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Logarithmic sheaves attached to arrangements of hyperplanes

By

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1. Introduction

Any divisor D on a nonsingular variety X defines a sheaf of logarithmic differential forms $\Omega^1_X(\log D)$. Its equivalent definitions and many useful properties are discussed in a fundamental paper of K. Saito [Sa]. This sheaf is locally free when D is a strictly normal crossing divisor, and in this situation it is a part of the logarithmic De Rham complex used by P. Deligne to define the mixed Hodge structure on the cohomology of the complement $X \setminus D$. In the theory of hyperplane arrangements this sheaf arises when D is a central arrangement of hyperplanes in $\mathbb{C}^{n+1}$. In exceptional situations this sheaf could be free (a free arrangement), for example, when $n = 2$ or the arrangement is a complex reflection arrangement. Many geometric properties of the vector bundle $\Omega^1_X(\log D)$ were studied in the case when D is a generic arrangement of hyperplanes in $\mathbb{P}^n$ [DK1]. Among these properties is a Torelli type theorem which asserts that two arrangements with isomorphic vector bundles of logarithmic 1-forms coincide unless they osculate a normal rational curve. In this paper we introduce and study a certain subsheaf $\tilde{\Omega}^1_X(\log D)$ of $\Omega^1_X(\log D)$. This sheaf contains as a subsheaf (and coincides with it in the case when the divisor D is the union of normal irreducible divisors) the sheaf of logarithmic differentials considered earlier in [CHKS]. Its double dual is isomorphic to $\Omega^1_X(\log D)$. The sheaf $\tilde{\Omega}^1_X(\log D)$ is locally free only if the divisor D is locally formally isomorphic to a strictly normal crossing divisor. This disadvantage is compensated by some good properties of this sheaf which $\Omega^1_X(\log D)$ does not posses in general. For example, one has always a residue exact sequence

$$0 \to \Omega^1_X \to \tilde{\Omega}^1_X(\log D) \to \nu_*\mathcal{O}_{D'} \to 0,$$

where $\nu : D' \to D$ is a resolution of singularities of D. Also, in the case when D is an arrangement of m hyperplanes in $\mathbb{P}^n$, the sheaf $\tilde{\Omega}^1_{\mathbb{P}^n}(\log D)$ admits a simple projective resolution

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1)^{m-n-1} \to \mathcal{O}_{\mathbb{P}^n}^{m-1} \to \tilde{\Omega}^1_{\mathbb{P}^n}(\log D) \to 0.$$

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In particular, its Chern polynomial does not depend on the combinatorics of the arrangement. This allows us to introduce the notion of a stable (resp. semi-stable, unstable) arrangement and define a map from the space of semi-stable arrangements to the moduli space of coherent torsion-free sheaves on $\mathbb{P}^n$ with fixed Chern numbers. All generic arrangements are semi-stable (and stable when $m \geq n+2$), and the Torelli Theorem mentioned above shows that the variety of semi-stable arrangements admits a birational morphism onto a subvariety of the moduli space of sheaves. We extend the Torelli theorem proving the injectivity on the set of semi-stable arrangements which contain a generic arrangement not osculating a normal rational curve and conjecture that the same is true for all semi-stable arrangements whose dual configurations of points in $\check{\mathbb{P}}^n$ does not lie on the set of nonsingular points of a stable normal rational curve. We check the conjecture in the case of ≤ 6 lines in the plane.

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2. The sheaf of logarithmic 1-forms

Let X be a nonsingular n-dimensional algebraic variety over a field k of characteristic 0 and D be an effective reduced Cartier divisor on X. Let $\Theta_{X/k}$ be the tangent sheaf on X defined by $\Theta_{X/k}(U) = \mathrm{Der}_k(O_X(U))$, the $\mathcal{O}_X(U)$-module of k-derivation of the coordinate ring $\mathcal{O}_X(U)$. Let $\phi_U = 0$ be a local equation of D on U. Define a submodule of $\Theta_{X/k}(U)$

$$\Theta_{X/k}(\log \phi_U) = \{\partial \in \mathrm{Der}_k(O_X(U)) : \partial(\phi_U) \in (\phi_U)\}.$$

Since $\partial(a\phi_U) = \partial(a)\phi_U + a\partial(\phi_U)$, this definition does not depend on a choice of a local equation. Since $\phi_U = g_{UV}\phi_V$ in $U \cap V$ and $\partial(\phi_U) = \partial(g_{UV})\phi_V + g_{UV}\partial(\phi_V)$ we see that the modules $\Theta_{X/k}(U)$ can be glued together to define a subsheaf $\Theta_{X/k}(\log D)$ of $\Theta_{X/k}$ and an exact sequence

$$0 \to \Theta_{X/k}(\log D) \to \Theta_{X/k} \to \mathcal{J}_D(D) \to 0, \tag{2.1}$$

where $\mathcal{J}_D$ is an ideal sheaf on $\mathcal{O}_D$ generated in each $\mathcal{O}_D(U)$ by $\partial(\phi_U), \partial \in \mathrm{Der}_k(O_X(U))$. In other words,

$$\mathcal{J}_D = \mathrm{Jacobian}(D) \cdot \mathcal{O}_D,$$

where Jacobian(D) is the *Jacobian ideal sheaf* in $\mathcal{O}_X$ generated in each $\mathcal{O}_X(U)$ by ϕ_U and $\partial(\phi_U), \partial \in \mathrm{Der}_k(O_X(U))$ (see [La, p. 181]). We set

$$\Omega^1_{X/k}(\log D) := \Theta_{X/k}(\log D)^* = \mathcal{H}om_X(\Theta_{X/k}(\log D), \mathcal{O}_X)$$

and call it the *sheaf of logarithmic* 1*-forms* of D. Since $\Theta_{X/k}$ is locally free, dualizing (2.1), we get an exact sequence

$$0 \to \Omega^1_{X/k} \to \Omega^1_{X/k}(\log D) \xrightarrow{\alpha} \mathcal{E}xt^1_X(\mathcal{J}_D(D), \mathcal{O}_X) \to 0. \tag{2.2}$$

It follows from (2.1) that depth $\Theta_{X/k}(\log D)_x \geq 2$ for any closed point x. Thus the sheaf $\Theta_{X/k}(\log D)$ is reflexive, hence

$$\Theta_{X/k}(\log D)^{**} \cong \Omega^1_{X/k}(\log D)^* \cong \Theta_{X/k}(\log D).$$

Let D^s be the closed subscheme of D defined by the sheaf of ideals $\mathcal{J}_D$ so that $\mathcal{O}_{D^s} = \mathcal{O}_D/\mathcal{J}_D$. It is supported on the singular locus of D.

