

## RESEARCH ARTICLE

# K3 surfaces of Kummer type in characteristic two

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Email: [idolga@umich.edu](mailto:idolga@umich.edu)**Abstract**

We discuss K3 surfaces in characteristic two that contain the Kummer configuration of smooth rational curves.

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We will go the other way.

Vladimir Illych Lenin

**INTRODUCTION**

The Kummer surface  $\text{Kum}(A)$  of an abelian surface  $A$  over an algebraically closed field of characteristic  $p$  is defined to be the quotient of  $A$  by the negation involution  $\iota : a \mapsto -a$ . If  $p \neq 2$ , the abelian surface  $A$  has  $2^4$  two-torsion points that give rise to 16 ordinary double points on  $\text{Kum}(A)$ . A minimal resolution of singularities  $X$  of  $\text{Kum}(A)$  is a K3 surface containing a set  $\mathcal{A}$  of 16 disjoint smooth rational curves on it ( $(-2)$ -curves for short because their self-intersection is equal to  $-2$ ). Conversely, if  $k = \mathbb{C}$ , the field of complex numbers, a theorem of Nikulin asserts that a K3 surface containing a set of 16 disjoint  $(-2)$ -curves arises in this way from the Kummer surface of some complex abelian surface.<sup>†</sup>

Let  $A$  be a simple principally polarized abelian surface, hence isomorphic to the Jacobian variety  $\text{Jac}(C)$  of a curve of genus 2. The embedding of  $C$  into  $\text{Jac}(C)$  can be chosen in such a way that its image  $\Theta$  is invariant under the involution  $\iota$ . The linear system  $|2\Theta|$  defines a regular map  $\phi : A \rightarrow |2\Theta|^* \cong \mathbb{P}^3$  that factors through  $\text{Kum}(A)$ . This map embeds  $\text{Kum}(A)$  into  $\mathbb{P}^3$  as a quartic surface with 16 ordinary double points. These surfaces have been studied for 200 years; we refer to

<sup>†</sup> This fact usually assumes that the ground field is the field of complex numbers, but, as shown to me by the anonymous referee, it is true if  $p \neq 2$ .

[8] for the history. The restriction of the map  $\phi$  to a translate of  $\Theta$  by a 2-torsion point is a degree two map to a conic in  $\text{Kum}(A)$  ramified over six points. The set  $\mathcal{B}$  of proper transforms of the 16 conics in  $X$  also consists of disjoint  $(-2)$ -curves. The incidence relation between the two sets defines an abstract symmetric configuration  $(16_6)$ , the *Kummer configuration*.

If  $A$  is a nonsimple abelian surface, that is,  $A$  is isomorphic to the product  $E_1 \times E_2$  of elliptic curves, the symmetric principal polarization  $\Theta$  can be chosen to be equal to  $(E_1 \times \{0\}) \cup (\{0\} \times E_2)$ . The map  $\phi : A \rightarrow \mathbb{P}^3$  defined by  $|\Theta|$  is of degree 4 onto a smooth quadric  $Q$ . The union of the images of the translates of  $\Theta$  is the union of eight lines  $L_i, M_i$  on  $Q$ , four from each of the two rulings. The double cover  $X'$  of  $Q$  branched along these eight lines has 16 ordinary points, and it is birationally isomorphic to a K3 surface  $X$ . The surface contains a set  $\mathcal{A}$  of 16 disjoint  $(-2)$ -curves equal to the exceptional curves  $E_{ij}, 1 \leq i, j \leq 4$ , of a minimal resolution of singularities of  $X'$ . Another set of 16 disjoint  $(-2)$ -curves consists of reducible  $(-2)$ -curves  $\bar{L}_i + E_{ij} + \bar{M}_j$ , where  $\bar{L}_i, \bar{M}_j$  are reduced preimages of the lines  $L_i, M_j$ . The two sets  $(\mathcal{A}, \mathcal{B})$  form the Kummer configuration  $(16_6)$ .

A beautiful aspect of the geometry of the Kummer surfaces of Jacobians of curves of genus 2 is their relationship with the classical geometry of quadratic line complexes [7, 10.3]. A Kummer surface appears as the *singular surface* of a quadratic line complex  $\mathfrak{C}$ , the locus of points  $x \in \mathbb{P}^3$  such that the plane  $\Omega(x)$  of lines containing  $x$  intersects  $\mathfrak{C}$  along a singular conic. The set of irreducible components of these conics (which are lines in  $\mathfrak{C}$ ) is isomorphic to the Jacobian variety of a curve  $C$  of genus 2. The curve  $C$  is isomorphic to the double cover of the pencil of quadrics containing  $\mathfrak{C}$  ramified over the set of six singular quadrics. The set of singular points of  $\Omega(x) \cap \mathfrak{C}, x \in \mathbb{P}^3$ , is an octic surface in the Plücker space  $\mathbb{P}^5$  birationally isomorphic to the Kummer surface. It is nonsingular if the characteristic is different from 2.

A less-known construction, due to Kummer himself, relates the Kummer surface  $\text{Kum}(\text{Jac}(C))$  to the theory of congruences of lines in  $\mathbb{P}^3$ , irreducible surfaces in the Grassmannian  $G_1(\mathbb{P}^3)$ . The Kummer surface appears as the focal surface of a smooth congruence of lines  $S$  of order 2 and class 2. The congruence  $S$  is a quartic del Pezzo surface anticanonically embedded in a hyperplane in the Plücker space  $\mathbb{P}^5$ . Its realization as a congruence of lines chooses a smooth anti-bicanonical curve  $B \in |-2K_S|$  that touches all 16 lines on  $S$ . The double cover  $X$  of  $S$  branched along  $B$  is a K3 surface birationally isomorphic to the Kummer surface  $\text{Kum}(\text{Jac}(C))$  for some genus 2 curve  $C$ . The 16 lines on  $S$  split into the union of two sets  $\mathcal{A}, \mathcal{B}$  of disjoint  $(-2)$ -curves, which form the Kummer configuration.

Let us see what is going wrong if we assume that  $p = 2$ . First of all, there are no normal quartic surfaces with 16 nodes [3]. An abelian surface  $A$  has four, two, or one 2-torsion points depending on its  $p$ -rank  $r$  equal to 2, 1, 0, respectively. If  $r = 2$  (resp.  $r = 1$ , resp.  $r = 0$ ), the singular points of  $\text{Kum}(A) := A/(t)$  are four rational double points of type  $D_4$  (resp. two rational double point of type  $D_8$ , resp. one elliptic double point) [19]. In the first two cases, the Kummer surface is birationally isomorphic to a K3 surface, in the third case, it is a rational surface. The linear system  $|\Theta|$  still defines a degree two map onto a quartic surface in  $\mathbb{P}^3$ . The equations of these surfaces can be found in [22] if  $r = 2$  and in [12] for arbitrary 2-rank.

The relationship with the quadratic line complexes is studied in a recent paper of T. Katsura and S. Kondō [20]. In characteristic 2, a pencil of quadrics in  $\mathbb{P}^5$  with smooth base locus  $Y$  has three (instead of six) singular quadrics. The variety of lines in  $Y$  is isomorphic to the Jacobian variety of a genus 2 curve with an Artin–Schreier cover of  $\mathbb{P}^1$  of the form  $y^2 + a_3(t_0, t_1)y + a_6(t_0, t_1) = 0$ , where the zeros of the binary cubic  $a_3$  correspond to singular quadrics in the pencil [2]. Identifying one of the smooth members of the pencil with the Grassmannian  $G_1(\mathbb{P}^3)$ , one can consider, as in

the case  $p \neq 2$ , the base locus of the pencil as a quadratic line complex  $\mathcal{C}$ . The singular surface of the quadratic line complex is a Kummer quartic surface and the surface in  $\mathbb{P}^5$  of singular points of the conics  $\Omega(x) \cap Y$  is a singular octic surface birationally isomorphic to the Kummer surface. The equations of the quartic and the octic surfaces are provided in [2].

