WILD p-CYCLIC ACTIONS ON K3-SURFACES

I. DOLGACHEV * and J. KEUM**

0. Introduction

In this paper we study automorphisms g of order p of K3-surfaces defined over an algebraically closed field of characteristic $p > 0$. We divide all possible actions in the following cases according to the structure of the set of fixed points X^g: X^g is a finite set, X^g contains a one-dimensional part D which is a positive divisor of Kodaira dimension $\kappa(X, D) = 0, 1, 2$. In the latter case we prove that $X^g = D$ and D is connected. Accordingly we obtain the following results.

Theorem 1. *Suppose that the fixed locus X^g of g is finite. Then $|X^g| \leq 2$. If $X^g = \emptyset$, then $p = 2$ and $Y = X/(g)$ is a non-classical Enriques surface of μ_2-type and X is its K3-cover. If $|X^g| = 1$ then a minimal resolution $\tilde{Y}$ of Y is either a rational surface or a K3-surface. If $|X^g| = 2$, then $\tilde{Y}$ is a K3-surface. The cases when $\tilde{Y}$ is a K3-surface can occur only if $p \leq 5$.*

Suppose now that X^g is one-dimensional. The next results cover three different cases corresponding to possible values of the Kodaira dimension (or D-dimension) $\kappa(X, X^g)$ of the divisor X^g of fixed points.

Theorem 2. *Suppose that $\kappa(X, X^g) = 0$. Then X^g is a nodal cycle, i.e. the union of smooth rational curves whose intersection matrix is negative definite (of type A-D-E). Moreover, if we contract X^g to get a singular normal surface X', then the quotient $X'/(g)$ is a rational surface with one rational singular point or one elliptic singular point. If $p \geq 5$ then the former case does not occur.*

Theorem 3. *Suppose that $\kappa(X, X^g) = 1$. Then $p \leq 11$ and we have the following:*

(1) There exists a divisor D with support equal to X^g such that the linear system $|D|$ defines an elliptic or quasi-elliptic fibration $\phi : X \to \mathbb{P}^1$ (the second option can occur only if $p \leq 3$).

(2) The induced map g^* on the base curve $\mathbb{P}^1$ is of order p unless $p = 2$ and ϕ is an elliptic fibration.

(3) If $p > 3$ and ϕ is an elliptic fibration, the type of the fibre D in Kodaira's notation is II if $p = 11$, III or II^* if $p = 7$, and IV or III^* if $p = 5$.

(4) If ϕ is a quasi-elliptic fibration, the type of the fibre D is IV or II^* if $p = 3$, and of type II, III, II^*, I_{2k}^*, $0 \leq k \leq 8$ if $p = 2$.

Note that the assumption $p \neq 2$ in (2) is esential as we show by an example.

* Research supported in part by a NSF grant.

** Supported by Korean Ministry of Education(BSRI 1438) and KOSEF-GARC.

Theorem 4. *Suppose that $\kappa(X, X^g) = 2$. Then X^g is equal to the support of some nef divisor D with $D^2 > 0$. Assume that it is chosen to be minimal with this property. Let $d = \dim H^0(X, \mathcal{O}_X(D - X^g))$ and $N = \frac{1}{2}D^2 + 1$. Then $p(N - d - 1) \leq 2N - 2$. In particular, if X^g is irreducible and $p \neq 2$, then the pair $(p, N) = (3,2), (3,3), (3,4)$, or $(s, 2), s \geq 5$. Moreover, if $p \neq 2$, X^g is a singular irreducble rational curve.*

The last theorem is essentially a result from the earlier work of Oguiso [Og2] where it was assumed that X^g is irreducible. In this case explicit computations show that $s = 5$. Oguiso also obtains the defining equation of X in a suitable weighted projective space. We note here that Oguiso's method does not apply to other cases, as it uses the projective model of X determined by the linear system $|X^g|$ on which g acts linearly.

Note that if an automorphism g of a K3-surface over k with positive characteristic p has an order of a power of p, then g is symplectic, i.e. acts on the space of global 2-forms trivially. Over the field of complex numbers $\mathbb{C}$, Nikulin ([Ni1], §5) determined all finite abelian symplectic automorphism groups of K3-surfaces. In particular, he proved that if g is a symplectic automorphism of order $n \geq 2$ of a K3-surface over $\mathbb{C}$, then the fixed locus X^g is nonempty and finite and the pair $(n, |X^g|)$ is one of the following : $(2,8), (3,6), (4,4), (5,4), (6,2), (7,3)$ and $(8,2)$. This result, being deduced from the fact that the quotient surface $X/(g)$ becomes a K3-surface with only rational double points of type A_n, can be extended to the case of positive characteristic p as long as $p \geq 3$ and the order n of the automorphism is coprime to p.

Also for comparison, note that involutions g of prime order p on a K3-surface over a field of characteristic prime to p with non-isolated fixed points can be completely classified. We have $p = 2$ or 3. If $p = 2$ the classification can be found in [Ni2]. If $p = 3$, then $X/(g)$ is a nonsingular quadric surface and the map $\pi : X \to X/(g)$ is branched along a smooth curve of bidegree $(3,3)$. This easily follows from the Hurwitz formula for the cyclic cover π of nonsingular surfaces.

In section 1 we give some general results about wild cyclic actions, and, in particular, answer a question from [Pe1]. In section 2 we study the case of isolated fixed points where our main tool is Grothendieck's spectral sequence for cohomology of G-equivariant sheaves from [Gr]. Section 3 is devoted to the case of non-isolated fixed points where we establish some general results in this case, proving for example that the fixed locus must be connected. We also show that the quotient surface $Y = X/(g)$ is a normal anticanonical surface with rational singularities. The other sections 4-6 are devoted to more detailed study of three different cases corresponding to the possible Kodaira dimension of the divisor of fixed points. At the end of section 5 we give some applications of our results to the theory of elliptic fibrations on a K3-surface.

A part of this paper was written during the second author's stay at Nagoya University June of 1998 on JSPS fellowship. He thanks S. Kondō and the University for hospitality, and JSPS for financial support. He also thanks K. Oguiso for bringing to his attention the paper [Og1] which contains the key idea for the proof of Lemma 5.2. Discussions with S. Mukai and S. Mori are very much appreciated.

1. Preliminaries on wild cyclic actions

.

Let G be a group acting on a topological space X, and let $Y = X/G$ be the quotient space and $\pi : X \to Y$ the quotient map. Consider the category $\mathcal{S}(X, G)$ of abelian G-sheaves on X and the category $\mathcal{A}$ of abelian groups. The functor

$$\mathcal{S}(X, G) \to \mathcal{A}, \quad \mathcal{F} \mapsto \Gamma(X, \mathcal{F})^G$$

can be represented as a composition of functors in two different ways:

$$\mathcal{S}(X,G) \to \mathcal{S}(Y) \to \mathcal{A}, \quad \mathcal{F} \mapsto \pi_*^G\mathcal{F} \mapsto \Gamma(Y, \pi_*^G\mathcal{F}),$$

$$\mathcal{S}(X,G) \to \mathcal{A} \to \mathcal{A}, \quad \mathcal{F} \mapsto \Gamma(X,\mathcal{F}) \mapsto \Gamma(X,\mathcal{F})^G,$$

where $\pi_*^G\mathcal{F}$ is the sheaf $U \mapsto \Gamma(\pi^{-1}(U),\mathcal{F})^G$. This gives two spectral sequences for the compositions of functors (see [Gr], Theorem 5.2.1):

$$'E_2^{p,q}(\mathcal{F}) = H^p(Y, \mathcal{H}^q(G,\mathcal{F})) \Rightarrow H^n, \tag{1.1}$$

$$''E_2^{p,q}(\mathcal{F}) = H^p(G, H^q(X,\mathcal{F})) \Rightarrow H^n. \tag{1.2}$$

Here the sheaves $\mathcal{H}^q(G,\mathcal{F})$ can be defined on open subsets of Y by

$$\mathcal{H}^q(G,\mathcal{F})(U) = H^q(G, \mathcal{F}|\pi^{-1}(U)). \tag{1.3}$$

We will apply this to the situation when X is an irreducible algebraic variety over a field k, G is a finite group of its automorphisms and $\mathcal{F}$ is the structure sheaf $\mathcal{O}_X$.

Proposition 1.1. *Let G be a finite group acting freely on an irreducible algebraic variety X over a field k. Then*

$$\mathcal{H}^i(G,\mathcal{O}_X) = 0, i > 0.$$

Proof. Let $\pi : X \to Y$ be the quotient map. It is a finite and hence affine morphism. So, without loss of generality we may assume that $X = \mathrm{Spec}(A)$ is an k-algebra and $Y = \mathrm{Spec}(B)$, where $B = A^G$ is the subalgebra of invariant elements. We need to show that $H^i(G,A) = 0$ for $i > 0$. Since G acts freely, X is a principal fibre bundle over Y in the sense that A is flat over A^G and the canonical morphism $G \times X \to X \times_Y X$ defined by the action is an isomorphism (see [Mu], Proposition 0.9). Consider the group algebra $B[G]$ of G over B as a B-module. By the above we have an isomorphism of A-modules

$$B[G] \otimes_B A \cong (k[G] \otimes_k B) \otimes_B A \cong k[G] \otimes_k A \cong A \otimes_B A.$$

Since A is flat over B, this implies that A is isomorphic to $B[G]$ as a B-module. As a G-module, A is induced from the trivial $\{1\}$-module B. By Shapiro's Lemma, we get

$$H^i(G,A) \cong H^i(\{1\},B) = 0, \quad i > 0.$$

This proves the assertion.

Corollary 1.2. *Let G be a finite group acting on an irreducible variety X. Then the sheaves $\mathcal{H}^q(G,\mathcal{O}_X), q > 0$, are equal to zero over the quotient of the open subset of X where G acts freely.*

Now let $G = (g)$ be a cyclic group of order n. Recall that the cohomology of a cyclic group $G = (g)$ of order n with coefficient in a G-module M are computed as follows:

$$H^{2i+1}(G,M) = \mathrm{Ker}T/\mathrm{Im}(g-1), \quad H^{2i}(G,M) = \mathrm{Ker}(g-1)/\mathrm{Im}T, i > 0, \quad H^0(G,M) = \mathrm{Ker}(g-1),$$

where $T = 1 + g + \ldots + g^{n-1}$ and $g - 1$ are elements of the group algebra $\mathbb{Z}[G]$ acting naturally on M. Globalizing we get the homomorphisms of $\mathcal{O}_Y$-modules

$$T : \pi_*(\mathcal{O}_X) \to \mathcal{O}_Y, \quad g - 1 : \pi_*(\mathcal{O}_X) \to \pi_*(\mathcal{O}_X)$$

and the isomorphisms of $\mathcal{O}_Y$-modules

$$\mathcal{H}^{2i+1}(G,\mathcal{O}_X) = \mathrm{Ker}T/\mathrm{Im}(g-1), \quad \mathcal{H}^{2i}(G,\mathcal{O}_X) = \mathrm{Ker}(g-1)/\mathrm{Im}T, \quad i > 0. \tag{1.4}$$

Of course, $\mathcal{H}^0(G,\mathcal{O}_X) = \mathcal{O}_Y$.

Proposition 1.3. *Let $G = (g)$ be a cyclic group acting on a Cohen-Macaulay algebraic variety X. Assume that the quotient $Y = X/G$ is also Cohen-Macaulay (e.g. $\dim X = 2$). Then we have an exact sequence of sheaves on Y:*

$$0 \to \omega_Y \to (\pi_*\omega_X)^G \to \mathcal{E}xt^1_{\mathcal{O}_Y}(\mathcal{H}^2(G, \mathcal{O}_X), \omega_Y) \to 0$$

where ω_X and ω_Y are the canonical dualizing sheaves of X and Y, respectively.