Consider the exact sequence

$$0 \to \mathcal{J}_D(D) \to \mathcal{O}_D(D) \to \mathcal{O}_{D^s}(D) \to 0.$$

Applying the functor $\mathcal{H}om_X(?, \mathcal{O}_X)$ we get an exact sequence

$$0 \to \mathcal{E}xt^1_X(\mathcal{O}_D(D), \mathcal{O}_X) \to \mathcal{E}xt^1_X(\mathcal{J}_D(D), \mathcal{O}_X) \to \mathcal{E}xt^2_X(\mathcal{O}_{D^s}(D), \mathcal{O}_X) \to 0.$$

Let ω_Z denote the dualizing sheaf of a projective Cohen-Macaulay algebraic variety Z, the canonical sheaf $\mathcal{O}_Z(K_Z)$ if Z is nonsingular. By the Duality Theory,

$$\mathcal{E}xt^1_X(\mathcal{O}_D, \omega_X) \cong \omega_D \cong \omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_D(D).$$

Therefore,

$$\mathcal{E}xt^1_X(\mathcal{O}_D, \mathcal{O}_X) \cong \mathcal{E}xt^1_X(\mathcal{O}_D, \omega_X) \otimes_{\mathcal{O}_X} \omega_X^{-1} \cong \mathcal{O}_D(D). \tag{2.3}$$

This proves the following:

Proposition 2.1. *The sheaf* $\mathcal{E}xt^1_X(\mathcal{J}_D(D), \mathcal{O}_X)$ *from the exact sequence* (2.2) *fits in the following exact sequence*

$$0 \to \mathcal{O}_D \to \mathcal{E}xt^1_X(\mathcal{J}_D(D), \mathcal{O}_X) \to \mathcal{E}xt^2_X(\mathcal{O}_{D^s}(D), \mathcal{O}_X) \to 0.$$

It is known (see [Ei, Proposition 18.4 and Theorem 18.7]) that, for any coherent sheaf $\mathcal{F}$ on X supported on a closed subset of codimension c,

$$\mathcal{E}xt^q_X(\mathcal{F}, \mathcal{O}_X) = 0, \quad q < c. \tag{2.4}$$

Corollary 2.1. *Assume that* $codim_X D^s \geq 3$. *Then*

$$\mathcal{E}xt^1_X(\mathcal{J}_D(D), \mathcal{O}_X) \cong \mathcal{O}_D,$$

and we have an exact sequence

$$0 \to \Omega^1_{X/k} \to \Omega^1_{X/k}(\log D) \to \mathcal{O}_D \to 0.$$

Now let us recall the definition of the *adjoint ideal sheaf* adj(D) of D (see [La, p. 179]). Let $\mu : X' \to X$ be a birational morphism such that the proper inverse transform D' of D is nonsingular (a *log resolution* of D). Write $\mu^*(D) = D' + F$ for some divisor F on X' supported on the exceptional locus of μ. We have

$$\text{adj}(D) = \mu_*(\mathcal{O}_{X'}(K_{X'/X} - F)),$$

where $K_{X'/X} = K_{X'} - \mu^*(K_X)$ is the relative canonical divisor of μ.

Lemma 2.1. *Let*

$$\mathfrak{c}_D = adj(D) \cdot \mathcal{O}_D.$$

Then

(i) $\mathcal{J}_D \subset \mathfrak{c}_D$;

(ii) *if* $\nu : D' \to D$ *is a resolution of singularities of* D*, then*

$$\mathfrak{c}_D \otimes \omega_D = \nu_* \omega_{D'};$$

(iii) *if* $\nu : D' \to D$ *is the normalization morphism with smooth* D'*, then* $\mathfrak{c}_D$ *is the conductor ideal sheaf, i.e. the annihilator sheaf of* $\nu_*\mathcal{O}_{D'}/\mathcal{O}_D$;

(iv) $adj(D) = \mathcal{O}_X$ *if and only if* D *is normal and has at most rational singularities.*

Proof. See [La, pp. 179–181]. □

Proposition 2.2. *Let* $\nu : D' \to D$ *be a resolution of singularities of* D*. The sheaf* $\mathcal{E}xt^1_X(\mathcal{J}_D(D), \mathcal{O}_X)$ *from exact sequence* (2.2) *fits in the following exact sequence*

$$0 \to \nu_*\mathcal{O}_{D'} \to \mathcal{E}xt^1_X(\mathcal{J}_D(D), \mathcal{O}_X) \xrightarrow{\phi} \mathcal{E}xt^2_X((\mathfrak{c}_D/\mathcal{J}_D)(D), \mathcal{O}_X). \tag{2.5}$$

The map ϕ *is surjective if* $R^i\nu_*\mathcal{O}_{D'} = 0, i > 0$.