The main drawback of this nice extension of the theory of Kummer surfaces to characteristic 2 is that the Kummer configuration and the relationship between six points in  $\mathbb{P}^1$  gets lost. In the present paper, we will present another approach whose goal is to reconstruct these relationships. Although we lose the relationship to curves of genus two, we will restore the relationships with the Kummer configuration  $(16_6)$ , sets of six points in  $\mathbb{P}^1$ , and the theory of congruences of lines in  $\mathbb{P}^3$ . The situation is very similar to what happens with del Pezzo surfaces of degree two (resp. one). The Geiser (resp. Bertini) involution defines a separable Artin–Schreier double cover whose branch curve is a smooth conic (resp. a rational quartic curve) instead of a plane quartic curve (resp. a standard genus 4 curve on a singular quadric). The connection to these curves is lost, but their attributes such as 28 bitangents (resp. 120 tritangent planes) survive (see [11]).

The paper should be considered as a lengthy footnote to [20]. I am thankful to the authors for a helpful discussion. I am also grateful to the anonymous referee for valuable comments and for detecting computational errors.

## 1 | K3 SURFACES OF KUMMER TYPE

Let  $k$  be an algebraically closed field of characteristic  $p \geq 0$ . We define a *K3 surface of Kummer type* to be a K3 surface  $X$  that contains two sets  $\mathcal{A}$  and  $\mathcal{B}$  of 16 disjoint  $(-2)$ -curves (or their degenerations), such that any  $A \in \mathcal{A}$  intersects  $n$  curves from  $\mathcal{B}$ , and vice versa, every curve  $B$  from  $\mathcal{B}$  intersects  $n$  curves  $A$  from  $\mathcal{A}$ . In other words, the two sets  $(\mathcal{A}, \mathcal{B})$  form a symmetric abstract configuration  $(16_n)$ . We call the number  $n$  the *index* of  $X$ .

A classical example of a K3 surface of Kummer type of index 6 is a minimal smooth model of the Kummer surface  $\text{Kum}(A)$  of a principally polarized abelian surface in characteristic  $p \neq 2$ .

As we discussed in the introduction, in characteristic 2, the Kummer surfaces are still defined but they are not of Kummer type. We also explained how the geometry of the Kummer surface of a principally polarized abelian surface  $A$  in characteristic  $p \neq 2$  is related to the geometry of the sets of six points in  $\mathbb{P}^1$ . Namely, the double cover of  $\mathbb{P}^1$  ramified over a set of six points is a smooth genus two curve  $C$ , and one can associate to  $C$  the Kummer surface  $\text{Kum}(\text{Jac}(C))$  of the Jacobian variety  $\text{Jac}(C)$ . When  $A$  is not a simple abelian surface but rather the product  $E_1 \times E_2$  of two elliptic curves, we replace six points on  $\mathbb{P}^1$  with six points on a stable rational curve  $C$  consisting of two irreducible components with three points on each component. The double cover of  $C$  of degree 2 ramified over six points (and the intersection point of the components) is isomorphic to the union of two elliptic curves  $E_1$  and  $E_2$  intersecting at one point. Its generalized Jacobian variety is isomorphic to  $E_1 \times E_2$ .

The following example of a K3 surface of Kummer type of index 10 is less known.

**Example 1.1.** A *Traynard surface* is a quartic surface in  $\mathbb{P}^3$  over an algebraically closed field  $k$  of characteristic  $p \neq 2$  with two sets of disjoint lines  $\mathcal{A}$  and  $\mathcal{B}$  that form a symmetric configuration  $(16_{10})$ . These surfaces were constructed by Traynard [28] (see [13], where the surfaces are named after Traynard). Not being aware of Traynard's work, W. Barth and I. Nieto rediscovered the Traynard surfaces in [1]. The surfaces are embedded Kummer surfaces of simple abelian surfaces  $A$

with polarization of type (1,3). The negation involution acts on the linear space  $H^0(A, \mathcal{O}_A(2\Theta))$ , where  $\Theta$  is a symmetric polarization divisor. The eigensubspace  $V$  with eigenvalue equal to  $-1$  is of dimension 4. The linear system  $|V| \subset |2\Theta|$  has base points at all 2-torsion points of  $A$  and defines a finite map of degree 2 of the blow-up of these points to  $\mathbb{P}^3$  with image a smooth quartic surface  $X$ . The images of the exceptional curves over the torsion points form a set  $\mathcal{A}$  of 16 lines on  $X$ . The unique symmetric theta divisor  $\Theta$  is a curve of genus 4, it passes through 10 torsion points, and the images of the translates of  $\Theta$  by 2-torsion points provide another set  $\mathcal{B}$  of 16 disjoint lines on  $X$ .

The next proposition is due to N. Shepherd–Barron [26, Corollary 13].

**Proposition 1.2.** *Let  $X$  be a K3 surface over a field of characteristic 2. Suppose that  $X$  contains  $\geq 13$  disjoint  $(-2)$ -curves. Then, it is unirational, and, in particular, a supersingular surface.*

The following is an example of a supersingular K3 surface of Kummer type of index 4 in characteristic 2 [6].

**Example 1.3.** Let  $X$  be a supersingular K3 surface with the Artin invariant  $\sigma_0$  equal to 1. The isomorphism class of  $X$  is unique. The surface contains a quasi-elliptic pencil with five reducible fibers of type  $\tilde{D}_4$  and 16 disjoint sections. The union of nonmultiple irreducible components of four reducible fibers is a set  $\mathcal{A}$  of 16 disjoint  $(-2)$ -curves. Another set is formed by the 16 sections. Each section intersects one nonmultiple component in each fiber, and this easily gives that the sets  $\mathcal{A}, \mathcal{B}$  form a symmetric configuration of type  $(16_4)$ . So, the surface is of K3 type of index 4 in five different ways.

In the next sections, we give three different constructions of a three-dimensional family of supersingular K3 surfaces of Kummer type and index 6 in characteristic 2. Its general member is a supersingular K3 surface with Artin invariant  $\sigma_0$  equal to 4.

## 2 | WEDDLE SURFACES

There is an explicit relationship between sets of six points in  $\mathbb{P}^1$  and Kummer surfaces. One uses the Veronese map to put the six points  $p_1, \dots, p_6$  on a twisted cubic  $R_3$  in  $\mathbb{P}^3$ . The discriminant surface of the web  $L$  of quadric surfaces through this set of six points is isomorphic (if  $p \neq 2$ ) to  $\text{Kum}(\text{Jac}(C))$ . The curve  $C$  is isomorphic to the double cover of  $R_3$  branched over the six points  $p_1, \dots, p_6$ .

In the case  $p \neq 2$ , the *Weddle surface*  $W$  is defined to be the locus of singular points of quadrics from the web  $L$ . Equivalently, it can be defined as the closure of the locus of points  $x \in \mathbb{P}^3$  such that the projections of the points  $p_1, \dots, p_6$  from  $x$  lie on a conic.

We may choose the projective coordinates to assume that

$$\begin{aligned} p_1 &= [1, 0, 0, 0], & p_2 &= [0, 1, 0, 0], & p_3 &= [0, 0, 1, 0], \\ p_4 &= [0, 0, 0, 1], & p_5 &= [1, 1, 1, 1], & p_6 &= [a, b, c, d], \end{aligned} \tag{1}$$

where the point  $p_6$  does not lie in any plane spanned by three of the points  $p_i, i < 6$ . Then, the equation of the Weddle surface is

$$\det \begin{pmatrix} ayzw & x & 1 & a \\ bxzw & y & 1 & b \\ cxyw & z & 1 & c \\ dxyz & w & 1 & d \end{pmatrix} = 0 \tag{2}$$

[15, §97]. One checks that, in all characteristics, a quartic surface  $W$  given by this equation has ordinary double points  $p_1, \dots, p_6$ . The surface  $W$  contains the lines  $\langle p_i, p_j \rangle$  and the twisted cubic  $R_3$  through  $p_1, \dots, p_6$ , all with multiplicity 1.