Proof. Let $\phi : \omega_Y \to (\pi_*\omega_X)^G$ be the natural homomorphism induced by the inverse image of a differential. This map is injective since the map $\pi : X \to Y$ is separable. By Grothendieck's duality theory (see [Ha], V.2.4),

$$\pi_*(\omega_X) = \mathcal{H}om_{\mathcal{O}_Y}(\pi_*(\mathcal{O}_X), \omega_Y) \tag{1.5}$$

and the image of ϕ is the subsheaf of G-invariant $\mathcal{O}_Y$-module homomorphisms which are composed of the trace map $T : \pi_*(\mathcal{O}_X) \to \mathcal{O}_Y$ and a map $\mathcal{O}_Y \to \omega_Y$ (see [Pe3]). Consider the natural exact sequence

$$0 \to \mathrm{Ker}(T)/\mathrm{Im}(g-1) \to \pi_*(\mathcal{O}_X)/\mathrm{Im}(g-1) \to \mathrm{Im}(T) \to 0. \tag{1.6}$$

Clearly, the dual sheaf of $\pi_*(\mathcal{O}_X)/\mathrm{Im}(g-1)$ is equal to the subsheaf of G-invariant $\mathcal{O}_Y$-module homomorphisms $\pi_*(\mathcal{O}_X) \to \omega_Y$. Since the sheaf $\mathcal{H}^1(G, \mathcal{O}_X) = \mathrm{Ker}(T)/\mathrm{Im}(g-1)$ is a torsion $\mathcal{O}_Y$-module, we get an isomorphism

$$\mathcal{H}om_{\mathcal{O}_Y}(\mathrm{Im}(T), \omega_Y) \cong \mathcal{H}om_{\mathcal{O}_Y}(\pi_*(\mathcal{O}_X)/\mathrm{Im}(g-1), \omega_Y).$$

Using the obvious exact sequence

$$0 \to \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_Y, \omega_Y) \to \mathcal{H}om_{\mathcal{O}_Y}(\mathrm{Im}(T), \omega_Y) \to \mathcal{E}xt^1_{\mathcal{O}_Y}(\mathcal{O}_Y/\mathrm{Im}(T), \omega_Y) \to 0$$

and the fact that the map ϕ is the dual of the trace homomorphism T ([Pe3], Proposition (2.4)), we get the exact sequence from the assertion of the proposition.

Corollary 1.4 (see [Pe3], Thm.2.7). *Assume additionally that g acts freely outside a closed subset of codimension ≥ 2 or n is invertible in k. Then*

$$\omega_Y \cong \pi_*(\omega_X)^G.$$

Proof. If n is invertible in k, the trace homomorphism is surjective so, $\mathcal{H}^2(G, \mathcal{O}_X) = 0$. If g acts freely outside a closed subset of codimension ≥ 2 then $\mathcal{H}^2(G, \mathcal{O}_X) \cong \mathcal{O}_Z$ for some closed subscheme Z of codimension ≥ 2. Therefore, by the property of the dualizing sheaf, we have $\mathcal{E}xt^1(\mathcal{O}_Z, \omega_Y) = 0$.

The next proposition answers a question from [Pe1]:

Proposition 1.5.

$$\chi(Y, \mathcal{H}^1(G, \mathcal{O}_X)) = \chi(Y, \mathcal{H}^2(G, \mathcal{O}_X)).$$

Proof. We shall consider the operators T and $g-1$ as endomorphisms of the $\mathcal{O}_Y$-Module $\pi_*(\mathcal{O}_X)$. Set

$$\mathcal{K}_i = \mathrm{Ker}(g-1)^{p-i}/\mathrm{Ker}(g-1)^{p-i} \cap \mathrm{Im}(g-1), \quad \mathcal{L}_i = \mathcal{O}_Y/\mathrm{Im}(g-1)^{p-i} \cap \mathcal{O}_Y.$$

Obviously,

$$\mathcal{K}_1 = \mathcal{H}^1(G, \mathcal{O}_X), \quad \mathcal{L}_1 = \mathcal{H}^2(G, \mathcal{O}_X),$$

$$\mathcal{K}_{p-1} = \mathcal{L}_{p-1} = \mathcal{O}_Y/\mathrm{Im}(g-1) \cap \mathcal{O}_Y.$$

We claim the existence of the exact sequence

$$0 \to \mathcal{K}_{i+1} \to \mathcal{K}_i \to \mathcal{L}_i \to \mathcal{L}_{i+1} \to 0, \tag{1.7}$$

where the map $\phi_i : \mathcal{K}_i \to \mathcal{L}_i$ is defined by applying the operator $(g-1)^{p-i-1}$ and the other maps are the natural inclusion $\mathcal{K}_{i+1} \to \mathcal{K}_i$ and the surjection $\mathcal{L}_i \to \mathcal{L}_{i+1}$. To check the exactedness, we may assume that $X = \mathrm{Spec}(A), Y = \mathrm{Spec}(A^G)$ are affine. Let $K_i = \mathcal{K}_i(Y), L_i = \mathcal{L}_i(Y)$. If $a \in \mathrm{Ker}(g-1)^{p-i}$ is a representative of an element from K_i, then $(g-1)((g-1)^{p-i-1}a) = (g-1)^{p-i}(a) = 0$. Therefore $(g-1)^{p-i-1}(a) \in A^G = B$. Also, if $a \in (g-1)(A) \cap \mathrm{Ker}(g-1)^{p-i}$, then $(g-1)^{p-i-1}(a) \in (g-1)^{p-i}(A)$, so the map is well-defined.

Let us find its kernel. In the above notation, suppose $(g-1)^{p-i-1}(a) = (g-1)^{p-i}(x)$ for some $x \in A$. Then $(g-1)^{p-i-1}(a-(g-1)x) = 0$. Replacing a with $a-(g-1)(x)$ we may assume that $a \in \mathrm{Ker}(g-1)^{p-i-1}$ and hence defines an element of K_{i+1}. This gives us a homomorphism $\mathrm{Ker}(\phi_i) \to K_{i+1}$. It is clearly bijective. Since $(g-1)^{p-i-1}(\mathrm{Ker}(g-1)^{p-i}) = (g-1)^{p-i-1}(A) \cap B$, the cokernel of ϕ_i is equal to L_{i+1}.

Now the assertion follows easily from the exact sequence (1.7). We have

$$\chi(Y, \mathcal{K}_i) - \chi(Y, \mathcal{L}_i) = \chi(Y, \mathcal{K}_{i+1}) - \chi(Y, \mathcal{L}_{i+1}).$$

Starting from $i = p-1$ and "going down", we get

$$\chi(Y, \mathcal{H}^1(G, \mathcal{O}_X)) - \chi(Y, \mathcal{H}^2(G, \mathcal{O}_X)) = \chi(Y, \mathcal{K}_1) - \chi(Y, \mathcal{L}_1) = \chi(Y, \mathcal{K}_2) - \chi(Y, \mathcal{L}_2) = 0.$$

Now let us remind some facts about the Artin-Schreier coverings of algebraic surfaces (see [Ta]).

Recall first a well-known fact from algebra: any cyclic extension of degree p of a field K of characteristic $p > 0$ is the decomposition field of an equation $x^p - x = a$ for some $a \in K$. We are trying to globalize this fact.

Let $\pi : X \to Y = X/G$ be as in Proposition 1.3 with $G = (g)$ of order $p = \mathrm{char}\, k$.

Proposition 1.6. *There is a canonical filtration of $\mathcal{O}_Y$-modules for $\pi_*(\mathcal{O}_X)$:*

$$\pi_*(\mathcal{O}_X) = \mathcal{F}_{p-1} \supset \ldots \mathcal{F}_1 \supset \mathcal{F}_0 = \mathcal{O}_Y,$$

whose quotients $\mathcal{L}_i = \mathcal{F}_i/\mathcal{F}_{i-1}, i = 1, \ldots, p-1$, are ideal sheaves in $\mathcal{O}_Y$ satisfying the inclusions:

$$\mathcal{L}_i \subset \mathcal{L}_1, \quad \mathcal{L}_1^i \subset \mathcal{L}_i.$$

If Y is nonsingular, all $\mathcal{F}_i$ and $\mathcal{L}_1$ are locally free.

Proof. Allmost all the assertions are proven in [Ta]. For completeness sake we briefly remind his proof. First we may assume that Y and X are affine, say $Y = \mathrm{Spec}(B), X = \mathrm{Spec}(A)$. The group algebra $k[G] \cong k[t]/(t^p - t)$ and the coaction of G on A is given by a homomorphism $\sigma : A \to k[G] \otimes A = A[t]/(t^p - t)$. For any $a \in A$ we can write

$$\sigma(a) = a_0 + a_1 t + \ldots + a_{p-1} t^{p-1}$$

and the axioms of the action imply that the homomorphisms $\delta_i : A \to A, \quad a \to a_i$, satisfy $i!\delta_i = \delta_1^i, \delta_0 = \mathrm{id}_A$. By definition of the action

$$g(a) = a + \delta_1(a) + \ldots + \delta_{p-1}(a).$$

We set

$$F_i = \{a \in A : \delta_{i+1}(a) = 0\}, \quad L_i = \delta_i(F_i).$$

Since, for any $a \in F_i$,

$$\sigma(\delta_i(a)) = \delta_i(a) + \delta_1(\delta_i(a)) + \ldots + \delta_{p-1}(\delta_i(a)) = \delta_i(a),$$

we see that L_i is an ideal in $B = A^G$. Moreover, we have an isomorphism of B-modules

$$\delta_i : F_i/F_{i-1} \to L_i.$$

After globalizing we get the filtration $(\mathcal{F}_i)$ and the corresponding quotient ideal sheaves $\mathcal{L}_i$. We refer to [Ta] for the proof of locally freeness of $\mathcal{F}_i$ and $\mathcal{L}_1$ in the case when Y is nonsingular. It remains to check the inclusions for the ideal sheaves. Again it is enough to consider the affine case. The first inclusion $L_i \subset L_1$ is obvious. By induction, it suffices to show that $L_1 L_{i-1} \subset L_i$. Let $x \in F_1, y \in F_{i-1}$, we have

$$g(xy) = g(x)g(y) = (x + \delta_1(x))(y + \delta_1(y) + \ldots + \delta_{i-1}(y)) = \delta_1(x)\delta_{i-1}(y) + z \in F_i$$

where $z \in F_{i-1}$. Clearly, $\delta_1(x)\delta_{i-1}(y) = \delta_i(xy) \in L_i$ proving the assertion.

Corollary 1.7. *Outside a finite set S of points in Y the sheaves $\mathcal{L}_i$ are locally free and*

$$\mathcal{L}_i = \mathcal{L}_1^{\otimes i}.$$

There exists an open affine covering $\{U_i\}$ of $Y \setminus S$ such that

$$\pi^{-1}(U_i) \cong \mathrm{Spec}(\mathcal{O}_Y(U_i)[t_i]/(t_i^p - a_i t_i - b_i))$$

for some $a_i, b_i \in \mathcal{O}_Y(U_i)$. The elements a_i define a section of $\mathcal{L}_1^{-p+1}$ which is a $(p-1)$-th power of a section s of $\mathcal{L}_1^{-1}$. The group G acts on X by the formula $t_i \to t_i + s_i$, where $s_i = s|U_i$.

Proof. The first assertion is obvious, just take the open subset of the smooth locus of Y where the primary components of the ideal sheaves $\mathcal{L}_i$ are all of codimension 1. For the remaining statements see [Ta], Lemma 1.2.