Proof. It follows from part (ii) of Lemma 2.1 that $\mathfrak{c}_D$ restricts to $\mathcal{O}_D$ on the nonsingular locus of D, and so the sheaf $\mathcal{J}_D$. This implies that $\mathfrak{c}_D/\mathcal{J}_D$ is supported on the closed subset of codimension ≥ 2 in X. By (2.4),

$$\mathcal{E}xt^1_X(\mathfrak{c}_D/\mathcal{J}_D, \mathcal{O}_X) = 0.$$

This gives an exact sequence

$$\begin{aligned} 0 &\to \mathcal{E}xt^1_X(\mathfrak{c}_D(D), \mathcal{O}_X) \to \mathcal{E}xt^1_X(\mathcal{J}_D(D), \mathcal{O}_X) \\ &\to \mathcal{E}xt^2_X((\mathfrak{c}_D/\mathcal{J}_D)(D), \mathcal{O}_X) \to \mathcal{E}xt^2_X(\mathfrak{c}_D(D), \mathcal{O}_X). \end{aligned} \tag{2.6}$$

By the adjunction formula, $\omega_D = \omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_D(D)$. Applying part (ii) of Lemma 2.1, we get

$$\mathfrak{c}_D(D) = \nu_*\omega_{D'} \otimes \omega_X^{-1}.$$

Hence

$$\mathcal{E}xt^i_X(\mathfrak{c}_D(D), \mathcal{O}_X) = \mathcal{E}xt^i_X(\nu_*\omega_{D'}, \omega_X). \tag{2.7}$$

Since $\nu_*\omega_{D'}$ does not depend on a choice of a resolution of singularities we may assume that ν comes from a log resolution $\mu : X' \to X$ of D, i.e. D' is the proper inverse transform $\mu^{-1}(D)$ of D and $\nu = \mu|D'$. We have

$$\mathcal{E}xt^1_{X'}(\omega_{D'}, \omega_{X'}) \cong \mathcal{O}_{D'},\ \mathcal{E}xt^i_{X'}(\omega_{D'}, \omega_{X'}) = 0,\ i \neq 1.$$

Applying Grauert-Riemenschneider's vanishing theorem

$$\nu_*\omega_{D'} \cong \omega_D, \ R^q\nu_*\omega_{D'} = 0, \ q > 0,$$

and the Duality Theorem for projective morphisms [Ha, Theorem 11.1], we obtain an isomorphism

$$\mathcal{E}xt^1_X(\nu_*\omega_{D'}, \omega_X) \cong \nu_*\mathcal{O}_{D'}, \ \mathcal{E}xt^i_X(\nu_*\omega_{D'}, \omega_X) = R^{i-1}\nu_*\mathcal{O}_{D'}, \ i \geq 2.$$

Now the assertion follows from (2.7) and exact sequence (2.6). □

Note that the condition $R^i\nu_*\mathcal{O}_{D'} = 0, i > 0$ is satisfied in one of the following cases

- D is a normal variety with rational singularities;
- D has smooth normalization.

Definition 2.1. Use (2.5) to identify $\nu_*\mathcal{O}_{D'}$ with a subsheaf of $\mathcal{E}xt^1_X(\mathcal{J}_D(D), \mathcal{O}_X)$ and set

$$\tilde{\Omega}^1_{X/k}(\log D) = \alpha^{-1}(\nu_*\mathcal{O}_{D'}),$$

where α is defined in (2.2).

By definition, we have an exact sequence

$$0 \to \Omega^1_{X/k} \to \tilde{\Omega}^1_{X/k}(\log D) \xrightarrow{\text{res}} \nu_*\mathcal{O}_{D'} \to 0. \tag{2.8}$$

We call this sequence the *residue* exact sequence. The reason for this name will be explained in the following example.

Also we have an exact sequence

$$0 \to \tilde{\Omega}^1_{X/k}(\log D) \to \Omega^1_{X/k}(\log D) \xrightarrow{\phi} \mathcal{E}xt^2_X((\mathfrak{c}_D/\mathcal{J}_D)(D), \mathcal{O}_X), \tag{2.9}$$

where the map ϕ is surjective if $R^i\nu_*\mathcal{O}_{D'} = 0$ for $i > 0$.

Since $\mathcal{E}xt^2_X((\mathfrak{c}_D/\mathcal{J}_D)(D), \mathcal{O}_X)$ is supported at a closed subset of codimension ≥ 2, we have

$$\tilde{\Omega}^1_{X/k}(\log D)^{**} \cong \Omega^1_{X/k}(\log D)^{**} = \Omega^1_{X/k}(\log D).$$

Proposition 2.3. *Suppose* $(\mathfrak{c}_D/\mathcal{J}_D)_x = \{0\}$ *for any point* $x \in D$ *with* $\dim \mathcal{O}_{D,x} = 1$. *Then*

$$\tilde{\Omega}^1_{X/k}(\log D) \cong \Omega^1_{X/k}(\log D). \tag{2.10}$$

The converse is true if $R^i\nu_*\mathcal{O}_{D'} = 0, i > 0$, *for some resolution of singularities* $\nu : D' \to D$.

Proof. If the condition is satisfied, the sheaf $\mathfrak{c}_D/\mathcal{J}_D$ is supported on a closed subset of D of codimension ≥ 3. By (2.4), $\mathcal{E}xt^2_X((\mathfrak{c}_D/\mathcal{J}_D)(D), \mathcal{O}_X) = 0$

and the first assertion follows from exact sequence (2.9). The same exact sequence implies that $\mathcal{E}xt^2_X((\mathfrak{c}_D/\mathcal{J}_D)(D), \mathcal{O}_X) = 0$ if (2.10) holds and $R^i\nu_*\mathcal{O}_{D'} = 0, i > 0$. Passing to stalks at points $x \in D$ of codimension 1, we use that $\mathrm{Ext}^2_A(M, A) = 0$ for a module M over a regular local ring of dimension 2 supported on the closed point implies $M = 0$. This easily follows from the fact that $\mathrm{Ext}^2_A(A/\mathfrak{m}, A) \neq 0$, where $A/\mathfrak{m}$ is the residue field of A. This proves the second assertion. □

Definition 2.2. A divisor D on X is called a *normal crossing divisor* at a point $x \in D$ if $\mathcal{O}_{D,x}$ is formally (or étale) isomorphic to the quotient of $\mathcal{O}_{X,x}$ by an ideal generated by $t_1 \dots t_k$, where $t_1, \dots, t_k$ is a subset of the set of local parameters in $\mathcal{O}_{X,x}$. We say that D is a normal crossing divisor in codimension $\leq k$ if D is a normal crossing divisor at any point x with $\dim \mathcal{O}_{X,x} \leq k$. A normal crossing divisor is a divisor which is normal crossing at each point.