Conversely, counting parameters, we obtain that a general quartic surface in  $\mathbb{P}^3$  containing six lines  $\langle p_i, p_j \rangle$  and the twisted cubic  $R_3$  passing through the points  $p_1, \dots, p_6$  is given by (2)

If  $p = 2$ , the symmetric matrix of the polar bilinear form of quadrics from the web  $L$  is an alternating form, so the discriminant surface is given by the pfaffian; hence, it is a quadric surface. However, a quadric with polar bilinear form of corank 2 can still have an isolated singular point. In fact, one checks that the geometric description of the Weddle surface still holds in characteristic 2.

Another peculiarity of the case  $p = 2$  is that  $W$  has an additional singular point

$$P := [\sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d}].$$

A direct computation of the resolution of this singular point shows that it is a rational double point of type  $D_4$ .

**Proposition 2.1.** *A minimal nonsingular model of the Weddle surface is a supersingular K3 surface of Kummer type and index 6.*

*Proof.* Since singular points of  $W$  are rational double points, its minimal nonsingular model  $X$  is a K3 surface. The proper transforms  $E_{ij}$  of the lines  $\ell_{ij} = \langle p_i, p_j \rangle$  and the proper transform  $E_0$  of the twisted cubic  $R_3$  are a set of 16 disjoint smooth rational curves on  $X$ . Let  $E_i, i = 1, \dots, 6$ , be the exceptional curves over the nodes of  $W$ , and  $\ell_{ijk}$  be the residual line in the intersection of  $W$  with the plane  $\Pi_{ijk} = \langle p_i, p_j, p_k \rangle$ . The plane  $\Pi_{lmn}$  with  $\{i, j, k\} \cap \{l, m, n\} = \emptyset$  intersects  $\Pi_{ijk}$  along a line  $\ell$ . It intersects  $W$  at three points on lines  $\ell_{ij}, \ell_{ik}, \ell_{jk}$  and  $\ell_{lm}, \ell_{ln}, \ell_{mn}$ . It follows that  $\ell$  coincides with the line  $\ell_{ijk}$ . Thus, we find another set of disjoint  $(-2)$ -curves  $E_{ijk}$ , the proper transforms of the lines  $\ell_{ijk}$ . It is immediate to check that the set  $\mathcal{A}$  of 16  $(-2)$ -curves  $E_0, E_{ij}$  and the set  $\mathcal{B}$  of  $(-2)$ -curves  $E_i, E_{ijk}$  form an abstract symmetric configuration  $(16_6)$  isomorphic to the Kummer configuration. The surface  $X$  has 20  $(-2)$ -curves; 16 of them come from the Kummer configuration, and four come from the resolution of  $P$ . This implies that the Picard number of  $X$  is equal to 22, and hence,  $X$  is a supersingular K3 surface.  $\square$

It was noticed by Hutchinson [16] that the Weddle surface admits a cubic Cremona involution:

$$T : [x, y, z, w] \mapsto \left[ \frac{a}{x}, \frac{b}{y}, \frac{c}{z}, \frac{d}{w} \right].$$

This works also in characteristic 2, but instead of eight fixed points of  $T$  outside  $W$ , there is a unique fixed point lying on  $W$ . This is the singular point  $P$ . Under the involution  $T$ , the curves  $R_3$  and the line  $\ell_{56}$  interchange. This allows us to find the parametric form of  $R_3$ :

$$[s, t] \mapsto \left[ \frac{a}{s+ta}, \frac{b}{s+tb}, \frac{c}{s+tc}, \frac{d}{s+td} \right].$$

We identify  $T$  with its biregular lift to the nonsingular model  $X$  of  $W$ . Then  $T(E_1) = E_{234}$ ,  $T(E_2) = E_{134}$ ,  $T(E_3) = E_{124}$ ,  $T(E_4) = E_{123}$ . Any other line  $\ell_{ijk}$  intersects two opposite edges of the coordinate tetrahedron, hence they form three orbits with respect to  $T$ .

**Lemma 2.2.** *The surface  $X$  contains a quasi-elliptic pencil invariant with respect to  $T$ . It has three reducible fibers of type  $\tilde{D}_4$  and eight reducible fibers of type  $\tilde{A}_1^*$ . The involution  $T$  fixes one of the fibers of type  $\tilde{D}_4$  and switches other fibers in pairs.*

*Proof.* Let  $F = 2E_0 + E_1 + E_2 + E_3 + E_4$ . It is immediate to check that  $F$  is an effective nef divisor of arithmetic genus one and type  $\tilde{D}_4$  (type  $I_0^*$  in Kodaira's notation), hence  $|F|$  is a genus one pencil on  $X$ . The image of  $F$  under the involution  $T$  is equal to  $G = 2E_{56} + E_{234} + E_{123} + E_{124} + E_{134}$ . Since  $G \cdot F = 0$ ,  $G$  is another fiber of type  $\tilde{D}_4$  and the pencil  $|F|$  is  $T$ -invariant. The invariant fiber is the unique member of  $|F|$  that contains the singular point  $P$ . It is of type  $\tilde{D}_4$ . We also have six more reducible fibers  $E_{ijk} + E'_{ijk}$ , where  $E_{ijk}$  is different from the components of  $G$ . Suppose that the pencil  $|F|$  is an elliptic pencil. Adding up the Euler–Poincaré characteristics of the reducible fibers, we see that the sum is greater or equal than to  $3 \times 6 + 2 \times 6 = 30 > 24$ . This contradiction shows that the genus one pencil  $|F|$  must be quasi-elliptic, also that there must be two more fibers of types  $\tilde{A}_1^*$  (of type III in Kodaira's notation) [17].  $\square$

Since our family of supersingular K3 surfaces depends on three parameters (the projective equivalence classes of six points  $p_1, \dots, p_6$ ), the Local Torelli Theorem for supersingular K3 surfaces [25, §10, Theorem 2] suggests that the Artin invariant  $\sigma_0$  of  $X$  is equal to four. The previous lemma can be used to confirm this.

**Proposition 2.3.** *The Artin invariant  $\sigma_0$  of a general  $X$  is equal to 4.*

*Proof.* Recall that the Artin invariant  $\sigma_0$  of a supersingular K3 surface is equal to half of the rank of the elementary abelian 2-group equal to the discriminant group of the Picard lattice. We use the pencil  $|F|$  from Lemma 2.2. The sublattice  $M$  spanned by irreducible components of fibers modulo the divisor class of a fiber is isomorphic to the orthogonal sum of three copies of the root lattices  $D_4$  and eight copies of the root lattice of type  $A_1$ . The pencil has eight disjoint sections  $E_{k5}, E_{k6}$ ,  $k = 1, 2, 3, 4$ ; hence, the Mordell–Weil group  $\text{MW}(|F|)$  of the quasi-elliptic fibration  $|F|$  contains a subgroup isomorphic to  $(\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$ . The Shioda–Tate determinant formula [27, 6.6.1]

$$|\text{discr}(\text{Pic}(X))| \cdot \#\text{MW}(|F|)^2 = |\text{discr}(M)|$$

gives  $\sigma_0 \leq 4$ . Since our family is three-dimensional, we get  $\sigma_0 \geq 4$ . This proves the assertion.  $\square$

*Remark 2.4.* It is known that the sum of 16 disjoint  $(-2)$ -curves on a K3 surface in characteristic  $p \neq 2$  is divisible by 2, and the corresponding  $\mu_2$ -cover is birationally isomorphic to an abelian

surface. I believe that in our case, the sum  $\Sigma$  of 16 disjoint curves  $R, E_{ij}$  or  $E_i, E_{ijk}$  is not divisible by 2 if  $X$  is a general member of the family.<sup>†</sup> If it were, the Kummer  $\mu_2$ -cover of  $X$  with the branch divisor  $\Sigma$  is a nonnormal surface (since  $c_2(\Omega_X(\Sigma)) = -8$ ). It is known that every supersingular surface of Artin invariant  $\sigma_0 \leq 3$  is birationally isomorphic to the quotient of the self-product of a rational cuspidal curve by the infinitesimal group scheme  $\mu_2$  [21]. The action depends on two parameters and the quotient surface has 16 rational double points of type  $A_1$  and one point of type  $D_4$ . I do not know whether the surfaces can be realized as degenerations of Weddle quartic surfaces.