We set

$$\mathcal{L} = (\mathcal{L}_1)^{**} = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{L}_1, \mathcal{O}_Y), \mathcal{O}_Y).$$

This is a reflexive sheaf of rank 1 on Y, hence locally free on the smooth locus of Y. Let B be the positive Weil divisor corresponding to $\mathcal{L}^{-1} = \mathcal{L}^*$ so that

$$\mathcal{L} = \mathcal{O}_Y(-B), \quad \mathcal{L}^{-1} = \mathcal{O}_Y(B).$$

We shall call it the *branch divisor* of the cover $\pi : X \to Y$.

Corollary 1.8. *Ouside a finite set of points on Y, we have*

$$\mathcal{H}^2(G,\mathcal{O}_X)=\mathcal{O}_Y/\mathrm{Im}(T)\cong\mathcal{O}_{(p-1)B}.$$

Proof. It follows from the definition that $\mathcal{L}_{p-1}=\mathrm{Im}T$. Take the open subset Y' where $\mathcal{L}_1=\mathcal{L}$, then $\mathcal{L}_{p-1}|Y'=\mathcal{O}_{Y'}(-(p-1)B')=\mathrm{Im}(T)|Y'$, where $B'=B|Y'$. This proves the assertion.

Corollary 1.9. *We have the following formula for the canonical sheaf ω_X of X:*

$$\omega_X=\pi^*(\omega_Y\otimes\mathcal{L}^{-p+1}).$$

Proof. Since ω_Y and $\mathcal{L}$ are both reflexive sheaves corresponding to some Weil divisors, it is enough to verify this formula over the complement of a finite set of points in Y. By Corollary 1.7, X is a positive divisor in $Y\times\mathbb{A}^1$ with associated invertible sheaf isomorphic to $p_1^*(\mathcal{L}^{1-p})$, where $p_1:Y\times\mathbb{A}^1\to Y$ is the first projection. Since $\omega_{Y\times\mathbb{A}^1}=p_1^*(\omega_Y)$ and $\pi=p_1|X$ the formula is just the adjunction formula.

Remark 1.10. A simple construction of separable p-covers $f:X\to Y$ which can be locally given as in Corollary 1.7 over the whole Y (Artin-Schreier covers of simple type in terminology of [Ta]) is given as follows. Let $\mathcal{L}$ be an invertible sheaf on Y with a non-zero section a of $\mathcal{L}^{1-p}$ and a section b of $\mathcal{L}^{-p}$. We have the following exact sequence of sheave in flat topology of Y:

$$0\to\alpha_a\to\mathcal{L}^{-1}\to\mathcal{L}^{-p}\to 0,$$

where the map $\mathcal{L}^{-1}\to\mathcal{L}^{-p}$ is given locally by $x\to x^p-ax$ and α_a is its kernel. Taking flat cohomology we get an exact sequence

$$H^0(X,\mathcal{L}^{-p})\to H^1_{fl}(X,\alpha_a)\to H^1(X,\mathcal{L}^{-1}).$$

The image of the section b defines an α_a-torsor which is separable p-cover ramified outside the divisor of zeroes of a. If $H^1(X,\mathcal{L}^{-1})=0$ every Artin-Schreier cover of simple type can be obtained in this way. The sheaf $\mathcal{L}$ is the sheaf $\mathcal{L}_1$ from Corollary 1.7.

2. The case of isolated fixed points.

In this section only we allow X to be an "orbitfold K3-surface". Recall that this means that $\omega_X\cong\mathcal{O}_X, H^1(X,\mathcal{O}_X)=0$ and $\sigma^*(\omega_X)=\omega_{\tilde{X}}$, where $\sigma:\tilde{X}\to X$ is the minimal resolution. Let $G=(g)$ be a group of order $p=\mathrm{char}\,k$ acting non-trivially on X. As before Y denotes the quotient X/G and $\tilde{Y}$ its minimal resolution.

Lemma 2.1. *Assume that G acts with finitely many isolated fixed points. Then*

$$\omega_Y\cong\mathcal{O}_Y.$$

Proof. Since we are in characteristic $p>0$ the isomorphism $\omega_X\cong\mathcal{O}_X$ is in fact an isomorphism of G-sheaves. By Corollary 1.4, we have

$$\omega_Y\cong(\pi_*\mathcal{O}_X)^G\cong\mathcal{O}_Y.$$

Applying the duality theorem on Y we get

Corollary 2.2.

$$\dim_k H^2(Y, \mathcal{O}_Y) = 1.$$

Lemma 2.3. *The target cohomology H^1 and H^2 in the spectral sequences (1.1) and (1.2) are computed as follows:*

$$H^1 \cong k,$$

$$H^2 \cong k^s, s \in \{1, 2\}.$$

Proof. Recall that every spectral sequence gives the standard five term sequence

$$0 \to E_2^{1,0} \to H^1 \to E_2^{0,1} \to E_2^{2,0} \to H^2. \tag{2.1}$$

Let us apply it to our second spectral sequence (1.2). Since $E_2^{0,1} = H^0(G, H^1(X, \mathcal{O}_X)) = 0$, we get

$$H^1 \cong H^1(G, H^0(X, \mathcal{O}_X)) = H^1(G, k) = \mathrm{Hom}(G, k) = k.$$

Also, in our situation

$$E_2^{p,q} = H^p(G, H^q(X, \mathcal{O}_X)) = 0, \quad q \neq 0, 2.$$

By [CE], Theorem XV.5.11, we have an exact sequence

$$\ldots E_2^{n,0} \to H^n \to E_2^{n-2,2} \to E_2^{n+1,0} \to H^{n+1} \to E_2^{n+1-2,2} \to \ldots. \tag{2.2}$$

Thus the exact sequence (2.1) gives us the exact sequence

$$0 \to E_2^{2,0} \to H^2 \to E_2^{0,2} \to E_2^{3,0} = k.$$

Since G acts trivially on $H^0(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = k$, and $T(k) = 0$, we get

$$E_2^{2,0} = H^2(G, H^0(X, \mathcal{O}_X)) = k/T(k) = k, \quad E_2^{0,2} = H^0(G, H^2(X, \mathcal{O}_X)) = k.$$

This proves the assertion for H^2.

Now we are ready to prove Theorem 1. In fact we prove a more general result since we are not assuming that X is nonsingular.

Theorem 2.4. *Suppose that the fixed locus X^g of g is finite. Then $|X^g| \leq 2$ and we have the following possible cases*

(i) $X^g = \emptyset$: Then $p = 2$ and $\tilde{Y}$ is an Enriques surface and $\tilde{X}$ is its K3-cover.

(ii) $|X^g| = 1$: Y has one Gorenstein elliptic singularity and Y is a rational surface, or Y has one double rational point and $\tilde{Y}$ is either a K3-surface or an Enriques surface (the latter case does not happen if X is nonsingular).

(iii) $|X^g| = 2$: Then Y has two rational double points and $\tilde{Y}$ is a K3-surface. The cases when $\tilde{Y}$ is a K3-surface (resp. Enriques surface) can occur only if $p \leq 5$ (resp. $p = 2$).

Proof. We apply the first spectral sequence (1.1). By Proposition 1.1 for $i > 0$ the sheaf $\mathcal{H}^i(G, \mathcal{O}_X)$ is concentrated at a finite set of points, so that $E_2^{p,q} = H^p(Y, \mathcal{H}^q(G, \mathcal{O}_X)) = 0$ when $q > 0, p > 0$ or $p > 2, q = 0$. By [CE], Proposition XV.5.9, we have the following exact sequence

$$0 \to H^1(Y, \mathcal{O}_Y) \to H^1 \to H^0(Y, \mathcal{H}^1(G, \mathcal{O}_X)) \to H^2(Y, \mathcal{O}_Y) \to$$

$$\to H^2 \to H^0(Y, \mathcal{H}^2(G, \mathcal{O}_X)) \to H^2(Y, \mathcal{H}^1(G, \mathcal{O}_X)) = 0. \tag{2.3}$$

Note first that for $i = 1, 2$

$$H^0(Y, \mathcal{H}^i(G, \mathcal{O}_X)) \cong \oplus_{x \in X^g} H^i(G, \mathcal{O}_{X,x}).$$

As we explained in the proof of Proposition 1.5, for any isolated fixed point $x \in X$ the cohomology $H^1(G, \mathcal{O}_{X,x})$ is a non-trivial finite-dimensional vector space over k. We also know that $\dim H^2(Y, \mathcal{O}_Y) = 1$ and it follows from (2.3) and Lemma 2.3 that $\dim H^1(Y, \mathcal{O}_Y) \leq 1$. Let us consider different cases corresponding to all possible values of $\dim_k H^1(Y, \mathcal{O}_Y)$.

Case 1: $\dim_k H^1(Y, \mathcal{O}_Y) = 1$.

Using (2.3), we get $H^0(Y, \mathcal{H}^1(G, \mathcal{O}_X)) = 0$ or k so that G has either acts freely or has one fixed point.

Case 2: $\dim_k H^1(Y, \mathcal{O}_Y) = 0$.

We get $H^0(Y, \mathcal{H}^1(G, \mathcal{O}_X)) = k$ or k^2. In this case, X^g consists of one or two points.

Now let us investigate each of the two cases in more detail.

Case 1: If $H^0(Y, \mathcal{H}^1(G, \mathcal{O}_X)) = 0$, the map $\pi : X \to Y$ is étale, and hence, the lift of the action to $\tilde{X}$ is free and $\tilde{Y} = \tilde{X}/G$ is an Enriques surface. Since $K_{\tilde{Y}} = 0$, this can happen only when $p = 2$. (see [CD], Theorem 1.1.3). Since the K3-cover is separable $\tilde{Y}$ is a non-classical Enriques surface of μ_2-type and $p = 2$ (see loc. cit., p.77).

If $H^0(Y, \mathcal{H}^1(G, \mathcal{O}_X)) = k$, then the image $y = \pi(x)$ of the unique fixed point $x \in X^g$ is a rational singularity ([Pe1], Theorem 6). Let $V = Y \setminus \{y\}$ and $U = \pi^{-1}(V)$. Since $\pi^*(\omega_V) = \omega_U = \mathcal{O}_U$ and the homomorphism $\mathrm{Pic}(V) \to \mathrm{Pic}(U)$ is injective (its kernel is isomorphic to $H^1(G, k^*) = 0$), we see that $\omega_V = \mathcal{O}_V$. Since Y is normal, this implies that $\omega_Y = \mathcal{O}_Y$. So, y is a rational double point. This implies that the canonical class of $\tilde{Y}$ is trivial and $H^1(\tilde{Y}, \mathcal{O}_{\tilde{Y}}) = 0$. By the classification of algebraic surfaces, $\tilde{Y}$ must be a non-classical Enriques surface. This could happen only if $p = 2$. Let us show that this is impossible if X is nonsingular.

Let $\tilde{X} \to \tilde{Y}$ be the K3-cover of $\tilde{Y}$. It is a principal bundle with respect to a group scheme $\mathcal{A}$ of order 2 isomorphic to either $\mathbb{Z}/2$ or α_2 (see [CD], Chapter 1, §3). Restricting the cover over the complement of the exceptional locus $E \subset \tilde{Y}$, we get a principal $\mathcal{A}$-cover of $Y \setminus \{y\}$. Its pre-image under the map $p : X \setminus \{x\} \to Y \setminus \{y\}$ is a principal $\mathcal{A}$-cover of $X \setminus \{x\}$. Since x is a nonsingular point, we can apply the purity theorem ([CD], 0.1.10) to extend this cover to a principal $\mathcal{A}$-cover of X. Since $H^1(X, \mathcal{O}_X) = 0$ this cover must be trivial (use loc. cit, Proposition 0.1.7 and 0.1.9). This implies that $X \setminus \{x\} \cong \tilde{X} \setminus E'$, where E' is the pre-image of E in $\tilde{X}$. Now, if $\mathcal{A} = \mathbb{Z}/2$, $\tilde{X}$ is a nonsingular K3-surface and E' is the disjoint union of two curves. Resolving the points of indeterminacy of the rational map $\tilde{X} \to X$ by blowing up some points on E', we get a birational morphism $\tilde{X}' \to X$ which has disconnected pre-image over x. This contradicts Zariski's Connectedness Theorem. Next we assume that $\mathcal{A} = \alpha_2$. In this case $X \setminus E'$ is an inseparable cover of $\tilde{Y} \setminus E$. Hence the l-adic Euler characteristic with compact support e_c^l of $\tilde{X} \setminus E'$ is equal to

$$e_c^l(\tilde{Y} \setminus E) = e_c^l(\tilde{Y}) - e_c^l(E) = 12 - e_c^l(E) \leq 10.$$

On the other hand, $e_c^l(X \setminus \{x\}) = e_c^l(X) - 1 = 23$. This contradiction proves the claim.