It is clear from the definition that a normal crossing divisor in codimension ≤ 1 is just a reduced divisor. A normal crossing divisor in codimension ≤ 2 is a divisor which is, in codimension ≤ 2, formally isomorphic to the product of an affine space and an ordinary double point.

Corollary 2.2. *Suppose D is a normal crossing divisor in codimension ≤ 2. Then*

$$\tilde{\Omega}^1_{X/k}(\log D) \cong \Omega^1_{X/k}(\log D).$$

The converse is true if $R^i\nu_*\mathcal{O}_{D'} = 0, i > 0$, and for any point $x \in D$ of codimension 1 the formal neighborhood of the pair (D, X) at x is given by the equation $u^a - v^b = 0$, where u, v are local parameters of $\mathcal{O}_{X,x}$.

Proof. If D is a normal crossing divisor in codimension ≤ 2 then a local computation shows that condition (ii) in Proposition 2.3 is satisfied. To prove the converse we may assume that X is two-dimensional with local parameters u, v at a point x and D is given by local equation $f(u, v) = u^a - v^b = 0$ at x. Then

$$\begin{aligned}\text{length } \mathcal{O}_{D,x}/\mathcal{J}_{D,x} &= \text{length } \mathcal{O}_{X,x}/(f'_u, f'_v, f) = \text{length } \mathcal{O}_{X,x}/(u^{a-1}, v^{b-1}) \\ &= (a-1)(b-1).\end{aligned}$$

Now we use a well-known Jung-Milnor formula from the theory of curve singularities (see an algebraic proof in [Ri])

$$\mu = 2\delta - r + 1. \tag{2.11}$$

Here

$$\mu = \text{length } \mathcal{O}_{X,x}/(f'_u, f'_v), \ \delta = \text{length } \mathcal{O}_{D,x}/\mathfrak{c}_{D,x}$$

and r is the number of local branches of D at x. Write $a = md, b = nd$, where $(m, n) = 1$. Then

$$u^a - v^b = (u^m)^d - (v^n)^d = \prod_{i=1}^{d}(u^m - \epsilon^i v^n),$$

where ϵ is a primitive dth root of unity. It follows that $d = r$ is the number of branches. By Proposition 2.3, $\delta = \mu$, hence by (2.11), we get

$$(a-1)(b-1) = (md-1)(nd-1) = d-1.$$

This can happen only if $d = m = n = 1$ or $m = n = 1, d = 2$. In the first case D is nonsingular at x. In the second case, D is a normal crossing at x. □

Remark 1. It follows from a result of Zariski [Za] that the singularities $f = u^a - v^b = 0$ are characterized by the condition that $f \in (f'_u, f'_v)$, or equivalently, length $\mathcal{O}_{D,x}/\mathcal{J}_{D,x} =$ length $\mathcal{O}_{X,x}/(f'_u, f'_v)$.

Definition 2.3. Let Y be a nonsingular subvariety of a nonsingular variety X and D be a reduced divisor on X. We say that Y intersects D transversally if $\mathcal{T}or_1^X(\mathcal{O}_Y, \mathcal{O}_D) = 0$ and for any resolution of singularities $f : D' \to D$ the morphism $D' \times_X Y \to D \times_X Y = Y \cap D$ is a resolution of singularities.

Proposition 2.4. *Let Y be a nonsingular subvariety of X with the sheaf of ideals $\mathcal{I}$. Assume that Y intersects transversally D. There is an exact sequence*

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \tilde{\Omega}^1_{X/k}(\log D) \otimes_{\mathcal{O}_X} \mathcal{O}_Y \to \tilde{\Omega}^1_{Y/k}(\log D \cap Y) \to 0.$$

Proof. We have a standard exact sequence

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega^1_{X/k} \otimes_{\mathcal{O}_X} \mathcal{O}_Y \to \Omega^1_{Y/k} \to 0. \tag{2.12}$$

Consider the residue exact sequence for (X, D) and tensor it with $\mathcal{O}_Y$. Using the condition $\mathcal{T}or_1^X(\mathcal{O}_Y, \mathcal{O}_D) = 0$, we get an exact sequence

$$0 \to \Omega^1_{X/k} \otimes_{\mathcal{O}_X} \mathcal{O}_Y \to \tilde{\Omega}^1_{X/k}(\log D) \otimes_{\mathcal{O}_X} \mathcal{O}_Y \to \nu_*\mathcal{O}_{D'} \otimes_{\mathcal{O}_X} \mathcal{O}_Y \to 0.$$

Now consider the following commutative diagram

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{P} & \longrightarrow & i^*(\nu_*\mathcal{O}_{D'}) & \longrightarrow & \pi_*\mathcal{O}_{(D\cap Y)'} & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{R} & \longrightarrow & i^*\tilde{\Omega}^1_{X/k}(\log D) & \longrightarrow & \tilde{\Omega}^1_Y(\log Y \cap D) & \longrightarrow & \mathcal{S} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{I}/\mathcal{I}^2 & \longrightarrow & i^*\Omega^1_{X/k} & \longrightarrow & \Omega^1_{Y/k} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

Here $i : Y \hookrightarrow X$ is the inclusion morphism, and $\pi : (D \cap Y)' \to D \cap Y$ is a resolution of singularities which we can choose to be a composition of a resolution of singularities of $D' \times_X Y$ and the projection $D' \times_X Y \to D \times_X Y =$

$D \cap Y$. The middle horizontal exact sequence is obtained by dualizing a natural homomorphism

$$\Theta_{Y/k}(\log D \cap Y) \to \Theta_{X/k}(\log D) \otimes_{\mathcal{O}_X} \mathcal{O}_Y.$$

In the row above it, we have a natural morphism of sheaves

$$\alpha : \nu_* \mathcal{O}_{D'} \otimes_{\mathcal{O}_X} \mathcal{O}_Y \to \nu_* \mathcal{O}_{(D \cap Y)'}$$

which is the composition of an isomorphism $\nu_* \mathcal{O}_{D'} \otimes_{\mathcal{O}_X} \mathcal{O}_Y \to \nu_* \mathcal{O}_{D' \times_X Y}$ and a natural morphism $\nu_* \mathcal{O}_{(D' \times_X Y)} \to \pi_* \mathcal{O}_{(D \cap Y)'}$. By the transversality assumption, $D' \times_X Y \cong (D \cap Y)'$, hence α is an isomorphism. This implies that $\mathcal{P} = \mathcal{Q} = 0$ and the assertion follows. □