As in the case where  $p \neq 2$ , we can consider a rational map  $\tau : \mathbb{P}^3 \dashrightarrow L^* = \mathbb{P}^3$  given by the web of quadrics  $L$ . The map factors through a separable Artin–Schreier cover of degree 2, the covering involution is defined by an element of order 2 in the normal subgroup  $G \cong (\mathbb{Z}/2\mathbb{Z})^5$  of the Weyl group  $W(D_6)$  that leaves invariant the projective orbit of the ordered point set  $(p_1, \dots, p_6)$  in the Coble–Cremona action of  $W(D_6)$  on the GIT-quotient  $P_3^6$  of six ordered points in  $\mathbb{P}^3$  (see [4, §36], [5]). It follows that the cover  $\mathbb{P}^3$  is birationally isomorphic to a hypersurface

$$y_4^2 + q_2(y_0, y_1, y_2, y_3)y_4 + q_4(y_0, y_1, y_2, y_3) = 0$$

in the weighted projective space  $\mathbb{P}(1, 1, 1, 1, 2)$ , where  $Q := V(q_2)$  is the pfaffian hypersurface of the linear system  $L$ . The Weddle surface is the ramification locus of this cover, and it is birationally isomorphic to an inseparable  $\mu_2$ -cover of the quadric  $V(q_2)$  defined by the invertible sheaf  $\mathcal{O}_Q(B)$ , where  $B = V(q_2, q_4)$ .

The rational map  $\tau$  lifts to a separable regular degree 2 map  $\tilde{\tau} : \text{Bl}_{p_1, \dots, p_6}(\mathbb{P}^3) \rightarrow \mathbb{P}^3$ . Its Stein factorization consists of a birational morphism  $\text{Bl}_{p_1, \dots, p_6}(\mathbb{P}^3) \rightarrow Y$  that blows down the proper transforms of  $R_3, \ell_{ij}$  and an Artin–Schreier finite map  $f : Y \rightarrow \mathbb{P}^3$ . The known formula for the canonical sheaf of an Artin–Schreier cover gives that the branch divisor of  $f$  is the union of the quadric  $Q$  and six planes, the images of the exceptional divisors of the blow-up. It follows that the branch divisor of the inseparable double cover  $X \rightarrow Q$  is equal to the union of six conics, the images of the curves  $E_1, \dots, E_6$ . They intersect at one point  $x_0$ , the image of  $E_0$ .

Let  $\pi_{x_0} : Q \dashrightarrow \mathbb{P}^2$  be the projection of  $Q$  to a plane with center at  $x_0$ . The projections of the conics are six lines  $V(l_1), \dots, V(l_6)$  in general linear position. The projections of the conics  $\tau(\ell_{ijk})$  are conics, each passing through six of the 15 intersection points  $q_{ij} = V(l_i) \cap V(l_j)$ . The differential  $d\Phi$  of the curve  $\Phi = V(l_1 \cdots l_6)$  has 21 zeros counted with multiplicities and each two conics intersect at two zeros outside the curve  $\Phi$  [29]. In fact, the 10 conics intersect at two of these points, the projections of the lines on  $Q$  passing through  $x_0$ .

The additional zero of  $d\Phi$  comes with multiplicity four. It is equal to the projection of the image on  $Q$  of the singular point of the Weddle surface  $W$ .

By choosing projective coordinates in the plane such that

$$l_1 = x, l_2 = y, l_3 = z, l_4 = x + y + z, l_5 = a_1x + a_2y + a_3z, l_6 = b_1x + b_2y + b_3z,$$

one directly checks that the condition that there exists a conic through the intersection points  $P_{ij} = V(l_i, l_j), 1 \leq i < j \leq 3$  and  $P_{mn} = V(l_m, l_n), 4 \leq m < n \leq 6$  is equivalent to the condition that the six lines are dual to six points lying on a smooth conic  $C$  in the dual plane. Also, an explicit computation shows that the line joining the two common points of the 10 conics coincides with

<sup>†</sup> In a recent preprint of Yuya Matsumoto [24, Theorem 4.8], it has been proven that my belief is wrong. In fact, any K3 surface of Kummer type admits an inseparable double cover ramified over 16 disjoint  $(-2)$ -curves.

the line  $\ell$  dual to the conic  $C$ . So, the double plane model of  $X$ :

$$w^2 + l_1 \cdots l_6 = 0$$

is an analog in characteristic 2 of the double plane model of a Kummer quartic surface, where instead of the dual line  $\ell$ , we have a contact conic to the lines.

*Remark 2.5.* In [29] I. Shimada gave a classification of supersingular K3 surfaces birationally isomorphic to an inseparable double plane with the branch curve  $V(\Phi)$  of degree 6. In this classification, it is assumed that the differential  $d\Phi$  has 21 simple zeros. Our surface does not appear in his list because  $d\Phi$  has a multiple zero corresponding to the singular point  $P$  of  $W$  of type  $D_4$ .

### 3 | SIX POINTS IN $\mathbb{P}^1$

We learned that a set of six points in  $\mathbb{P}^1$  in arbitrary characteristic leads to a K3 surface of Kummer type of index 6 birationally isomorphic to a Weddle surface. We see the projective equivalence class of six points in different ways: the six points  $p_1, \dots, p_6$  on a twisted cubic  $R_3$ , the six points dual to the lines  $V(l_i)$  in its double plane birational model, six intersection points of the lines lying on a conic, and six intersection points of the lines  $V(l_i)$  with the line  $\ell$ . It is classically known that the GIT-quotient  $P_1^6 := (\mathbb{P}^1)^6 // \text{PGL}_2(k)$  with respect to the democratic linearization is isomorphic to the Segre cubic primal  $\Sigma_3$  representing the unique projective isomorphism class of a cubic hypersurface in  $\mathbb{P}^4$  with 10 ordinary nodes [5, Chapter 1, §3], [7, Theorem 9.4.10]. An equation of  $\Sigma_3$  in all characteristics can be chosen to be the following:

$$x_1 x_2 x_4 - x_0 x_3 x_4 - x_1 x_2 x_3 + x_0 x_1 x_3 + x_0 x_2 x_3 - x_0^2 x_3 = 0. \quad (3)$$

The 10 singular points of  $\Sigma_3$  are:

$$\begin{aligned} & [0, 0, 0, 0, 1], [0, 0, 0, 1, 0], [0, 0, 1, 0, 0], [0, 1, 0, 0, 0], [0, 1, 0, 1, 1], \\ & [0, 0, 1, 1, 1], [1, 1, 0, 0, 0], [1, 0, 1, 0, 0], [1, 0, 0, 0, 1], [1, 1, 1, 1, 0]. \end{aligned} \quad (4)$$

If  $p \neq 2$ , one can transform equation (3) to the familiar form

$$x_0^3 + \cdots + x_4^3 - (x_0 + \cdots + x_4)^3 = 0,$$

which exhibits obvious  $\mathfrak{S}_6$ -symmetry of the equation.