Case 2: Arguing as in the previous case we have the following possible cases:

(a) $\tilde{Y}$ is a rational surface and Y has one non-rational Gorenstein singularity. An easy argument using the Leray spectral sequence for the minimal resolution $\tilde{Y} \to Y$ shows that this is an elliptic singularity.

(b) $\tilde{Y}$ is a K3-surface and Y has one rational double point with $H^0(Y, \mathcal{H}^1(G, \mathcal{O}_X)) = k^2$
(c) $\tilde{Y}$ is a K3-surface and Y has two rational double points with $H^0(Y, \mathcal{H}^1(G, \mathcal{O}_X)) = k^2$.
(d) $\tilde{Y}$ is a K3-surface and Y has one rational double point with $H^0(Y, \mathcal{H}^1(G, \mathcal{O}_X)) = k$.

Now, each of the rational double points on Y has a smooth covering of order p étale above a punctured neighborhood of the point. By Artin ([Ar2], Corollary 2.7), this is possible only if $p \leq 5$. This finishes the proof.

Remark 2.5. If $p \geq 3$, then the nonemptyness of X^g can be seen more easily; the algebraic Euler characteristic $\chi(X, \mathcal{O}_X)$ is equal to 2, which is not divisible by p.

Remark 2.6. In the case $p = 2$ Artin shows in [Ar1] that the completion of the local ring of the image $y \in Y$ of an isolated fixed point $x \in X$ is isomorphic to the ring $k[[x, y, z]]/(z^2 + abz + a^2y + b^2x)$, where $a, b \in k[[x, y]]$ are relatively prime nonunits. Also it follows that the image of the trace map $\mathrm{Im}(T)$ equals the ideal (a, b, z). Thus, by Proposition 1.5,

$$\dim H^1(G, \mathcal{O}_{X,x}) = \dim H^2(G, \mathcal{O}_{X,x}) = \dim \mathcal{O}_{Y,y}/(a, b, z).$$

This shows that in cases 2(c),(d) the completion of the local ring $\mathcal{O}_{Y,y}$ is isomorphic to the ring $k[[x, y, z]]/(z^2 + xyz + x^2y + y^2x)$. This is a rational double point of type $D_4^{(1)}$ from Artin's list in [Ar2]. In case 2(b), we get the ring $k[[x, y, z]]/(z^2 + x^2yz + x^4y + y^2x)$, or $k[[x, y, z]]/(z^2 + x^2yz + x^5 + y^3)$. This is a double rational point of type $D_8^{(2)}$ or $E_8^{(4)}$, respectively.

The case 2(a) does not occur in characteristic 2. However it may occur in characteristic 3 as the following example from [Pe1] shows. The group $G = \mathbb{Z}/3\mathbb{Z}$ acts on $k[[u, v]]$ by $\sigma(u) = u + y^i, \sigma(v) = v + u$. The ring of invariants is isomorphic to the ring $k[[x, y, z]]/(z^3 + y^{2i}z^2 - y^{3i+1} - x^2)$. The ideal $\mathrm{Im}(T)$ equals (x, y^i, z). When $i = 1$ we get a rational double point of type $E_6^{(1)}$. When $i = 2$ we get an elliptic singularity with $\dim H^2(G, \mathcal{O}_{X,x}) = 2$.

Finally note that a rational singularity of Y need not be a double point. This may occur already when $p = 3$ (see Example 13 from [Pe1]).

Remark 2.7. We remark that even if $|X^g| = 1$, or 2, the *fixed point scheme* X^g can not be isomorphic to $\mathrm{Spec}(k)$, locally. More generally, if a finite abelian p-group acts on a nonsingular variety of positive dimension defined over k of characteristic p by k-automorphisms, then its fixed point scheme can not contain any isolated closed point (see [ABK], Theorem 3.1). This can be seen by noting that at an isolated fixed point the linear action of the group on the tangent space of the fixed point is unipotent, and hence fixes a nonzero tangent vector as well.

Examples 2.8. 1. Let $X \subset (\mathbb{P}^1)^4$ be a complete intersection of two hypersurfaces $(F_1 = 0)$ and $(F_2 = 0)$ of multidegree $(1, 1, 1, 1)$. Let g be the involution on $(\mathbb{P}^1)^4$, $g(x, y, z, w) = (y, x, w, z)$. Suppose that $g^*F_i = F_i$, $i = 1, 2$. Then $|X^g| = 8$ if char $k = 0$ and $|X^g| = 2$ if char $k = 2$.

2. Let $X \subset \mathbb{P}^4$ be a complete intersection of a quadric and a cubic, both invariant under the action $g(x_0, x_1, x_2, x_3, x_4) = (x_0, x_2, x_1, x_4, x_3)$ of order 2. Suppose that the cubic has the term x_0^3. Then $|X^g| = 1$ if char $k = 2$.

3. Let X be the Fermat quartic $x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0$ in $\mathbb{P}^3$. Let g be the automorphism of order 3, $g(x_0, x_1, x_2, x_3) = (x_0, x_2, x_3, x_1)$. Then $|X^g| = 6$ if char $k = 0$ and $|X^g| = 1$ if char $k = 3$.

4. Let X be the hypersurface in $(\mathbb{P}^1)^3$ of multidegree $(2, 2, 2)$, invariant under the action of $g : (x, y, z) \to (y, z, x)$. Then $|X^g| = 6$ if char $k = 0$ and $|X^g| = 2$ if char $k = 3$.

5. Let $X \subset \mathbb{P}^5$ be a complete intersection of three quadrics, invariant under the action

$$g(x_0, x_1, x_2, x_3, x_4, x_5) = (x_0, x_2, x_3, x_4, x_5, x_1)$$

of order 5. Suppose that one of the three quadrics has the term x_0^2. Then $X^g = \{(0,1,1,1,1,1)\}$ if char $k = 5$.

6. Let $X \subset \mathbb{P}^6$ be a complete intersection of three quadrics and a hyperplane, invariant under the action of order 7, $g(x_0,x_1,x_2,x_3,x_4,x_5,x_6) = (x_1,x_2,x_3,x_4,x_5,x_6,x_0)$. Then $X^g = \{(1,1,1,1,1,1,1)\}$ if char $k = 7$.

3. The case of non-isolated fixed points.

From now on we assume that the set X^g of fixed points of the involution g contains a one-dimensional part. Denote it by F. Recall from Section 1 that we have defined the branch divisor B. Its support B_{red} is equal, outside a finite set of points, to the support of the sheaf $\mathcal{H}^2(G,\mathcal{O}_X)$.

Lemma 3.1.

$$B_{red} = \pi(F).$$

Proof. Let U be the open subset where B_{red} is equal to the support of $\mathcal{H}^2(G,\mathcal{O}_X)$. Over U, we have by Corollary 1.2, $B_{red} \subset \pi(X^g)$. Let $x \in X^g$, then g acts on the local ring $\mathcal{O}_{X,x}$ sending its maximal ideal $\mathfrak{m}$ to itself. Since T acts trivially on constants, the image of T on $\mathcal{O}_{X,x}$ is contained in $\mathfrak{m}\cap\mathcal{O}_{Y,\pi(x)}$. Thus $\pi(x)$ belongs to B_{red}. Now two one-dimensional closed subsets B_{red} and $\pi(F)$ are equal on U, hence are equal over the whole Y since $Y \setminus U$ is of codimension ≥ 2.

Proposition 3.2. *Let ω_Y be the dualizing sheaf of Y. Then*

$$\omega_Y = \mathcal{O}_Y(-(p-1)B).$$

Moreover

$$\mathcal{O}_{(p-1)B} \cong \mathcal{O}_Z,$$

where Z is the closed subscheme of Y defined by the ideal $\mathrm{Im}(T)$.

Proof. As we saw in the proof of Lemma 2.1, $(\pi_*\omega_X)^G = \mathcal{O}_Y$ as $G-\mathcal{O}_Y$-modules. Applying Proposition 1.3, we obtain the exact sequence

$$0 \to \omega_Y \to \mathcal{O}_Y \to \mathcal{E}xt^1_{\mathcal{O}_Y}(\mathcal{H}^2(G,\mathcal{O}_X),\omega_Y) \to 0.$$

Let Z be the closed subscheme of Y defined by the ideal sheaf $\mathrm{Im}(T)$. Then $\mathcal{H}^2(G,\mathcal{O}_X) = \mathcal{O}_Z$ and, by the duality $\mathcal{E}xt^1_{\mathcal{O}_Y}(\mathcal{H}^2(G,\mathcal{O}_X),\omega_Y) \cong \omega_Z$. Let W be the closed subscheme of Y defined by the ideal sheaf $\omega_Y \subset \mathcal{O}_Y$. Restricting to the smooth locus Y' of Y we find that $Z' = Z|Y'$ and $W' = W|Y'$ are Cartier divisors and hence

$$\mathcal{O}_{W'} \cong \omega_{Z'} = \omega_{Y'}(Z')/\omega_{Y'} \cong \omega_{Y'}(Z') \otimes \mathcal{O}_{Z'}.$$

Since $\omega_{Y'}(Z')$ is an invertible sheaf on Y', we see that the subschemes Z' and W' are identical. The corresponding Cartier divisor extends uniquely to a Weil divisor $Z = W$ on Y. Since $(p-1)B = Z$ on a set with complement of codimension ≥ 2 we see that $Z = (p-1)B$.

Corollary 3.3. *Let K_Y be the canonical divisor of Y. Then*

$$K_Y \sim -(p-1)B.$$

Proof. By Corollary 1.9, we have

$$\pi^*(\omega_Y \otimes \mathcal{O}_Y((p-1)B)) \cong \mathcal{O}_X.$$

Since $\pi^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ is injective, we get the desired formula.

Proposition 3.4. *We have*

$$H^1(Y, \mathcal{O}_Y) = 0.$$

Proof. Consider the exact sequence

$$0 \to \mathcal{O}_Y \to \pi_*(\mathcal{O}_X) \to \mathcal{E} \to 0. \tag{3.1}$$

Since $\pi : X \to Y$ is a finite morphism, we have $H^1(X, \mathcal{O}_X) = H^1(Y, \pi_*(\mathcal{O}_X)) = 0$, and hence we infer from (3.1) that $H^1(Y, \mathcal{O}_Y) = 0$ if $H^0(Y, \mathcal{E}) = 0$. Aplying Proposition 1.6 we find that $\mathcal{E}$ admits a filtration with quotients isomorphic to ideal sheaves $\mathcal{L}_i$ and $\mathcal{L}_i \subset \mathcal{L}_1$. So, it is enough to show that $H^0(Y, \mathcal{L}_1) = 0$. But this is easy. Over a complement to a finite set of points, $\mathcal{L}_1 = \mathcal{O}_Y(-B)$ for some positive divisor B. Obviously such a sheaf has no nonzero sections.