Example 2.1. In the case when D is a *strictly normal crossing divisor*, i.e. the union of smooth divisors $D_i, i = 1, \ldots, m$, which intersect transversally at each point, the sheaf $\Omega^1_{X/k}(\log D)$ and its exterior powers $\Omega^r_{X/k}(\log D)$ are well-known tools for defining the mixed Hodge structure on the complement $X \setminus D$. The sheaf $\Omega^1_{X/k}(\log D)$ is isomorphic to a subsheaf of the sheaf of rational differentials with poles on D_i of order at most one. If $z_i = 0, i = 1, \ldots, s$, is a local equation of D_i at a point x in the intersection $D_1 \cap \ldots \cap D_s$, then $\Omega^1_{X/k}(\log D)$ is locally free at x and is generated in an open neighborhood of x by meromorphic differential forms $d \log z_1, \ldots, d \log z_s, dz_{s+1}, \ldots, dz_n$. Let $\epsilon_i : D_i \to X$ be the closed embedding. The map of sheaves

$$\text{res} : \tilde{\Omega}^1_{X/k}(\log D) \to \nu_* \mathcal{O}_{D'} \cong \bigoplus_{i=1}^{s} \epsilon_{i*} \mathcal{O}_{D_i}$$

is given by the residue map

$$\text{res} \left(\sum_{i=1}^{s} a_i d \log z_i + \sum_{s+1}^{n} b_i dz_i \right) = (a_1 + (z_1), \ldots, a_s + (z_s), 0, \ldots, 0).$$

Since a normal crossing divisor is locally formally isomorphic to a simple normal crossing divisor, it follows that the sheaf $\Omega^1_{\mathbb{P}^n}(\log D)$ is locally free if D is a normal crossing divisor.

3. The logarithmic sheaf of a hyperplane arrangement

This is a special case of the construction from the previous section. First we assume that X is the projective space $\mathbb{P}^n$ over k and D is a hypersurface $V(f)$, where f is a homogeneous element of degree m in the polynomial algebra $S = k[T_0, \ldots, T_n]$. Let

$$\Omega^1_{S/k} = SdT_0 + \cdots + SdT_n \cong S(-1)^{n+1}$$

and

$$\text{Der}_{S/k} = S \tfrac{\partial}{\partial T_0} + \cdots + S \tfrac{\partial}{\partial T_n} \cong S(1)^{n+1}$$

be the graded S-module of differentials and the graded S-module of derivations, dual to each other. Recall that $S(a)_i = S_{a+i}$. Let $E = \sum_{i=0}^n T_i \frac{\partial}{\partial T_i}$ be the Euler derivation. It defines a homomorphism of $E : \Omega^1_{S/k} \to S$ of graded modules. Let $\bar{\Omega}_{S/k}$ be its kernel. The corresponding sheaf on $\mathbb{P}^n$ is the sheaf $\Omega^1_{\mathbb{P}^n}$ of regular differential 1-forms. Its dual is the tangent sheaf $\Theta_{\mathbb{P}^n}$ associated to the cokernel of the homomorphism $S \to \mathrm{Der}_{S/k}, a \mapsto aE$. Let

$$\mathrm{Der}_{S/k}(\log f) = \{\partial \in \mathrm{Der}_{S/k} : \partial(f) \in (f)\}.$$

Obviously, $E \in \mathrm{Der}_{S/k}(\log f)$. For any $\partial \in \mathrm{Der}_{S/k}(\log f)$, there exists a unique $p \in S$ such that $\partial(f) - pE(f) = 0$. Thus

$$\mathrm{Der}_{S/k}(\log f) = SE \oplus \mathrm{Der}^0_{S/k},$$

where $\mathrm{Der}^0_{S/k}$ is the kernel of the map $\mathrm{Der}_{S/k} \to S(m), \partial \mapsto \partial(f)$. Clearly,

$$\widetilde{\mathrm{Der}}^0_{S/k} \cong \Theta_{\mathbb{P}^n}(\log V(f)),$$

where $\tilde{}$ denotes the sheaf associated to a graded S-module. Since $f \in J_f$, the ideal sheaf $\tilde{J}_f$ on $\mathbb{P}^n$ can be considered as an ideal sheaf in $V(f)$ and it coincides with $\mathcal{J}_{V(f)}$ defined in the previous section.

From now on we will consider the case when $f = f_1 \cdots f_m$ is the product of distinct linear forms. The divisor $\mathcal{A} = V(f)$ is called an *arrangement of hyperplanes*. We set

$$\Omega^1(\mathcal{A}) := \Omega^1_{\mathbb{P}^n}(\log \mathcal{A}), \quad \tilde{\Omega}^1(\mathcal{A}) := \tilde{\Omega}^1_{\mathbb{P}^n}(\log \mathcal{A}).$$

It is customary in the theory of hyperplane arrangements to grade $\Omega^1_{S/k}$ and its dual by assigning the grade zero to each dT_i and $\frac{\partial}{\partial T_i}$. So their sheaf of logarithmic differentials is equal to $\Omega^1(\mathcal{A})(1)$.