If  $p = 2$ , the symmetry is not obvious. Fix the set of reference points  $p_1, \dots, p_5$  in  $\mathbb{P}^3$  from (1). An explicit rational parameterization

$$\phi : \mathbb{P}^3 \dashrightarrow \Sigma_3 \subset \mathbb{P}^4 \quad (5)$$

is given by quadrics through the reference points:

$$[t_0, t_1, t_2, t_3] = [t_3(t_0 + t_1), t_3(t_1 + t_3), t_2(t_0 + t_1), t_2(t_1 + t_3), (t_0 + t_2)(t_1 + t_3)].$$

The action of the symmetric group  $\mathfrak{S}_6$  on  $\Sigma_3$  is induced by its rational action on  $\mathbb{P}^3$ . The transpositions (12),(23),(34),(45) of  $\mathfrak{S}_6$  are realized by projective transformations of  $\mathbb{P}^3$  that leave invariant the set of five reference points. The transposition (56) acts by the standard Cremona involution  $[t_0, t_1, t_2, t_3] \mapsto [\frac{1}{t_0}, \frac{1}{t_1}, \frac{1}{t_2}, \frac{1}{t_3}]$ . The action of  $\mathfrak{S}_6$  on  $\Sigma_3$  is induced by a five-dimensional linear representation of  $\mathfrak{S}_6$ :

$$\begin{aligned} (12) : [x_0, \dots, x_4] &\mapsto [x_0, x_0 + x_1, x_2, x_2 + x_3, x_0 + x_2 + x_4], \\ (23) : [x_0, \dots, x_4] &\mapsto [x_0 + x_1, x_1, x_0 + x_1 + x_4, x_1 + x_3, x_0 + x_2], \\ (34) : [x_0, \dots, x_4] &\mapsto [x_2, x_3, x_0, x_1, x_0 + x_2 + x_4], \\ (45) : [x_0, \dots, x_4] &\mapsto [x_0, x_1, x_0 + x_2, x_1 + x_3, x_0 + x_1 + x_4], \\ (56) : [x_0, \dots, x_4] &\mapsto [x_2, x_0 + x_1 + x_2 + x_4, x_0, x_3 + x_4, x_4]. \end{aligned}$$

As in the case  $p \neq 2$ , the linear representation is an irreducible representation of  $\mathfrak{S}_6$  corresponding to the partition  $\lambda = (3, 3)$ .

If  $p \neq 2$ , the dual hypersurface  $\Sigma_3^*$  is a quartic hypersurface. The group  $\mathfrak{S}_6$  acts linearly in the dual projective space via its action on the partial derivatives of  $\Sigma_3$ . It defines an irreducible linear representation of  $\mathfrak{S}_6$  corresponding to the partition (2,2,2). In appropriate dual coordinates,  $\Sigma_3^*$  can be given by the following equations in  $\mathbb{P}^5$ :

$$\left( \sum_{i=0}^5 y_i^2 \right)^2 - 4 \sum_{i=0}^5 y_i^4 = \sum_{i=0}^5 y_i = 0, \tag{6}$$

which exhibit obvious  $\mathfrak{S}_6$ -symmetry. The quartic hypersurface  $\Sigma_3^*$  is isomorphic to the Igusa compactification of the moduli space  $\mathcal{A}_2(2)$  of principally polarized abelian surfaces with a level two structure. For this reason, in modern literature, the quartic  $\Sigma^*$  is called the *Igusa quartic*, although, in classical literature, it was known as the *Castelnuovo quartic*. For any smooth point  $x \in \Sigma_3^*$ , the tangent hyperplane at  $x$  cuts out  $\Sigma_3^*$  along a quartic surface with 16 ordinary nodes. This is the Kummer surface of the Jacobian variety of the genus two curve associated to the corresponding point from  $\Sigma_3$  [4, p. 141].

The double cover of  $\mathbb{P}^4$  branched along  $\Sigma_3^*$  admits a modular interpretation as the GIT-quotient  $P_2^6 := (\mathbb{P}^2)^6 // \text{PGL}_3(k)$  [4, 5, Chapter 1]. Its equation in  $\mathbb{P}(1, 1, 1, 1, 1, 2)$  is

$$w^2 + F_4(y_0, \dots, y_4) = 0, \tag{7}$$

where we rewrite equations (6) by eliminating  $y_5$ .<sup>†</sup> The involution  $w \mapsto -w$  corresponds to the association involution, and its locus of fixed points is the GIT-quotient of the subvariety of  $(\mathbb{P}^2)^6$  of ordered sets of points lying on a conic.

If  $p = 2$ , the GIT-quotient  $P_2^6$  is still defined. It is isomorphic to a hypersurface  $\mathcal{V}$  in  $\mathbb{P}(1, 1, 1, 1, 1, 2)$  given by equation:

$$w^2 + w(y_2y_3 + y_1y_4 + y_0(y_0 + y_1 + y_2 + y_3 + y_4)) + y_0y_1y_4(y_0 + y_1 + y_2 + y_3 + y_4) = 0. \tag{8}$$

<sup>†</sup> In modern literature, the fourfold given by (7) is known as the Coble fourfold.

The projection to the  $y$ -coordinates defines a separable double cover  $\mathcal{V} \rightarrow \mathbb{P}^4$  branched over the quadric  $Q = V(q)$ , where

$$q = y_2y_3 + y_1y_4 + y_0(y_0 + y_1 + y_2 + y_3 + y_4).$$

The involution  $w \mapsto w + q$  has the same geometric meaning as in the case  $p \neq 2$ . The locus of fixed points  $F$  of the involution is isomorphic to the inseparable double cover of  $Q \subset \mathbb{P}^4$ . As in the case  $p \neq 2$ ,  $F$  is singular over the preimages of 15-lines. They represent the closed semistable orbits of point sets of the form  $(a, a, b_1, b_2, b_3, b_4)$ . Each line contains three points representing the closed semistable orbits of point sets  $(a, a, b, b, c, c)$ . The incidence relation between the lines and the points is the famous Cremona–Richmond symmetric configuration  $(15_3)$ .

The duality fails if  $p = 2$ : the Hessian of the cubic polynomial defining  $\Sigma_3$  is identically zero. The group  $\mathfrak{S}_6$  still acts on  $\mathcal{V}$  via its action on  $P_2^6$  defining a linear representation in the dual space  $\mathbb{P}^4$  corresponding to the partition  $(2,2,2)$ :

$$(12) : (y_0, \dots, y_4) \mapsto (y_0, y_1, y_2, y_0 + y_1 + y_3, y_0 + y_2 + y_4),$$

$$(23) : (y_0, \dots, y_4) \mapsto (y_0, y_3, y_4, y_1, y_2),$$

$$(34) : (y_0, \dots, y_4) \mapsto (y_1, y_0, y_2, y_3, y_0 + y_1 + y_2 + y_3 + y_4),$$

$$(45) : (y_0, \dots, y_4) \mapsto (y_0, y_2, y_1, y_4, y_3),$$

$$(56) : (y_0, \dots, y_4) \mapsto (y_0, y_1, y_0 + y_1 + y_2, y_3, y_0 + y_3 + y_4).$$

The quadric  $Q$  is invariant with respect to the representation. Since the partition  $(2,2,2)$  is not 2-regular, the linear representation is reducible [18]. In fact, one observes that the vector  $(1,1,1,1,1)$  is invariant.

**Theorem 3.1.** *Let  $x \in \Sigma_3 \subset \mathbb{P}^4$  be a nonsingular point, and  $Q_x$  be the polar quadric of  $\Sigma_3$  with pole at  $x$ . The preimage  $X$  of  $Q_x$  under the map  $\phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^4$  is isomorphic to the Weddle surface associated with six points  $(p_1, \dots, p_5, p_6 = \phi^{-1}(x))$ .*

*Proof.* Since the map  $\phi$  from (5) is given by the web of quadrics  $L$ , the pre-image of  $Q_x$  is a quartic surface  $W$  in  $\mathbb{P}^3$  with double points at  $p_1, \dots, p_5$ . Since  $Q_x$  is tangent to  $\Sigma_3$  at the point  $x$ , the quartic acquires an additional double point at  $p_6 = \phi^{-1}(x)$ . The images of the lines  $\ell_i = \langle p_i, p_6 \rangle, i = 1, \dots, 5$ , are lines on  $\Sigma_3$  passing through  $x$ . It is known that the polar quadric  $Q_x$  intersects  $\Sigma_3$  at points  $y$  such that the tangent hyperplane of  $\Sigma_3$  at  $y$  contains  $x$  [7, Theorem 1.1.5]. This implies that the five lines  $\phi(\ell_i)$  are contained in  $Q_x$ , and hence, the lines  $\ell_i$  are contained in  $W$ . Since  $Q_x$  passes through singular points of  $\Sigma_3$ , the lines  $\langle p_i, p_j \rangle, 1 \leq i < j \leq 5$  are also contained in  $W$ .