Proposition 3.5. *For any $i > 0$ we have*

$$\dim H^0(Y, \mathcal{H}^i(G, \mathcal{O}_X)) = \dim H^1(Y, \mathcal{H}^i(G, \mathcal{O}_X)) = 1.$$

Proof. Applying the exact sequence (2.1) and Lemma 2.3 (both do not use the assumption that G acts with finitely many fixed points) we find that $\dim H^0(Y, \mathcal{H}^1(G, \mathcal{O}_X)) = 1$. Since, for any $q \geq 0, p > 1$, $'E_2^{p,q} = H^p(Y, \mathcal{H}^q(G, \mathcal{O}_X)) = 0$, we can apply Proposition XV.5.5 from [CE] to find an exact sequence

$$0 \to E_\infty^{1,1} \to H^2 \to E_\infty^{0,2} \to 0.$$

It is easy to see that $E_\infty^{p,q} = E_2^{p,q}$ for $(p, q) = (1, 1), (0, 2)$. This gives us the exact sequence

$$0 \to H^1(Y, \mathcal{H}^1(G, \mathcal{O}_X)) \to H^2 \to H^0(Y, \mathcal{H}^2(G, \mathcal{O}_X)) \to 0. \tag{3.2}$$

Applying Proposition 3.2 and duality, we get

$$\chi(Y, \mathcal{H}^2(G, \mathcal{O}_X)) = \chi(\mathcal{O}_Z) = -\chi(\omega_Z) = \chi(\mathcal{O}_{(p-1)B}) =$$

$$\chi(Y, \mathcal{O}_Y) - \chi(Y, \mathcal{O}_Y(-(p-1)B)) = \chi(Y, \mathcal{O}_Y) - \chi(Y, \omega_Y) = 0.$$

Thus, applying Proposition 1.5, we get $\dim H^1(Y, \mathcal{H}^1(G, \mathcal{O}_X)) = 1$, and the exact sequence (3.2) together with Lemma 2.3 proves the assertion.

Corollary 3.6. *The fixed locus X^g is connected and its image in Y is equal to the support of the branch divisor $B \in |-\frac{1}{p-1}K_Y|$ with*

$$\dim H^i((p-1)B, \mathcal{O}_{(p-1)B}) = 1, i = 0, 1.$$

Let us summarize what we know about the quotient surface Y.

Theorem 3.7. *Assume that G acts on X with non-isolated fixed points. Then the quotient surface $Y = X/G$ is a normal surface with effective anticanonical divisor. It is rational and has at most rational singularities.*

Proof. Let $\sigma : \tilde{Y} \to Y$ be a resolution of singularities and $\tilde{X}$ be the normalization of $\tilde{Y}$ in the field of rational functions of X. Let $Z \to \tilde{X}$ be its resolution of singularities. Then the composition $f : Z \to \tilde{X} \to X$ is a resolution of singularities of a nonsingular surface. This easily implies that $R^1 f_* \mathcal{O}_Z = 0$ and this gives immediately that $H^1(Z, \mathcal{O}_Z) \cong H^1(X, \mathcal{O}_X) = H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$. Now the group G acts on $\tilde{X}$ with the quotient equal to $\tilde{Y}$. One can extend the proof of Proposition

3.4 to this situation to prove that $H^1(\tilde{Y}, \mathcal{O}_{\tilde{Y}}) = 0$. This means that singularities of Y are rational. Also it is known that

$$K_{\tilde{Y}} = \sigma^*(K_Y) - \Delta \tag{3.3}$$

for some positive divisor Δ supported on the exceptional locus of σ. Thus $-K_{\tilde{Y}}$ is effective, and $\tilde{Y}$ is of Kodaira dimension equal to $-\infty$. Since $H^1(\tilde{Y}, \mathcal{O}_{\tilde{Y}}) = 0$, $\tilde{Y}$ is a nonsingular rational surface.

Recall that a normal surface S is called *anticanonical* if $-K_S$ is effective. Thus our quotient surface Y is a normal rational anticanonical surface. It follows from the formula (3.3) that its minimal resolution is a nonsingular rational anticanonical surface. Now we can invoke the classification of such surfaces (see [Sa1, Sa2]). First, they are divided into the classes corresponding to different anti-Kodaira dimension $\kappa^{-1}(S)$, i.e. the D-dimension of Y, where $D = -K_S$. For any anticanonical surface, $\kappa^{-1}(S) \geq 0$, so we have three possible cases for our Y: $\kappa^{-1}(Y) = 0, 1, 2$.

Next, we use the Zariski decomposition

$$-K_S = P + N, \tag{3.4}$$

where P is a nef $\mathbb{Q}$-divisor, and N is an effective $\mathbb{Q}$-divisor such that either $N = 0$ or its irreducible components N_i define a negative definite intersection matrix and $P \cdot N_i = 0$ for each i. Here we use the standard intersection theory on a normal surface due to Mumford. We set

$$\nu(S, -K_S) = \begin{cases} 0 & \text{if } P \equiv 0 \\ 1 & \text{if } P^2 = 0, P \not\equiv 0 \\ 2 & \text{if } P^2 > 0 \end{cases},$$

$$d(S) = P^2.$$

We have

$$d(S) = \begin{cases} 0 & \text{if } \kappa^{-1}(S) = 0, 1 \\ > 0 & \text{if } \kappa^{-1}(S) = 2 \end{cases}.$$

By contracting irreducible curves C with $C^2 < 0, C \cdot K_S < 0$ we arrive at a minimal normal anticanonical surface S_0. Also by resolving rational double points (which do not affect the canonical class) we may assume that either S_0 is nonsingular or contains at least one singularity which is not a rational double point.

Remark 3.8. The occurence of singularities of the quotient surface depends on the linear part of the action of g. For example, if $p = 3$ and the linear part of g in a neighborhhod of a point $x \in F$ has one Jordan block then the image of x is a nonsingular point (see Corollary 5.15 in [Pe2]).

Theorem 3.9 ([Sa1]). *Let S be a nonsingular anticanonical rational surface. There exists a birational morphism $\phi : S \to S_0$ where S_0 is a nonsingular anticanonical surface with $\nu(S) = \nu(S_0, -K_{S_0})$ and Zariski decomposition $-K_{S_0} = P_0 + N_0$ where $n_0\phi^*(P_0) = nP$ for some positive integers n and n_0. We have four possibilities:*

(i) $\nu(S, -K_S) = \kappa^{-1}(S) = 0$:$P_0 = 0$;

(ii) $\nu(S, -K_S) = 1, \kappa^{-1}(S) = 0$: $P_0 = -C_0$ where C_0 is an indecomposable curve of canonical type (i.e. each its irreducible component R satisfies $R \cdot K_{S_0} = C_0 \cdot R = 0$ and C_0 is not a sum of two curves satisfying such property) with normal sheaf $\mathcal{N}_{C_0}$ of infinite order in $\mathrm{Pic}(C_0)$*;*

(iii) $\nu(S, -K_S) = \kappa^{-1}(S) = 1$: there exists a minimal elliptic (or quasi-elliptic) fibration on S_0 for which $-mK_{S_0}$ is linearly equivalent to a fibre for some $m > 0$.

(iv) $\nu(S, -K_S) = \kappa^{-1}(S) = 2$.

Note that in our situation the surface Y is often singular so there is no hope for a complete classification and even when singularities are quotient singularities this is sill hopeless (see [Ni3]). In the case when Y is Gorenstein (i.e. ω_Y is locally free), and we are in case (iv) , it is obtained from a Del Pezzo surface V by blowing down (-2)-curves. Notice $-K_V$ is divisible by $p-1 > 1$ in $\mathrm{Pic}(V)$ only when $p = 3$ and $V = Y$ is a quadric.

Let us return to our situation. We have

Lemma 3.10. *Let D be a nef divisor on a K3-surface. If $|D|$ has a fixed part, then there exists an irreducible curve E of arithmetic genus 1 such that $D \cdot E = 1$ and $D = aE + \Gamma$, where $a \geq 2$ and $|E|$ is a free pencil and Γ is a smooth rational curve with $E \cdot \Gamma = 1$. If $|D|$ has no fixed part and $D^2 = 0$, then $D = aE$ for some free pencil $|E|$. Moreover, if $D^2 > 0$ and $|D|$ has no fixed part, then $|D|$ has no base-points and the corresponding map $\phi_{|D|}$ is either of degree 2 or birational onto a surface with at most double rational points as singularities.*

Proof. This is a well-known result of B. Saint-Donat (see, for example, [Re], 3.8, 3.15).

Theorem 3.11.
$$\nu(Y, -K_Y) = \kappa^{-1}(Y) = \kappa(X, X^g).$$

Proof. We put $F = X^g$. Let B be the branch divisor of $\pi : X \to Y$ so that $aF = \pi^*(B)$ for some $a > 0$. From the projection formula we find

$$\mathcal{O}_Y(nB) \subset \pi_*\pi^*(\mathcal{O}_Y(nB)) = \mathcal{O}_Y(nB) \otimes \pi_*(\mathcal{O}_X).$$

Thus

$$\dim H^0(X, \mathcal{O}_X(naF)) = \dim H^0(Y, \pi_*\pi^*(\mathcal{O}_Y(nB))) = \dim H^0(Y, \mathcal{O}_Y(nB) \otimes \pi_*(\mathcal{O}_X)).$$

By Corollary 1.7, outside a finite set of points, $\pi_*(\mathcal{O}_X)$ admits a filtration with successive quotients equal to $\mathcal{L}^i, i = 0, \ldots, p-1$, for some invertible sheaf $\mathcal{L}$. Also we know that $\mathcal{L} \cong \mathcal{O}_Y(-B)$. This immediately implies that

$$\dim H^0(Y, \mathcal{O}_Y(nB)) \leq \dim H^0(X, \mathcal{O}_X(naF)) \leq p \dim H^0(Y, \mathcal{O}_Y(nB)).$$

In particular,
$$\kappa^{-1}(Y) = \kappa(Y, B) = \kappa(X, F).$$

Let $F = P + N$ be the Zariski decomposition of F. Let m be an integer such that $mP \in \mathrm{Pic}(X)$. Then $mF = mP + mN$ where mP is a nef divisor. Obviously, mN is the negative part of the Zariski decomposition of mF. Thus, mN is contained in the fixed part of $|mF|$. Assume $P^2 = 0$, i.e. $\nu(X, F) = 1$. By Lemma 3.10, mP has no fixed part and is composed of a pencil. Thus mP is the moving part of mF and $\kappa(X, F) = \kappa(X, mP) = 1$. Assume $P^2 > 0$. It follows from Lemma 3.10 that $|2mP|$ has no fixed part. So, $\nu(X, F) = \kappa(X, F) = \kappa(X, mP) = 2$. On the other hand, by Lemma 2.5 of [Sa1] $P = \pi^*(P'), N = \pi^*(N')$ where $B = P' + N'$ is the Zariski decomposition of B. This implies that $\nu(Y, B) = \nu(X, F)$ and finishes the proof.

So, we see that there are three different cases to consider according to three possible values of the Kodaira dimension of the divisor F of fixed points of G. This will be dealt with in the next sections.