Let $L_i = V(f_i), i = 1, \ldots, m$, so that $\mathcal{A} = L_1 \cup \ldots \cup L_m$. The normalization $\mathcal{A}'$ of $\mathcal{A}$ is isomorphic to the disjoint union of the L_i's. Thus it is smooth and the normalization morphism $\nu : \mathcal{A}' \to \mathcal{A}$ can be taken for a resolution of singularities of $\mathcal{A}$. We have

$$\nu_* \mathcal{O}_{\mathcal{A}'} = \bigoplus_{i=1}^m \epsilon_{i*} \mathcal{O}_{L_i}, \tag{3.1}$$

where $\epsilon_i : L_i \hookrightarrow \mathbb{P}^n$ is the inclusion morphism. Since $\omega_{L_i} = \mathcal{O}_{L_i}(-n)$, we have

$$\nu_* \omega_{\mathcal{A}'} = \nu_* \nu^* \mathcal{O}_{\mathbb{P}^n}(-n) = (\nu_* \mathcal{O}_{\mathcal{A}'})(-n) = \bigoplus_{i=1}^m \epsilon_{i*} \mathcal{O}_{L_i}(-n).$$

Thus

$$\mathfrak{c}_{\mathcal{A}}(\mathcal{A}) = \nu_* \omega_{\mathcal{A}'} \otimes \omega^{-1}_{\mathbb{P}^n} = \left(\bigoplus_{i=1}^m \epsilon_{i*} \mathcal{O}_{L_i}(-n) \right) \otimes \mathcal{O}_{\mathbb{P}^n}(n+1) = \bigoplus_{i=1}^m \epsilon_{i*} \mathcal{O}_{L_i}(1).$$

Since the normalization morphism is finite we have

$$R^i\nu_*\mathcal{O}_{\mathcal{A}'} = 0,\ i > 0.$$

The following exact sequences are just exact sequences (2.8) and (2.9) rewritten in our special situation.

$$0 \to \Omega^1_{\mathbb{P}^n} \to \tilde{\Omega}^1(\mathcal{A}) \xrightarrow{\text{res}} \bigoplus_{i=1}^m \epsilon_{i*}\mathcal{O}_{L_i} \to 0, \tag{3.2}$$

$$0 \to \tilde{\Omega}^1(\mathcal{A}) \to \Omega^1(\mathcal{A}) \to \mathcal{E}xt^2_{\mathbb{P}^n}((\mathfrak{c}_{\mathcal{A}}/\mathcal{J}_{\mathcal{A}})(m), \mathcal{O}_{\mathbb{P}^n}) \to 0. \tag{3.3}$$

Theorem 3.1. *Assume $m \geq n+2$. The sheaf $\tilde{\Omega}^1(\mathcal{A})$ admits a projective resolution*

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1)^{m-n-1} \to \mathcal{O}^{m-1}_{\mathbb{P}^n} \to \tilde{\Omega}^1(\mathcal{A}) \to 0.$$

Proof. Let

$$i : \mathbb{P}^n \to \mathbb{P}^{m-1},\ (t_0, \ldots, t_n) \mapsto (f_1, \ldots, f_m).$$

It is a closed embedding with the image a linear subspace of dimension n. Let $z_0, \ldots, z_{m-1}$ be projective coordinates in $\mathbb{P}^{m-1}$ and $\mathcal{B}$ be the arrangement of the coordinate hyperplanes. Obviously, $i^*(\mathcal{B}) = \mathcal{A}$. We apply Proposition 2.4. Formula (3.1) allows us to check the transversality condition. Thus we have an exact sequence

$$0 \to \mathcal{I}/\mathcal{I}^2 \to i^*\Omega^1_{\mathbb{P}^{m-1}}(\log V(z)) \to \tilde{\Omega}^1(\mathcal{A}) \to 0.$$

The ideal sheaf $\mathcal{I}$ of $i(\mathbb{P}^n)$ in $\mathbb{P}^{m-1}$ is associated to a free $k[z_0, \ldots, z_{m-1}]$-module generated by the subspace of linear polynomials spanned by $m-1-n$ linear independent linear relations between the functions $f_1, \ldots, f_m$. Thus

$$\mathcal{I}/\mathcal{I}^2 \cong \mathcal{O}^{m-n-1}_{\mathbb{P}^n}(-1).$$

It is easy to check that

$$\Omega^1_{\mathbb{P}^{m-1}}(\mathcal{B}) \cong \mathcal{O}^{m-1}_{\mathbb{P}^{m-1}}$$

(see [DK1, Proposition 2.10]). □

Recall that an arrangement $\mathcal{A}$ is called a *generic arrangement* if it is a simple normal crossing divisor.

Proposition 3.1. *The following assertions are equivalent*
(i) $\tilde{\Omega}^1(\mathcal{A})$ *is locally free*;
(ii) $\mathcal{A}$ *is a generic arrangement.*

Proof. It follows from Example 2.1 that (ii) implies (i). Assume (i) holds. Applying the residue exact sequence (3.2), we find that the sheaf $\nu_*\mathcal{O}_{\mathcal{A}'}$ is locally generated by n elements. Suppose $\mathcal{A}$ is not a normal crossing divisor.

Then there exists a closed point $x \in \mathbb{P}^n$ such that there are $s > n$ hyperplanes L_i passing through x. Without loss of generality we may assume that $x = (1, 0, \dots, 0)$ and the hyperplanes are given by linear equations $g_1, \dots, g_m$ in inhomogeneous coordinates $z_1, \dots, z_n$. By (3.1)

$$(\nu_* \mathcal{O}_{\mathcal{A}'})_x \cong \bigoplus_{i=1}^{s} (k[z_1, \dots, z_s]/(g_i))_{(z_1,\dots,z_n)}.$$

We have a surjection $\mathcal{O}^n_{X,x} \to (\nu_* \mathcal{O}_{\mathcal{A}'})_x$. After tensoring with $k[z_1, \dots, z_n]_{(z_1,\dots,z_n)}/(z_1, \dots, z_n)$, we get a surjection of vector spaces $k^n \to k^s$. This contradiction proves the assertion. □

Proposition 3.2. *The following assertions are equivalent*
(i) $\tilde{\Omega}^1(\mathcal{A}) \cong \Omega^1(\mathcal{A})$;
(ii) $\mathcal{A}$ *is a normal crossing divisor in codimension* ≤ 2.