Let  $R_3$  be the unique twisted cubic through the six points  $p_1, \dots, p_6$ . Its image in  $\Sigma_3$  is the sixth line in  $\Sigma_3$  passing through  $x$ . By above, it is also contained in  $Q_x$ , and hence  $W$  contains  $R_3$ . It follows from Section 2 that  $W$  is a Weddle surface.  $\square$

## 4 | CONGRUENCES OF LINES AND QUARTIC DEL PEZZO SURFACES

A congruence of lines in  $\mathbb{P}^3$  is an irreducible surface  $S$  in the Grassmannian  $\mathbb{G} := G_1(\mathbb{P}^3)$  of lines in  $\mathbb{P}^3$ . A line  $\ell_s$  in  $\mathbb{P}^3$  corresponding to a point  $s \in S$  is called a ray of the congruence. The algebraic

cycle class  $[S]$  of  $S$  in the Chow ring  $A^*(\mathbb{G})$  is determined by two numbers, the order  $m$  and the class  $n$ . The order  $m$  (resp. the class  $n$ ) is equal to the number of rays passing through a general point  $x$  in  $\mathbb{P}^3$  (resp. contained in a general plane  $\Pi \subset \mathbb{P}^3$ ). We have  $[S] = m\sigma_x + n\sigma_\Pi$ , where  $\sigma_x$  (resp.  $\sigma_\Pi$ ) is the algebraic cycle class of an  $\alpha$ -plane  $\Omega(x)$  of lines through a point  $x \in \mathbb{P}^3$  (resp. of a  $\beta$ -plane  $\Omega(\Pi)$  of lines contained in a plane  $\Pi$ ). The degree of the surface  $S$  in the Plücker embedding  $\mathbb{G} \hookrightarrow \mathbb{P}^5$  is equal to  $m + n$ .

The universal family of rays  $Z_S = \{(x, s) \in \mathbb{P}^3 \times S : x \in \ell_s\}$  comes with two projections  $p_S : Z_S \rightarrow \mathbb{P}^3$  and  $q_S : Z_S \rightarrow S$ .

We assume that  $m = n = 2$  and  $S$  is smooth. Then  $S$  is a quartic del Pezzo surface in its Plücker embedding that coincides with its anticanonical embedding. It follows that  $S$  is contained in a hyperplane section  $H \cap \mathbb{G}$ , a linear complex of lines.

By the definition of the order of a congruence, the cover  $p_S : Z_S \rightarrow \mathbb{P}^3$  is of degree 2. It is known that  $S$  does not contain fundamental curves, that is, curves in  $\mathbb{P}^3$  over which the fibers are one-dimensional. Thus, the cover  $p_S$  is a finite cover over the complement of a finite set of points.

Let us assume now that  $p \neq 2$  and see later what happens in the case  $p = 2$ . Although the classical theory of congruences of lines assumes that the ground field is the field of complex numbers, all the facts are true only assuming that  $p$  does not divide the order and the class (see a brief exposition of the theory of congruences in [9, §2]). The cover  $p_S : Z_S \rightarrow \mathbb{P}^3$  is a Kummer-type double cover branched along the focal surface  $\text{Foc}(S)$  of  $S$ . The focal surface is a quartic Kummer surface with 16 nodes. The congruence is one of the six irreducible components of order 2 of the surface of bitangent lines to  $\text{Foc}(S)$ . If the Plücker equation of  $\mathbb{G}$  is taken to be  $\sum_{i=1}^6 x_i^2 = 0$ , the equations of the six congruences of bitangents are  $x_i = 0$ .

The preimage of a ray  $\ell_s$  under  $p_S$  in  $Z_S$  is equal to the union of the fiber  $q_S^{-1}(s)$  (which can be identified with  $\ell_s$ ) and a curve  $L_s$  that is projected to  $C(s) = S \cap \mathbb{T}_s(\mathbb{G})$  under the map  $q_S : Z_S \rightarrow S$ . The intersection points  $L_s \cap q_S^{-1}(s)$  are the preimages of the tangency points of  $\ell_s$  with  $\text{Foc}(S)$ . The map  $L_s \rightarrow C(s)$  is the normalization map, and the points in  $L_s \cap q_S^{-1}(s)$  correspond to the branches of  $C(s)$  at the singular point  $s \in C(s)$ . The locus of the pairs of points  $L_s \cap q_S^{-1}(s)$  defines a double cover  $q'_S : X \rightarrow S$  of the ramification divisor  $X$  of  $p_S$ . The ramification curve  $R$  of  $q'_S$  is the locus of the preimages in  $Z_S$  of points in  $\text{Foc}(S)$ , where a ray  $\ell_s$  is tangent to  $\text{Foc}(S)$  with multiplicity 4. The branch curve  $B$  of  $q'_S$  is the locus of points  $s \in S$  such that the curve  $C(s)$  has a cusp at  $s$ . It is known that  $B \in |-2K_S|$  [9, (2.9)]. It is a special canonical curve of genus 5, a *Humbert curve*. The curve  $B$  is cut out by a quadric in  $\mathbb{P}^5$ . The adjunction formula shows that  $X$  is a K3 surface.

The first projection  $p_S : X \rightarrow \text{Foc}(S)$  is a minimal resolution of singularities. The fibers  $E(x_i)$  over the singular points  $x_i$  of  $\text{Foc}(S)$  form a set  $\mathcal{A}$  of 16 disjoint  $(-2)$ -curves. Another set  $\mathcal{B}$  of 16 disjoint  $(-2)$ -curves is obtained as the intersection of the plane  $\Pi(x_i)$  swept by the rays from  $\Omega(x_i)$  with  $\text{Foc}(S)$ . The plane  $\Pi(x_i)$  is tangent to  $\text{Foc}(S)$  along a conic. In classical terminology, such a plane is a *trope* and the corresponding conic is a *trope-conic*. The map  $T(x_i) \rightarrow E(x_i)$  is defined by the deck transformation of the cover  $q_S : X \rightarrow S$ . It follows that each line on  $S$  splits under the cover  $q_S : X \rightarrow S$ . This is a remarkable property of the curve  $B$ : it is a curve in  $|-2K_S|$ , which is tangent to all lines contained in  $S$ . Assume now that  $p = 2$ . We still have a realization of a quartic del Pezzo surface  $S$  as a congruence of lines in  $\mathbb{P}^3$  of order 2. It is equal to a hyperplane section of a quadratic line complex  $\mathbb{C}$  which we may assume to be smooth. It follows that the order and the class of  $S$  is equal to 2.

By definition of the order of a congruence, the projection  $p_S : Z_S \rightarrow \mathbb{P}^3$  is a map of degree 2. Its general fiber is equal to the intersection of the smooth conic  $\Omega(x) \cap \mathbb{C}$  with a hyperplane section of  $\mathbb{G}$ . Since  $S$  is smooth, it consists of two points. This shows that the map  $p_S$  is separable.