4. $\kappa(X,F)=0$

In this case the Zariski decomposition of F consists only of the negative part N. Each component of F must be a smooth rational curve, and the sublattice of $\mathrm{Pic}(X)$ spanned by the classes of the components is a lattice of type A-D-E. It is known that we can blow down F to a double rational singular point x of an "orbitfold K3-surface " X'. The involution g acts on X' with one isolated fixed point x. Then we apply the results of section 2. By Theorem 2.4, we see that the quotient $Y' = X'/G$ has either one rational singular point or one elliptic singular point. A rational double point has a rational double point as a covering of order p which is étale above a punctured neighborhood of the point, only if $p \le 3$ (see [Ar2]). Actually, Artin shows that x must be of type $A_{8r-4n-1}$, $A_{8r-4n+1}$, A_2, $D_4^{(1)}$ if $p=2$, A_1, D_4 if $p=3$, and its corresponding rational double point in Y' is of type $D_{2n}^{(r)}(2r\ge n)$, $D_{2n+1}^{(r)}(2r\ge n)$, $E_6^{(1)}$, $E_7^{(3)}$ if $p=2$, $E_7^{(1)}$, $E_8^{(2)}$ if $p=3$, respectively.

This proves Theorem 2 from the introduction.

Example 4.1 Consider the K3-surface X over a field of characteristic 2 given by the equations

$$x_0^3+x_1^3+x_2^3+x_3^3+x_4^3=0, \quad x_0^2+x_1x_2+x_3x_4=0.$$

It has the involution defined by $(x_0,x_1,x_2,x_3,x_4)\to(x_0,x_2,x_1,x_4,x_3)$ and has the unique fixed point $(0,1,1,1,1)$ which happens to be an ordinary double point of X. The quotient surface Y is isomorphic to a surface of degree 3 in the weighted projective space $\mathbb{P}(1,1,1,2)$ given by the equation

$$x_0^3+x_1^3+x_2^3+x_1x_3+x_2(x_0^2+x_3)=0.$$

Recall that $\mathbb{P}(1,1,1,2)$ is isomorphic to the projective cone over the Veronese surface in $\mathbb{P}^5$. Our surface Y passes through the vertex of this cone. It is a rational surface with an elliptic singularity. Note that by adjunction formula $\omega_Y=\mathcal{O}_{\mathbb{P}}(-2)\otimes\mathcal{O}_Y$. There is no contradiction here with Lemma 2.1 since $\mathcal{O}_{\mathbb{P}}(-2)$ is not an invertible sheaf and $\mathcal{O}_{\mathbb{P}}(-2)\otimes\mathcal{O}_Y\neq\mathcal{O}_Y(-2)$.

5. $\kappa(X,F)=1$

Lemma 5.1. *There exists a divisor F' with the same support as F such that the linear system $|F'|$ defines an elliptic or quasi-elliptic fibration $\phi: X\to\mathbb{P}^1$.*

Proof. Let $F=P+N$ be the Zariski decomposition of F. As in the proof of Theorem 3.11 we write the Zariski decomposition of integral divisors $mF=P'+N'$, where P' is a nef divisor and N' is a sum of smooth rational curves whose intersection matrix is negative definite, and hence of type A-D-E. Since F is connected, all components of N' are contained in P'. In particular the support of P' is equal to F. It follows from Lemma 3.10 that $P'^2=0$ and $|P'|=|aE|$ for some pencil $|E|$ of curves of arithmetic genus 1. Since P' and F have the same support we are done.

Lemma 5.2. *The induced map g^* on the base curve $\mathbb{P}^1$ of the fibration $\phi: X\to\mathbb{P}^1$ is of order p unless $p=2$ and the fibration is elliptic.*

Proof. Assume first that $\phi: X\to\mathbb{P}^1$ is a quasi-elliptic fibration. In this case a general fibre is an irreducible curve of arithmetic genus 1 with a cusp. The closure of the cusps of irreducible fibres is a curve C, called the cuspidal curve. It is known that the restriction of ϕ to C is a purely inseparable cover of degree p. Thus $C\cdot E=p>1$. If fibres were preserved under g, the curve C would be contained in X^g. However by the previous lemma this is impossible.

Now suppose that $\phi: X\to\mathbb{P}^1$ is an elliptic fibration and that g^* acts as identity on the base curve $\mathbb{P}^1$. Then g becomes an automorphism of the elliptic curve $X/\mathbb{P}^1$ over the function field of

$\mathbb{P}^1$. The automorphism g induces an automorphism $\bar{g}$ of order p on the Jacobian of this elliptic curve. It fixes a specific fibre pointwisely. Let $j : J \to \mathbb{P}^1$ be the Jacobian surface. Note that J is a K3-surface (cf. [CD], Theorem 5.7.2). By Lemma 5.1, $\bar{g}$ can not fix a section of j pointwisely, and so is a translation by a p-torsion section P. We recall the height paring $< -, - >$ defined by Shioda on the Mordell-Weil group of an elliptic surface. Since P is a torsion section, $< P, P >= 0$. On the other hand, explicit formula for the height paring (Shioda [Sh], Theorem 8.6) tells that in our case

$$< P, P >= 4 + 2P \cdot O - \sum_v \mathrm{contr}_v(P),$$

where O is the zero-section of $j : J \to \mathbb{P}^1$, $P \cdot O$ the intersection number of the divisors P and O, the summation over all critical values of j, and $\mathrm{contr}_v(P)$ is a rational number determined by the incidence relation among the divisor P and irreducible components of the singular fibre $j^{-1}(v)$, e.g. it is equal to 0 if P meets the identity component of $j^{-1}(v)$. Assume for simplicity that $p \geq 5$. Since P is a torsion section of order $p \geq 5$, the number $\mathrm{contr}_v(P)$ is not zero only if the singular fibre $j^{-1}(v)$ is of type I_{ap}. Let $I_{a_1p}, \ldots, I_{a_rp}$ be all singular fibres of j of such type. Then by the formula for $\mathrm{contr}_v(P)$ ([Sh], p. 229 or [CZ]), we have

$$\sum_{v=1}^{r} (i_v(p - i_v)a_v/p) = 4 + 2P \cdot O, \tag{5.1}$$

where we assume that P meets the $i_v a_v$-th component of $j^{-1}(v)$ counted from the identity component. From the bounds for the second Chern number and Picard number of the surface J, we have

$$24 \geq \sum_{v=1}^{r} pa_v \quad \text{and} \quad 20 \geq \sum_{v=1}^{r} (pa_v - 1). \tag{5.2}$$

Under these constraints, some computation shows that the left hand side of (5.1) takes its maximum 4. On the other hand, we must have $P \cdot O \geq 1$, because the section P intersects the zero-section O at the fibre fixed pointwisely by $\bar{g}$. This is a contradiction to (5.1), and we have proved the assertion for $p \geq 5$. If $p = 3$, the number $\mathrm{contr}_v(P)$ is not zero only if the singular fibre $j^{-1}(v)$ is of type I_{ap}, IV or IV^*, and hence (5.1) and (5.2) can be written as follows:

$$2s/3 + 4t/3 + \sum_{v=1}^{r} (i_v(3 - i_v)a_v/3) = 4 + 2P \cdot O,$$

$$24 \geq 4s + 8t + 3\sum_{v=1}^{r} a_v \quad \text{and} \quad 20 \geq 2s + 6t + \sum_{v=1}^{r} (3a_v - 1),$$

where s(resp. t) is the number of fibres of j of type IV(resp. IV^*). A similar computation also leads to a contradiction in this case.

Next example shows that the assumption $p \neq 2$ is essential in the case of elliptic fibrations.

Example 5.3. Here $p = 2$. Consider a rational elliptic surface $f : V \to \mathbb{P}^1$ from the list of extremal rational surface in [La] which is given by the Weierstrass equation $y^2 + txy + ty = x^3$. It has three degenerate fibres of type A_5, A_1 and A_2^*. Let us apply the construction from Remark 1.10 by taking $\mathcal{L} = f^*(\mathcal{O}_{\mathbb{P}^1}(-1))$. Take its section a defined by the fibre V_0 of type A_2^* and take b to be the sum of two disjoint nonsingular fibres. Let $X' \to V$ be the corresponding cover. By the formula for the canonical class (Corollary 1.9) we get that $\omega_{X'} = 0$. It has an elliptic fibration and unramified outside V_0. The preimage of the other singular fibres are fibres of type A_5, A_5, A_1, A_1. They satisfy the condition (5.1). One can show by local computations using equations from Corollary 1.7 that X' has a rational double point of type $D_4^{(1)}$ over the singular point of V_0. After its resolution we get an elliptic K3-surface $\phi : X \to \mathbb{P}^1$ with 4 fibres of multiplicative type as above and one fibre of additive type E_6. The Mordell-Weil group of f is isomorphic to $\mathbb{Z}/6\mathbb{Z}$ (see for example [CD], Corollary 5.6.7). This shows that the 2-torsion part of the Mordell-Weil group of ϕ is is not trivial. The corresponding translation automorphism has fixed point set equal to the fibre of type E_6 and preserves the fibres of the elliptic fibration.

Lemma 5.4. *Assume $p \neq 2$. The genus 1 fibration $\phi : X \to \mathbb{P}^1$ has at least p degenerate fibres away from the fibre containing F.*

Proof. Since g acts on the base with one fixed point, we obtain that other singular fibres form the union of orbits of G each consisting of p fibres. So it is enough to show that ϕ contains more than one degenerate fibre. Assume first that ϕ is quasi-elliptic. Then the formula for the Euler characteristic of X from [CD], Proposition 5.1.6 gives that the unique degenerate fibre must have 21 irreducible components. The only type of fibre of additive type with so many components is of type D_{20} (Kodaira's I_{16}^*). However such a fibre may occur in a quasi-elliptic fibration only if $p = 2$(loc. cit., Corollary 5.2.4). Assume now that ϕ is an elliptic fibration with one degenerate fibre. Over the complement to one point of the base, ϕ defines an abelian group scheme $\mathcal{A}' \to \mathbb{A}^1$. For any prime $l \neq p$ its group of l-torsion sections defines an unramified extension of the affine line $\mathbb{A}^1$ of degree l^2. Since the algebraic fundamental group of the affine line in characteristic $p > 0$ is a profinite p-group, we obtain that all l-torsion points are defined over $k(\mathbb{P}^1)$. Thus ϕ has sections of any order $l \neq p$ which contradicts the Mordell-Weil Theorem.

Lemma 5.5. *Let $\tilde{Y}$ be a minimal resolution of the quotient surface $Y = X/ < g >$. Then $\tilde{Y}$ is a rational elliptic surface.*

Proof. We already know that Y is a rational surface (Theorem 3.7). Away from the branch divisor B, it has an elliptic fibration induced by the elliptic fibration $\phi : X \to \mathbb{P}^1$. Let $\sigma : \tilde{Y} \to Y$ be a minimal resolution of Y. Then it is a rational elliptic surface which however may not be (relatively) minimal. By blowing down the exceptional components of the fibre $\bar{E}$ arising from the fixed locus F of X, we obtain a minimal rational elliptic surface V. So, we are exactly in the situation described by Theorem 3.9, case (iii).

Corollary 5.6. *Under the assumptions of this section*

$$p \leq 11.$$

Proof. Assume $p > 11$. From Lemma 5.4 we infer that $\phi : X \to \mathbb{P}^1$ has mp singular fibres away from the fibre containing F. We use the well-known formula for the Euler characteristic of an elliptic surface ([CD], Proposition 5.1.6):

$$24 = e(X) = c_2(X) = \sum_{b \in \mathbb{P}^1} (e(X_b) + \delta(X_b)), \tag{5.3}$$

where $e(X_b)$ is the Euler (étale) characteristic of fibre X_b and $\delta(X_b)$ is a certain invariant of wild ramification. We have $\delta(X_b) \geq 0$ and $\delta(X_b) = 0$ unless $p = 2, 3$ and X_b is singular of additive type. Under our assumption, $\delta(X_b) = 0$. Let $f : V \to \mathbb{P}^1$ be the elliptic fibration of a minimal rational elliptic surface birationally equivalent to Y. Let X_{b_0} be the fibre of ϕ fixed pointwisely by g and

$V_0 = V_{b_0}$ be the corresponding fibre of f. The cover $\pi' : X \setminus X_{b_0} \to V \setminus V_0$ induced by $\pi : X \to Y$ is unramified and moreover splits over each fibre. We have the similar formula for V:

$$12 = e(V) = \sum_{b \in \mathbb{P}^1} e(V_b).$$

We have $e(X_b) = e(\pi(X_b))$ if $X_b \neq X_{b_0}$. Thus we get $p(\sum_{b \neq b_0} e(V_b)) \leq 24$ and $p > 11$ would imply that $\sum_{b \neq b_0} e(V_b) = 1$ and hence $e(V_0) = 11$. It follows from Kodaira's classification of fibres that V_0 contains 10 or 11 irreducible components, a contradiction to the fact that the Picard number of V is equal to 10.