Proof. This follows from Corollary 2.2 since, locally in codimension 2, the divisor D can be written by equation $u^a - v^a = 0$, where a is the number of hyperplanes in the arrangement $\mathcal{A}$ intersecting along a codimension 2 subspace. □

Corollary 3.1. *Suppose $\mathcal{A}$ is a normal crossing divisor in codimension* ≤ 2. *The following properties are equivalent*
(i) $\Omega^1(\mathcal{A})$ *is locally free*;
(ii) $\mathcal{A}$ *is a generic arrangement.*

Remark 2. Recall that an arrangement $\mathcal{A}$ is called *free* if the S-module $\mathrm{Der}_{S/k}(\log V(f))$ is free. Also $\mathcal{A}$ is called *locally free* if the sheaf $\Omega^1(\mathcal{A})$ is locally free. Of course, a free divisor is locally free but the converse is not true in general. If $n = 1$ any divisor is free but already in dimension 2 any reduced divisor is locally free but not necessary free. The assertion from Corollary 3.1 follows from [Zi] or [Yu], where it is proven that a free arrangement which is normal crossing in codimension ≤ 2 is a Boolean arrangement (i.e. consists of $n + 1$ linear independent hyperplanes). For any X from the lattice of the arrangement one considers the arrangement $\mathcal{A}_X$ of hyperplanes which contain X. It is known that an arrangement is locally free if and only if each $\mathcal{A}_X$ is free. The arrangement $\mathcal{A}$ is normal crossing if and only if each $\mathcal{A}_X$ is Boolean. Another simple proof of this fact follows easily from [MS], where the Chern polynomial of $\Omega^1(\mathcal{A})$ is computed for a locally free arrangement (see (4.5)).

4. Stability of Steiner sheaves

A coherent torsion-free sheaf $\mathcal{F}$ on $\mathbb{P}^n$ with a projective resolution

$$0 \to \mathcal{O}_{\mathbb{P}^n}(d)^a \to \mathcal{O}_{\mathbb{P}^n}(d+1)^b \to \mathcal{F} \to 0, \quad 0 < a < b,$$

is called a *Steiner sheaf* (see [DK1]).

Assume $m \geq n+2$. It follows from Theorem 3.1 that the sheaf $\mathcal{F} = \tilde{\Omega}^1(\mathcal{A})$ is a Steiner sheaf with the projective resolution

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1)^{m-n-1} \to \mathcal{O}_{\mathbb{P}^n}^{m-1} \to \mathcal{F} \to 0. \tag{4.1}$$

Let $\mathbb{P}^n = \mathbb{P}(V) = V \setminus \{0\}/k^*$ for some vector space V,

$$U = H^0(\mathbb{P}(V), \mathcal{F} \otimes \Omega^1_{\mathbb{P}(V)}(1)), \quad W = H^0(\mathbb{P}(V), \mathcal{F}).$$

One identifies U with $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}^{m-n-1})$ by tensoring (4.1) with $\Omega^1_{\mathbb{P}(V)}(1)$ and using the natural isomorphism $H^1(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}) \cong k$. Also one identifies W with $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}^{m-1})$. The map of sheaves $\mathcal{O}_{\mathbb{P}^n}(-1)^{m-n-1} \to \mathcal{O}_{\mathbb{P}^n}^{m-1}$ is defined by an injective linear map

$$t : V \to \mathrm{Hom}(U, W).$$

Conversely, one can reconstruct $\mathcal{F}$ from such a map as the differential $d_{-1,0}$ in the Beilinson spectral sequence (see [OSS]).

In our situation when $\mathcal{F} = \tilde{\Omega}^1(\mathcal{A})$, the proof of Theorem 3.1 shows that U is isomorphic the subspace of k^m which consists of relations between f_i's, W is isomorphic to the subspace of k^m equal to the kernel of the map $(a_1, \ldots, a_m) \to \sum a_i$. The linear map t is defined by the formula

$$t(v)((a_1, \ldots, a_m)) = \big(a_1 f_1(v), \ldots, a_m f_m(v)\big) \tag{4.2}$$

(cf. [DK1]). We will refer to $t_{\mathcal{A}} := t$ as the defining tensor of $\tilde{\Omega}^1(\mathcal{A})$. It could be considered as an element of the space $U^* \otimes V^* \otimes W$ and hence defines a divisor of multi-degree $(1,1,1)$ on $\mathbb{P}(U) \times \mathbb{P}(V) \times \mathbb{P}(W^*)$. We say that $t_{\mathcal{A}}$ is non-degenerate, if the divisor is a nonsingular subvariety. The following proposition follows easily from the definition.

Proposition 4.1. *$\tilde{\Omega}^1(\mathcal{A})$ is locally free if and only if the tensor $t_{\mathcal{A}}$ is non-degenerate.*

Let $\mathcal{F}$ be a torsion-free sheaf on $\mathbb{P}^n$. We identify its Chern classes with integers. It follows from (4.1) that the Steiner sheaf $\tilde{\Omega}^1(\mathcal{A})$ has the Chern polynomial

$$c_t(\tilde{\Omega}^1(\mathcal{A})) = 1/(1-t)^{m-1-n} = (1 + t + \cdots + t^n)^{m-1-n} \mod (t^{n+1}). \tag{4.3}$$

Twisting (4.1) by $\mathcal{O}_{\mathbb{P}^n}(1)$, we also get

$$c_t(\tilde{\Omega}^1(\mathcal{A})(1)) = (1+t)^{m-1} \mod (t^{n+1}) = \sum_{i=0}^{n} c_i(\Omega^1(\mathcal{A})) t^i (1+t)^{n-i}, \tag{4.4}$$

where the last equality uses a well-known relationship between the Chern polynomial of a sheaf and its Serre's twist. On the other hand, if $\Omega^1(\mathcal{A})$ is locally free, its Chern classes can be derived from [MS], Corollary 4.3:

$$P_{\mathcal{A}}(t) = (1+t) c_t(\tilde{\Omega}^1(\mathcal{A})(1)), \tag{4.5}$$

where $P_{\mathcal{A}}(t)$ is the Poincaré polynomial of the arrangement

$$P_{\mathcal{A}}(t) = \sum_{x \in \mathcal{L}} \mu(x)(-t)^{\mathrm{rank}(x)}.$$

Here $\mathcal{L}$ is the *lattice of the arrangement*, i.e. the partial ordered, by inclusion, set of non-empty subsets

$$L_I = L_{i_1} \cap \ldots \cap L_{i_s}, \quad I = \{i_1, \ldots, i_s\},$$

$\mu : \mathcal{L} \to \mathbb{Z}$ is the Möebius function of $\mathcal{L}$ defined by

$$\mu(L_\emptyset) = 1, \ \ \mu(L_I) = -\sum_{L_I \subset L_J} \mu(L_J),$$

and $\mathrm{rank}(L_I) = \mathrm{codim} L_I$.