Let  $Z_S \rightarrow Z'_S \rightarrow \mathbb{P}^3$  be its Stein factorization, where the first map is a birational morphism and the second map is a separable finite morphism of degree 2. Since  $H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(n)) = 0$ ,  $n \geq 0$ , the cover  $Z'_S \rightarrow \mathbb{P}^3$  is an Artin–Schreier cover. The known formula for the canonical class of the universal family of lines  $Z_G$  over  $\mathbb{G}$  [7, 10.1.1] easily gives that  $\omega_{Z_S} \cong p_S^* \mathcal{O}_{\mathbb{P}^3}(-2) \otimes q_S^* \omega_S(1)$ . The formula for the canonical sheaf of an Artin–Schreier cover shows that  $Z'_S$  is given by an equation:

$$x_4^2 + F_2(x_0, x_1, x_2, x_3)x_4 + F_4(x_0, x_1, x_2, x_3) = 0, \quad (9)$$

where  $Q = V(F_2)$  is a quadric and  $V(F_4)$  is a quartic surface. The quartic polynomial  $F_4$  is defined up to a replacement of  $F_4$  with  $A^2 + AF_2 + F_4$ , where  $A$  is a quadratic form.

*Remark 4.1.* As communicated to me by T. Katsura, one can give an explicit equation of the quadric  $V(F_2)$  in terms of the equation of a congruence of lines  $S$  of order 2 and class 2. If  $\mathbb{G}_1(\mathbb{P}^3) = V(x_1y_1 + x_2y_2 + x_3y_3) \subset \mathbb{P}^5$  and  $S$  is given by equations

$$\begin{aligned} a_1x_1y_1 + a_2x_2y_2 + a_3x_3y_3 + c_1y_1^2 + c_2y_2^2 + c_3y_3^2 &= 0, \\ \alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3 + \beta_1y_1 + \beta_2y_2 + \beta_3y_3 &= 0, \end{aligned}$$

then

$$\begin{aligned} F_2 &= (a_1 + a_3)(\alpha_2x_0x_2 + \beta_2x_1x_3) + (a_2 + a_3)(\alpha_1x_0x_1 + \beta_1x_2x_3) \\ &+ (a_1 + a_2)(\alpha_3x_0x_3 + \beta_3x_1x_2). \end{aligned} \quad (10)$$

The following proposition is an analog of the description of  $S$  as an irreducible component of the surface of bitangent lines to  $\text{Foc}(S)$ .

**Proposition 4.2.** *The congruence  $S$  is an irreducible component of the locus of points in  $\mathbb{G}$  parametrizing lines in  $\mathbb{P}^3$  that split under the cover  $p_S : Z_S \rightarrow \mathbb{P}^3$  into two irreducible components. In particular, no ray of the congruence is contained in the quadric  $Q = V(F_2)$ .*

*Proof.* The fiber  $q_S^{-1}(s)$  maps isomorphically to the ray  $\ell_s$  under the projection  $p_S : Z_S \rightarrow \mathbb{P}^3$ . Thus, the preimage  $p_S^{-1}(\ell_s)$  is equal to the union of  $q_S^{-1}(s)$  and a curve  $L_s$  whose points are the preimages of rays intersecting  $\ell_s$ . The image of  $L_s$  in  $S$  under the projection  $q_S : Z_S \rightarrow S$  is equal to the hyperplane section  $C(s) := \mathbb{T}_s \cap S$ . If  $\ell_s \subset Q$ , then the restriction of  $p_S$  over  $\ell_s$  is a purely inseparable cover, so the preimage of  $\ell_s$  does not split.  $\square$

Note the last assertion is an analog of the fact that  $\text{Foc}(S)$  does not contain lines.

I do not know how to describe explicitly the locus of splitting lines under a separable double cover. However, the condition for splitting of a line is clear. A separable cover  $y^2 + a_k(t_0, t_1)y + b_{2k}(t_0, t_1) = 0$  of a line with coordinates  $t_0, t_1$  is reducible if and only if  $b_{2k} = a_k c_k + c_k^2$  for some binary form  $c_k$  of degree  $k$ .

The preimage of a general plane  $\Pi$  in  $Z_S$  under the map  $p_S$  is a separable double cover given by the equation

$$w^2 + a_2(t_0, t_1, t_2)w + b_4(t_0, t_1, t_2) = 0,$$

where  $(t_0, t_1, t_2)$  are coordinates in  $\mathbb{P}^2$ . The double cover is isomorphic to a del Pezzo surface of degree 2. It is known that it has 28 lines that are split under the cover. They correspond to 56  $(-1)$ -curves on the del Pezzo surface. The splitting lines are discussed in [11], where they are called *fake bitangent lines*. The variety of splitting lines is a congruence in  $\mathbb{G}$  of class equal to 28. It is an analog in characteristic 2 of the congruence of bitangents of a Kummer surface. Its order is known to be equal to 12, and its class is equal to 28. Also is known that the congruence of bitangents of a Kummer surface consists of six irreducible congruences of order 2 and class 2, and 16  $\beta$ -planes  $\Omega(T)$ , where  $T$  is a trope. It is natural to conjecture that the congruence of splitting lines is also of degree 12 and  $S$  is one of its six irreducible components of order 2.

A general ray  $\ell_s$  intersects the quadric  $Q$  at two points, and the preimages of these two points in  $Z_S$  correspond to the branches of the singular point  $s \in C(s)$ . Let  $X \subset Z_S$  be the closure of the set of the branches of the curves  $C(s)$ . By Proposition 4.2, no ray is contained in  $X$ . This shows that the projection  $q'_S : X \rightarrow S$  is a separable Artin–Schreier finite cover of degree 2. This is the analog in characteristic 2 of the fact that the cover  $q'_S : X \rightarrow S$  is a finite cover of smooth surfaces of degree 2. In the blow-up plane model of  $S$ ,  $X$  is isomorphic to a surface of degree 6 in the weighted projective space  $\mathbb{P}(1, 1, 1, 3)$  given by equation:

$$x_3^2 + F_3(x_0, x_1, x_2)x_3 + F_6(x_0, x_1, x_2) = 0. \tag{11}$$

By the adjunction formula,  $\omega_X \cong \mathcal{O}_X$ . A ray  $\ell_s$  defines a cusp of  $C(s)$  at  $s$  if and only if it is tangent to the quadric  $Q$ . It is known that the lines in  $\mathbb{P}^3$  tangent to a smooth quadric surface are parametrized by the tangential quadratic line complex  $\mathcal{T}(Q)$ . It is singular along the locus of lines contained in  $Q$  [7, Proposition 10.3.23]. Since  $\mathcal{T}(Q) \cap S \in |-2K_S|$ , we see that  $\mathcal{T}(Q)$  is tangent to  $S$  along the curve  $B = V(F_3) \in |-K_S|$ . This differs from the case  $p \neq 2$ , where the branch curve  $B$  belongs to  $|-2K_S|$ .

Let  $L_i$  be one of 16 lines on the del Pezzo surface  $S \subset \mathbb{G}$ . A line in  $\mathbb{G}$  is a pencil of rays contained in a plane, that is,  $L_i = \Omega(x_i) \cap \Omega(\Pi_i)$  for some  $x_i$  in a plane  $\Pi_i$ . All rays  $\ell_s, s \in L_i$ , pass through  $x_i$ , hence the fiber  $E(x_i)$  of  $Z_S \rightarrow \mathbb{P}^3$  over  $x_i$  is equal to the fiber of the projections  $p_S : X \rightarrow \mathbb{P}^3$ . This implies that  $X$  is singular over the point  $x_i$ . So, we have 16 points  $x_i \in Q$ , over which the map  $p_S$  is not a finite morphism. The points are analogs of singular points of  $\text{Foc}(S)$ . We have also 16 planes  $\Pi(x_i)$ , they are swept by the rays  $\ell_s, s \in L_i$ . Each plane  $\Pi(x_i)$  intersects  $Q$  along a conic  $T(x_i)$ . They are characteristic two analogs of trope-conics of  $\text{Foc}(S)$ . Both curves  $E(x_i)$  and the proper transforms of  $T(x_i)$ 's in  $X$  are mapped to the line  $L_i$ , so the line  $L_i$  splits under the separable cover  $q_S : X \rightarrow S$ . This defines two sets  $\mathcal{A}$  (of curves  $E(x_i)$ ) and  $\mathcal{B}$  (of curves  $T(x_i)$ ) of disjoint  $(-2)$ -curves on  $X$ .