Remark 5.7. In fact, when $p > 3$, one can give a complete classification of possible degenerate fibres of the elliptic fibration $\phi : X \to \mathbb{P}^1$. We give it in A-D-E notation, recalling the dictionary for the types of fibres:

$$I_0 = A_0, I_1 = A_0^*, I_n = A_{n-1} (n > 1), \quad II = A_0^{**}, \quad III = A_1^*,$$

$$IV = A_2^*, \quad I_n^* = D_{n+4}, \quad IV^* = E_6, \quad III^* = E_7, \quad II^* = E_8.$$

We have (in each case the type of the fibre X_{b_0} appears first):

$p = 11$:

$$A_0^{**} + 22A_0^*, \quad A_0^{**} + 11A_0^{**};$$

$p = 7$:

$$E_8 + 14A_0^*, \quad E_8 + 7A_0^{**}, \quad A_1^* + 7A_1^*, \quad A_1^* + 7A_0^* + 7A_0^{**}, \quad A_1^* + 7A_0^* + 7A_1, \quad A_1^* + 21A_0^*.$$

$p = 5$:

$$E_7 + 5A_1^*, \quad E_7 + 5A_0^* + 5A_0^{**}, \quad E_7 + 5A_0^* + 5A_1, \quad E_7 + 15A_0^*, \quad A_2^* + 5A_2^*, \quad A_2^* + 5A_2 + 5A_0^*,$$

$$A_2^* + 5A_1^* + 5A_0^*, \quad A_2^* + 10A_0^{**}, \quad A_2^* + 5A_0^{**} + 5A_1, \quad A_2^* + 5A_0^{**} + 10A_0^*, \quad A_2^* + 5A_1 + 10A_0^*, \ A_2^* + 20A_0^*.$$

When $p = 3$ and the fibration is elliptic the situation is complicated because of the presence of wild ramification. However, if the fibration is quasi-elliptic we can have only the folowing posibilities:

$$A_2^* + 9A_2^*, \quad A_2^* + 3E_6, \quad E_8 + 6A_2^*$$

if $p = 3$. In the case $p = 2$ there are many more possibilities. We only indicate the possible type of the fixed fibre: $II, III, II^*, I_{2k}^*, 0 \leq k \leq 8$.

To prove the irredundancy of this list (in the case of elliptic fibrations), we provide an example for each case.

Examples 5.8. In each example below the K3-surface X is given by a Weierstrass equation, the automorphism g is given by $g(x, y, t) = (x, y, t + 1)$, and the fixed locus X^g is the support of the fibre at $t = \infty$.

$p = 11$:

11-(1) $y^2 = x^3 + x^2 + t^{11} - t$
A_0^{**} at $t = \infty$, $11A_0^*$ at $t^{11} - t = 0$ and $11A_0^*$ at $t^{11} - t = -4/27$.

11-(2) $y^2 = x^3 + t^{11} - t$
A_0^{**} at $t = \infty$ and $11A_0^{**}$ at $t^{11} - t = 0$.

$p = 7:$

7-(1) $y^2 = x^3 + x^2 + t^7 - t$
E_8 at $t = \infty$, $7A_0^*$ at $t^7 - t = 0$ and $7A_0^*$ at $t^7 - t = -4/27$.

7-(2) $y^2 = x^3 + t^7 - t$
E_8 at $t = \infty$ and $7A_0^{**}$ at $t^7 - t = 0$.

7-(3) $y^2 = x^3 + (t^7 - t)x$
A_1^* at $t = \infty$ and $7A_1^*$ at $t^7 - t = 0$.

7-(4) $y^2 = x^3 + (t^7 - t)(x + 1)$
A_1^* at $t = \infty$, $7A_0^{**}$ at $t^7 - t = 0$ and $7A_0^*$ at $t^7 - t = -4/27$.

7-(5) $y^2 = x^3 + x^2 + (t^7 - t)x$
A_1^* at $t = \infty$, $7A_1$ at $t^7 - t = 0$ and $7A_0^*$ at $t^7 - t = 1/4$.

7-(6) $y^2 = x^3 + (t^7 - t)x + 1$
A_1^* at $t = \infty$ and $21A_0^*$ at $(t^7 - t)^3 = -27/4$

$p = 5:$

5-(1) $y^2 = x^3 + (t^5 - t)x$
E_7 at $t = \infty$ and $5A_1^*$ at $t^5 - t = 0$.

5-(2) $y^2 = x^3 + (t^5 - t)(x + 1)$
E_7 at $t = \infty$, $5A_0^{**}$ at $t^5 - t = 0$ and $5A_0^*$ at $t^5 - t = -27/4$.

5-(3) $y^2 = x^3 + x^2 + (t^5 - t)x$
E_7 at $t = \infty$, $5A_1$ at $t^5 - t = 0$ and $5A_0^*$ at $t^5 - t = 1/4$.

5-(4) $y^2 = x^3 + (t^5 - t)x + 1$
E_7 at $t = \infty$ and $15A_0^*$ at $(t^5 - t)^3 = -27/4$.

5-(5) $y^2 = x^3 + (t^5 - t)^2$
A_2^* at $t = \infty$ and $5A_2^*$ at $t^5 - t = 0$.

5-(6) $y^2 = x^3 + (x + t^5 - t)^2$
A_2^* at $t = \infty$, $5A_2$ at $t^5 - t = 0$ and $5A_0^*$ at $t^5 - t = 4/27$.

5-(7) $y^2 = x^3 + (t^5 - t)x + (t^5 - t)^2$
A_2^* at $t = \infty$, $5A_1^*$ at $t^5 - t = 0$ and $5A_0^*$ at $t^5 - t = -4/27$.

5-(8) $y^2 = x^3 + (t^5 - t)^2 - 1$
A_2^* at $t = \infty$ and $10A_0^{**}$ at $(t^5 - t)^2 - 1 = 0$.

5-(9) $y^2 = x^3 + 3x^2 + 3(t^5 - t)x + (t^5 - t)^2$
A_2^* at $t = \infty$, $5A_1$ at $t^5 - t = 0$ and $5A_0^{**}$ at $(t^5 - t) - 1 = 0$.

5-(10) $y^2 = x^3 + (t^5 - t)x + (t^5 - t)(1 - t^5 + t)$
A_2^* at $t = \infty$, $5A_0^{**}$ at $t^5 - t = 0$ and $10A_0^*$ at $(t^5 - t)^2 + 1 = 0$.

5-(11) $y^2 = x^3 + x^2 + (t^5 - t)^2$
A_2^* at $t = \infty$, $5A_1$ at $t^5 - t = 0$ and $10A_0^*$ at $(t^5 - t)^2 = -4/27$.

5-(12) $y^2 = x^3 + x^2 + (t^5 - t)^2 - 1$

A_2^* at $t = \infty$, $10A_0^*$ at $(t^5 - t)^2 - 1 = 0$ and $10A_0^*$ at $(t^5 - t)^2 - 1 = -4/27$.

Let us give one application of so far obtained results to elliptic fibrations with a section on a K3-surface. Assume $f : X \to \mathbb{P}^1$ is such a fibration with a torsion section s of order $p = \mathrm{char}\ k$. Then the translation automorphism g is of order p and we can apply our results to it. Let s_0 be the identity section. Let $\mathcal{N} \to \mathbb{P}^1$ be the Neron model of f, the group scheme obtained by deleting from X the singular locus of every fibre. Let $\mathcal{N}^0$ be its connected component, the largest group subscheme of $\mathcal{N}$ with connected fibres. Then the section s is a section of the Neron model $\mathcal{N} \to \mathbb{P}^1$. If it intersects $\mathcal{N}^0$ at a fibre of multiplicative type (A_0^* or $A_n, n > 0$), the fibre must be pointwisely fixed. If it intersects a fibre of multiplcative type outside $\mathcal{N}^0$, then this fibre must be of type A_{ap-1}, on which the automorphism g has no fixed points. If s intersects a fibre of additive type (other singular types) and $p \neq 2, 3$, then s is a section of $\mathcal{N}^0$ and the locus of singular points of the fibre belongs to the fixed locus F. (Furthermore, F contains central components if $p > 2$ and the fibre is of type D_n, E_7, E_8, or if $p > 3$ and the fibre is of type D_n, E_6, E_7, E_8.) If $p = 2$, the section s may intersect a fibre of type A_1^*, E_7 or D_n outside $\mathcal{N}_0$. In this case the set of fixed points of g on the fibre is again non-empty.

If $p = 3$, the section s may intersect a fibre of type A_2^*, E_6 outside $\mathcal{N}^0$. In this case the set of fixed points of g on this fibre is equal to an isolated point if the type is A_2^* and to an isolated point on the central component if the type is E_6. Since we know that F is connected and the number of isolated fixed points is at most 2, we obtain the following information:

(i) The fibration f has no fibre of type $A_0^*, A_n, n \neq ap - 1$, if $p > 2$. (More generally, if $p > 2$, a translation automorphism by a p-torsion can not fix a fibre pointwisely, as was proved in the proof of Lemma 5.2.) This is still true even if $p = 2$. (See Corollary 5.11 below.)

(ii) If f has a fibre of type A_0^{**}, A_1^*, A_2^*, then it has at most two such fibres, and, if $p > 3$, then its other possible fibres are of type A_{ap-1};

(iii) if f has a fibre of type D_n, E_6, E_7 or E_8, and $p > 3$, then it has at most one such fibre and all other singular fibres are of type A_{ap-1}.

Corollary 5.9. *Assume $p = \mathrm{char}\ k > 19$. Then an elliptic fibration on a K3-surface has no p-torsion sections.*

Proof. Use the formula (5.3). In case (ii) from above, we get n fibres of type $A_{a_i p-1}$ and at most two fibres of type A_0^{**}, A_1^*, A_2^*. This gives $p(\sum_{i=1}^n a_i) \leq 20$, and hence $p \leq 19$.

In case (iii) from above, we get $p(\sum_{i=1}^n a_i) \leq 24$ and we get $p \leq 23$. In fact, if $p = 23$ we have one fibre of type A_{22} which has two many irreducible components. So $p \leq 19$ again.

Remark 5.10. If $p = 2$ and an elliptic fibration has a section, we have an obvious involution of order 2 which is the induced by the inversion automorphism $x \to -x$ of the general fibre. The fixed locus of this involution contains the identity section and any irreducible fibre of additive type. On each multiplicative fibre of type A_{2n} it has an isolated fixed point. This looks like a contradiction to the connectedness of X^g. However, it is not. If our results were true that we sincerely hope so, it implies that in the case of presence of multiplicative fibres with odd number of irreducible components there is another one-dimensional part of the fixed locus which intersects the multiplicative fibres at its fixed point. This is the curve R which is an inseparable cover of degree 2 of the base which after a base change defining this cover becomes a section of order 2. Recall that the Weierstrass equation for an elliptic curve E over a field K of characteristic 2 looks as follows:

$$y^2 + xy = x^3 + ax + b \quad \text{if } j(E) \neq 0,$$

$$y^2 + cy = x^3 + ax + b \quad \text{if } j(E) = 0.$$

The inversion automorphism is given by $(x,y)\to(x,y+x)$ in the former case and by $(x,y+c)$ in the latter case. From this we see that the point $(0,\sqrt{b})$ is the nontrivial 2-torsion point of $E(\bar{K})$ if $j(E)\neq 0$ and the group of 2-torsion points is trivial if $j(E)=0$. In our case when $K=k(\mathbb{P}^1)$, we see that $E(\bar{K})_2=\{0\}$ only when all fibres are of additive type. Otherwise, either $E(K)_2\neq\{0\}$ or $E(L)_2\neq 0$ for some purely inseparable double cover L of K. So we derive the following:

Corollary 5.11. *Let $f:X\to\mathbb{P}^1$ be an elliptic fibration of a K3-surface over a field of characteristic $p=2$. Assume there exists a non-trivial section of order 2. Then all fibres of multiplicative type contain even number of irreducible components.*

6. $\kappa(X,F)=2$.

