k^1$ defines a structure of elliptic surface on X'. Since the index of X'_η obviously does not exceed the index of X_η, we then conclude that X' is an elliptic K3 surface of index ≤ 2. It remains to apply Theorem 4.5.

Remarks. 1) In case $\mathrm{char}(k) = 0$ one can show that every Enriques surface is an elliptic surface (see [1], Chapter X). This apparently is also true in general.

2) One can show that for an elliptic Enriques surface X the torsion subgroup $\mathrm{Tors}(\mathrm{Pic}(F)) \simeq \mathbf{Z}/2\mathbf{Z}$. Moreover, since $H^1(X, \mathcal{O}_X) = 0$, it follows from Kummer and Artin-Schreier theory that the surface X' constructed in Theorem 4.4 is the unique finite abelian unramified cover of X. In characteristic zero this fact follows trivially from the simply-connectedness of K3 surfaces.

3) An elliptic Enriques surface X is said to be *special* (cf. [1], Chapter X) if

there exists a quasi-section of degree 2 which is a rational curve. In this case the double plane corresponding to the K3 surface X' can be constructed explicitly. It has the form $z^2 = F_3(x, y) \cdot F'_3(x, y)$, where $F_3(x, y) = 0$ and $F'_3(x, y) = 0$ are cubic curves. In the general case $\pi \leq 3$ for the surface X'.

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