**Theorem 4.3.** *Let  $\tilde{X}$  be a minimal resolution of singularities of  $X$ . Then  $\tilde{X}$  is a K3 surface of Kummer type of index 6 birationally isomorphic to a Weddle quartic surface.*

*Proof.* The known formula for the canonical class of a separable double cover  $X \rightarrow S$  gives  $\omega_X \cong q^* \omega_S(-K_S) \cong \mathcal{O}_X$ . Let us look at the singularities of  $X$ .

The surface  $X$  is an inseparable Kummer cover of the quadric  $Q$  defined by a section of  $\mathcal{L}^{\otimes 2}$ , where  $\mathcal{L} \cong \mathcal{O}_Q(2)$ . It is known that its set of singular points is equal to the support of the scheme of zeros of a section  $\nabla$  of  $\Omega_Q^1 \otimes \mathcal{L}^{\otimes 2}$ . This well-known fact follows from local computations: locally, the cover is given by  $z^2 + f(x, y) = 0$ , and local sections  $\nabla = df(x, y)$  glue together to a global section of  $\Omega_Q^1 \otimes \mathcal{L}^{\otimes 2}$ . An ordinary node is locally given by equation  $z^2 + xy = 0$ , hence the local  $\nabla$

is equal to  $ydx + xdy$ ; hence, it has a simple zero at the node. Any other singular point contributes more than one to  $c_2$ .

We have

$$c_2(\Omega_Q^1 \otimes \mathcal{L}^{\otimes 2}) = c_2(\Omega_Q^1) + c_1(\Omega_Q^1)c_1(\mathcal{L}^2) + c_1(\mathcal{L}^2)^2 = 20.$$

We skip the proof that  $X$  is normal (the direct proof is rather elaborate but the fact follows from the argument in the last paragraph of the proof). It has 16 singular points over the 16 points  $x_i$ . It is known that the automorphism group  $\text{Aut}(S)$  contains a subgroup isomorphic to  $(\mathbb{Z}/2\mathbb{Z})^{\oplus 4}$ , and that group acts transitively on the set of 16 lines [10, Theorem 3.1]. This implies that the 16 points are ordinary nodes. Since the total sum of the multiplicities of the remaining zeros of  $\nabla$  is equal to 4,  $X$  has only rational double points. This proves that a minimal resolution  $\tilde{X}$  of  $X$  is a K3 surface.

The sum of 16 lines on  $S$  is a divisor in the linear system  $| -4K_S |$ . The images on  $S$  of extra singular points of  $X$  lie outside the union of 16 lines. Thus, the self-intersection of its preimage on  $\tilde{X}$  is equal to 128. If  $n$  is the index of the configuration  $\mathcal{A}, \mathcal{B}$  of  $(-2)$ -curves, then this self-intersection must be equal to  $-64 + 32n$ . This implies that  $n = 6$ .

We have a Kummer configuration of 16 points and 16 conics on the quadric  $Q$ . Projecting  $Q$  from one of the points, we get six lines  $V(l_i)$ , the projections of the six conics containing the center of the projection map. The projections of the other 10 conics are conics in the plane passing through six intersection points of the lines. So, the surface  $X$  is birationally isomorphic to the double plane  $V(w^2 + l_1 \cdots l_6)$  as in the case of a Weddle surface. The 10 conics intersect at the projections of the rulings of  $Q$  containing the center of the projection map. We also get, as a bonus, that  $X$  contains a rational double point of type  $D_4$ .<sup>†</sup> □

## 5 | ROSENHAIN AND GÖPEL TETRADS

A Rosenhain tetrad of a quartic Kummer surface in characteristic  $p \neq 2$  is a subset of four nodes such that the planes containing three of the nodes are tropes [15, §50]. If one equips the set of 2-torsion points of  $\text{Jac}(C)$  with a structure of a symplectic four-dimensional linear space over  $\mathbb{F}_2$ , then a Rosenhain tetrad is the image of a translate of a nonisotropic plane. There are 80 Rosenhain tetrads. Each Rosenhain tetrad defines a symmetric configuration  $(4_3)$  between the sets of tropes and nodes. The union of two Rosenhain tetrads without common points forms a symmetric configuration  $(8_4)$ . This configuration is realized by eight vertices of a cube and eight faces of two tetrahedra inscribed in the cube.

The union of two Rosenhain tetrads can be illustrated by the following figure (see [23, 7.3]):

$$\begin{array}{ccc|ccc}
 \circ & \star & & \circ & \star & \\
 \circ & \star & & \circ & \star & \\
 \circ & \star & & \circ & \star & \\
 \circ & \star & & \circ & \star & 
 \end{array} \tag{12}$$

Here, circles correspond to the nodes, and the stars correspond to tropes. Each side of the diagram represents a Rosenhain tetrad. A point in a row  $i$  lies in the plane in the same row on the other

<sup>†</sup> I believe that the image of the singular point under the projection  $q_S$  is the strange point on the del Pezzo surface  $S$  defined in [10].

side of the diagram, and it also lies in the three planes on the same side of the diagram from different rows.

Let us see how to get this configuration with the absence of the Kummer surface. A quartic del Pezzo surface  $S$  contains 20 pairs of tetrads of disjoint lines with the intersection relation of each pair forming an abstract configuration  $(4_3)$ . We use a birational model of  $S$  as the blow-up of five points  $q_1, \dots, q_5$  in the plane, and denote by  $L_i$  the lines on  $S$  coming from the exceptional curves over the points  $p_i$ , 10 lines  $L_{ij}$  coming from the lines  $\langle q_i, q_j \rangle$ , and one line  $L_0$  coming from the conic through the five points. Then the 20 pairs are the following:

- 10 pairs

$$\{L_0, L_{ij}, L_{ik}, L_{jk}\}, \quad \{L_i, L_j, L_k, L_{lm}\},$$

- 10 pairs

$$\{L_i, L_{ij}, L_{ik}, L_{il}\}, \quad \{L_m, L_{jm}, L_{km}, L_{lm}\}.$$

Each tetrad of lines on  $S$  from above splits in  $X$  into eight disjoint  $(-2)$ -curves. The curves correspond to the first two columns in the diagram, and the other tetrad in the pair defines the third and the fourth columns. We get only 40 Rosenhain tetrads in this way. If  $p \neq 2$ , other 40 Rosenhain tetrads arise from different congruences of lines, which define different irreducible components of the surface of bitangent lines of the Kummer surface. I believe that the same is true if  $p = 2$ : other 40 tetrads arise from different components of the surface of splitting lines of the double cover (9).

Note that a configuration of type  $(4_3)$  is realized by two sets of lines among 20 lines on an octic model of the Kummer surface in characteristic two [20, Figure 2].

A Göpel tetrad is a subset of four nodes such that no three of them lie on a trope. There are 60 Göpel tetrads. They correspond to the translated isotropic planes in  $\mathbb{F}_2^4$ . To get them from a quartic del Pezzo surface  $S$ , one considers 30 subsets of four skew lines  $(L_i, L_j, L_{kl}, L_{km})$ , where  $\{i, j\} \cap \{k, l, m\} = \emptyset$ . The preimage of each subset in  $X$  defines a set of four tropes and four points  $x_i$  forming a Göpel tetrad. There will be 30 Göpel tetrads arising in this way. As in the case of Rosenhain tetrads, other 30 Göpel tetrads should arise from other irreducible components of the surface of splitting lines of (9).

Recall from [14, Theorem 1.20] that there are three abstract configurations of type  $(16_6)$ . The Kummer one is nondegenerate in the sense that any pair of trope-conics has two common vertices. It follows from our construction of Kummer configurations that they are nondegenerate. If  $p \neq 2$ , any nondegenerate Kummer configuration of points and planes of type  $(16_6)$  is realized on a Kummer quartic surface. As we see, in characteristic 2 this is not true anymore, and the Kummer surface should be replaced by a quadric surface.

## JOURNAL INFORMATION

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