As in the previous case, using the Zariski decomposition, we get that $nF=P'+N'$, where P' is a nef divisor with $P'^2>0$ and all components of N' are contained in P'. This shows that F is equal to the support of a nef divisor D with $D^2>0$. Replacing D by $2D$ we get a linear system without fixed part. Also by lemma 3.10, $|2D|$ has no base-points and defines a morphism of degree ≤ 2 onto a surface X' in $\mathbb{P}(H^0(X,\mathcal{O}_X(2D))^*)=\mathbb{P}^N$ with at most rational double points.

Theorem 6.1. *Let D be the smallest nef divsior with support on F such that the linear system has no fixed part. Let $N=\frac{D^2}{2}+1$ and $d=\dim H^0(X,\mathcal{O}_X(D-F))$. Then we have the following inequality*

$$p(N-d-1)\leq 2N-2.$$

Proof. This is a generalization of Oguiso's argument from [Og2], where this result was proven under the assumption that F is irreducible (and hence $D=F$). We refer for the details of the proof to Oguiso's paper and only sketch the main idea.

Let $\phi_D:X\to\mathbb{P}(H^0(X,\mathcal{O}_X(D))^*)=\mathbb{P}^N$ be the map given by the linear system $|D|$. By Lemma 3.10, it is a morphism of degree at most 2. Thus the image X' of X under $\phi_{|D|}$ is a surface of degree $D^2=2N-2$ or $D^2/2=N-1$. Since D is fixed by g pointwisely, the linear system is G-invariant and the induced action of G on X' is linear. The exact sequence

$$0\to\mathcal{O}_X(D-F)\to\mathcal{O}_X(D)\to\mathcal{O}_F(D)\to 0$$

shows that the image F' of F spans a linear subspace

$$H=\mathbb{P}(H^0(X,\mathcal{O}_X(D-F))^{\perp})\subset\mathbb{P}(H^0(X,\mathcal{O}_X(D))^*).$$

It is easy to see that one can choose a projection from $f:\mathbb{P}^N\to H$ given by a linear sysbsystem $|V|$ where $V=H^0(X,\mathcal{O}_X(D))^g$ is the linear subspace of g-invariant elements from $H^0(X,\mathcal{O}_X(D))$. The image $\bar{X}'=f(X')$ is an irreducible non-degenerate surface in H, and hence has degree $\geq\dim H-1=N-d-1$. On the other hand, the composition $f\circ\phi_{|D|}:X\to H$ factors through the quotient surface $Y=X/G$ and hence the degree of the map $f:X'\to\bar{X}'$ is divisible by p. Thus

$$p(N-d-1)\leq p\deg(\bar{X}')\leq\deg X'\leq 2N-2.$$

This proves the assertion.

Under additional assumption that F is irreducible, we see that $d=1$, and, when $p\neq 2$, this easily gives $(p,N)=(3,2),(3,3),(3,4)$, or $(s,2), s\geq 5$, and $\phi_{|2D|}$ is birational. The case $N=2$ (a double plane) requires additional consideration, and Oguiso showed that $s=5$.

Finally let us reprove, by our means, another result of Oguiso.

Theorem 6.2. *Assume F is irreducible and $p > 2$. Then F is a rational curve of arithmetic genus > 1.*

Proof. Consider the branch curve $B \subset Y$. By Corollary 3.3, $K_Y \sim -(p-1)B$ and, by Corollary 3.6, $H^1(((p-1)B, \mathcal{O}_{(p-1)B}) = k$. We know that $F = \pi^*(B)_{red}$. Since F is irreducible, B is also irreducible. Let $B = a\bar{B}$ as Weil diviors, where $\bar{B}$ is reduced. The restriction of π to F defines a map $\bar{\pi} : F \to \bar{B}$ of degree 1 or purely inseparable of degree p. Consider the exact sequence

$$0 \to \mathcal{O}_Y(-\bar{B})/\mathcal{O}_Y(-(a(p-1)\bar{B})) \to \mathcal{O}_{a(p-1)\bar{B}} \to \mathcal{O}_{\bar{B}} \to 0. \qquad (6.1)$$

Since $\bar{B}$ is irreducible and reduced, we have $h^0(\mathcal{O}_{\bar{B}}) = 1$. Thus if we show that

$$H^1(Y, \mathcal{O}_Y(-\bar{B})/\mathcal{O}_Y(-(a(p-1)\bar{B})) \neq 0$$

we will be able to infer from (6.1) that $H^1(\bar{B}, \mathcal{O}_{\bar{B}}) = 0$. This would imply that $\bar{B}$ is a smooth rational curve. Since F is (Zariski) homeomeorphic to $\bar{B}$, we see that the étale Euler characteristics of F and $\bar{B}$ are equal. This immediately implies that the normalization of F is a smooth rational curve. The arithmetic genus of F is of course greater than 1 since $F^2 > 0$ under the assumption that $\kappa(X, F) = 2$.

It remains to show that $H^1(Y, \mathcal{O}_Y(-\bar{B})/\mathcal{O}(-(a(p-1)\bar{B}))) \neq 0$. We have the obvious exact sequence

$$H^1(Y, \mathcal{O}_Y(-\bar{B})) \to H^1(Y, \mathcal{O}_Y(-\bar{B})/\mathcal{O}_Y(-(a(p-1)\bar{B}))) \to$$
$$H^2(Y, \mathcal{O}_Y(-a(p-1)\bar{B})) \to H^2(Y, \mathcal{O}_Y(-\bar{B})) \to 0.$$

Since $\bar{B}$ is irreducible and $\nu(Y, \bar{B}) = 2$, we can apply Theorem (5.1) from [Sa3] to get

$$H^1(Y, \mathcal{O}_Y(-\bar{B})) = 0.$$

By duality

$$H^2(Y, \mathcal{O}_Y(-a(p-1)\bar{B})) = H^0(Y, \mathcal{O}_Y(K_Y + (a(p-1)\bar{B})) = H^0(Y, \mathcal{O}_Y) = k,$$

$$H^2(Y, \mathcal{O}_Y(-\bar{B})) = H^0(Y, \mathcal{O}_Y(K_Y + \bar{B})) = H^0(Y, \mathcal{O}_Y((1 - a(p-1))\bar{B})) = 0.$$

Here we use that $p > 1$ so that $1 - a(p-1) < 0$. This proves the assertion.

Remark 6.3. Assume $p = 2$ and let $f : X \to \mathbb{P}^1$ be an elliptic fibration with a section. As we have explained in Remark 5.10, the fixed locus F of the inversion involution has the following structure:

Case 1 (j is constant): F is the union of the zero section and the supports of singular fibres (all of them are of additive type).

Case 2 (j is not constant)

(a): F is the union of two 2-torsion sections and the supports of the fibres of additive type.

(b): F is the union of the identity section, an irreducible curve C which covers the base inseparably of degree 2, and the supports of the fibres of additive type.

References

[Ar1] M. Artin, Wildly ramified $\mathbb{Z}/2$ actions in dimension two, *Proc. A.M.S*, 52 (1975),60-64.

[Ar2] M. Artin, Coverings of the rational double points in characteristic p, in *Complex Analysis and Algebraic Geometry*, Cambridge Univ. Press, (1977), 11-22.

[ABK] A. Assadi, R. Barlow, F. Knop, Abelian group actions on algebraic varieties with one fixed point, *Math. Z.* **191** (1992), 129-136.

[CE] A. Cartan, S. Eilenberg, *Homological Algebra.* Princeton Univ. Press.1956.

[CD] F. Cossec, I. Dolgachev, *Enriques Surfaces I*, Birkhäuser, 1989.

[CZ] D. Cox, S. Zucker, Intersection numbers of sections of elliptic surfaces, *Invent. Math.*, **53**, (1979), 1-44.

[G] A. Grothendieck, Sur quelques points d'algébre homologique, *Tohoku Math. J.* **9** (1957), 119-221.

[Ha] R. Hartshorne, *Residues and Duality*, Lecture Notes in Math. 20, Springer-Verlag, 1966.

[La] W. Lang, Extremal rational elliptic surfaces in characteristic p. II.: surfaces with three or fewer singular fibres, *Arkiv för Matematik*, 32 (1994),423-448.

[Mu] D. Mumford, *Geometric Invariant Theory.* Springer-Verlag. 1965

[Ni1] V. Nikulin, Finite groups of automorphisms of Kählerian surfaces of type K3, *Trans. Mosc. Math. Soc.* **38** (1980), 71-135.

[Ni2] V. Nikulin, Quotient-groups of groups of automorphisms of hyperbolic forms by subgroups generated by 2-reflections. Algebro-geometric applications, in " Current problems in mathematics", Vol. 18, pp. 3–114, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii, Moscow,1981 [English translation: J. Soviet Mathematics, 22 (1983),1401-1476].

[Ni3] V. Nikulin, Del Pezzo surfaces with log-terminal singularities. Matematicheskii Sbornik, 180 (1989), 226–243.

[Og1] K. Oguiso, Application of height paring, (in Japanese), *Proc. Minisymposium on algebraic geometry, University of Tokyo*, (1989), 52-66.

[Og2] K. Oguiso, A note on $\mathbb{Z}/p\mathbb{Z}$-actions on K3 surfaces in odd characteristic p, *Math. Ann.* **286** (1990), 735-752.

[Pe1] B. Peskin, On rings of invariants with rational singularities, *Proc. Amer. Math. Soc.* **87** no.4 (1983), 621-626.

[Pe2] B. Peskin, Quotient singularities and wild p-cyclic actions, *J. of Algebra*, 81 (1983), 72-99.

[Pe3] B. Peskin, On the dualizing sheaf of a quotient scheme, *Communications in Algebra*, 12 (1984),1855-1869.

[Re] M. Reid, Chapters on algebraic surfaces, in "Complex Algebraic geometry", IAS/Park City Math. Series, vol. 3, 1998.

[Sa1] F. Sakai, Anticanonical models of rational surfaces, *Math. Ann.* 269 (1984), 389-410.

[Sa2] F. Sakai, Classification of normal surfaces, in "Algebraic Geometry. Bowdoin, Part 1, pp. 451-467. Providence. 1985.

[Sa3] F. Sakai, Weil divisors on normal surfaces, *Duke Math. J.* 51 (1984), 877-887.

[Sh] T. Shioda, On the Mordell-Weil lattices, *Commentarii Mathematici Univ. Sancti Pauli* 39 no.2(1990), 211-240.

[Ta] Y. Takeda, Artin-Shreier coverings of algebraic surfaces, *J. Math. Soc. Japan* 41 (1989), 415-435.

I. Dolgachev
Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA.
e-mail: idolga@math.lsa.umich.edu

J. Keum
Department of Mathematics, Konkuk University, Seoul 143-701, Korea.
e-mail: jhkeum@kkucc.konkuk.ac.kr