For a generic arrangement, we have $P_{\mathcal{A}}(t) = (1+t)^m$ and formulas (4.4) and (4.5) agree.

Note that the Poincaré polynomial $\Pi_{\mathcal{A}}(t)$ of the corresponding central arrangement of affine hyperplanes in k^{n+1} is related to ours $P_{\mathcal{A}}(t)$ via the formula

$$\Pi_{\mathcal{A}}(t) = P_{\mathcal{A}}(t) - P_{\mathcal{A}}(-1)(-t)^{n+1}.$$

Example 4.1. Assume $n = 2$. Let $\mathcal{P}$ be the set of singular points of $\mathcal{A}$ (i.e. elements of $\mathcal{L}$ of rank 2). We have $\mu(x) = s(x) - 1$, where $s(x)$ is the number of lines through the point x. Then

$$P_{\mathcal{A}}(t) = 1 + mt + \sum_{x \in \mathcal{P}} (s(x) - 1)t^2.$$

Using (4.5), we get

$$\begin{aligned} c_1(\Omega^1(\mathcal{A})) &= m - 3, \\ c_2(\Omega^1(\mathcal{A})) &= \sum_{x \in \mathcal{P}} (s(x) - 1) - 2m + 3. \end{aligned} \tag{4.6}$$

It follows from (3.3) that

$$c_1(\tilde{\Omega}^1(\mathcal{A})) = c_1(\Omega^1(\mathcal{A}))$$

and

$$c_2(\Omega^1(\mathcal{A})/\tilde{\Omega}^1(\mathcal{A})) = c_2(\Omega^1(\mathcal{A})) - c_2(\tilde{\Omega}^1(\mathcal{A})) = \sum_{x \in \mathcal{P}} (s(x) - 1) - \binom{m}{2}. \tag{4.7}$$

The second Chern class of a sheaf $\mathcal{T}$ concentrated at a finite set of points is equal to $-h^0(\mathcal{T})$. Also, applying Theorem 3.1, we get

$$h^0(\tilde{\Omega}^1(\mathcal{A})) = m - 1, \quad h^1(\tilde{\Omega}^1(\mathcal{A})) = 0. \tag{4.8}$$

Now (3.3) gives

$$h^0(\Omega^1(\mathcal{A})) = m - 1 - \sum_{x \in \mathcal{P}} (s(x) - 1) + \tbinom{m}{2}, \quad h^1(\Omega^1(\mathcal{A})) = 0. \tag{4.9}$$

The rank $\mathcal{F}$ is the rank of the vector bundle obtained by restriction to some open subset of $\mathbb{P}^n$. Recall that $\mathcal{F}$ is called *semi-stable* (resp. *stable*) if for any proper subscheaf $\mathcal{F}' \subset \mathcal{F}$,

$$\frac{h_{\mathcal{F}'}(t)}{\text{rank } \mathcal{F}'} \leq \frac{h_{\mathcal{F}}(t)}{\text{rank } \mathcal{F}}, \quad \left(\text{resp.} \frac{h_{\mathcal{F}'}(t)}{\text{rank } \mathcal{F}'} < \frac{h_{\mathcal{F}}(t)}{\text{rank } \mathcal{F}}\right),$$

where $h_{\mathcal{F}}(t) = \chi(\mathbb{P}^n, \mathcal{F}(t))$ is the Hilbert polynomial of $\mathcal{F}(t)$ and the inequality means the inequality between the values of the polynomials for $t >> 0$.

Comparing the coefficients at t^{n-1}, we see that stability (resp. semi-stability) implies *slope-stability* $\mu(\mathcal{F}') < \mu(\mathcal{F})$ (resp.$\mu(\mathcal{F}') \leq \mu(\mathcal{F})$), where $\mu(\mathcal{F}) = \frac{c_1(\mathcal{F})}{\text{rank } \mathcal{F}}$ is the *slope* of $\mathcal{F}$. The slope-stability implies stability but slope-semi-stability does not imply semi-stability. In the case $n = 2$ and $\mathcal{F}$ is of rank r with Chern classes c_1 and c_2, we have

$$\frac{h_{\mathcal{F}}(t)}{r} = \frac{1}{2}t^2 + (\mu(\mathcal{F}) + 3)t + \frac{3}{2}\mu(\mathcal{F}) + \frac{1}{2r}(c_1^2 - 2c_2) + 1.$$

This shows that $\mu(\mathcal{F}) = \mu(\mathcal{F}')$ implies stability (resp. semi-stability) only if $\Delta(\mathcal{F}) < \Delta(\mathcal{F}')$ (resp. $\leq$), where

$$\Delta(\mathcal{F}) = \frac{1}{r}\left(c_2 - \frac{r-1}{2r}c_1^2\right) = \frac{1}{2r}(2c_2 - c_1^2) + \frac{1}{2}\mu(\mathcal{F})^2$$

is the *discriminant* of $\mathcal{F}$.

It is known that there is a coarse moduli space $\mathcal{M}_{\mathbb{P}^n}(r; c_t)$ of torsion-free semi-stable sheaves of rank r on $\mathbb{P}^n$ with fixed Chern polynomial c_t ([Ma]). It is a projective variety. If $n = r = 2$, we have

$$\dim \mathcal{M}_{\mathbb{P}^2}(2; c_1, c_2) = 4c_2 - c_1^2 - 3, \tag{4.10}$$

if the open subset of the moduli space representing stable sheaves is not empty. If any semi-stable sheaf is stable (e.g., if $(c_1, r) = 1)$), then $\mathcal{M}_{\mathbb{P}^n}(r; c_t)$ is a fine moduli space.

Proposition 4.2. *Assume $n > 1$. Any Steiner vector bundle $\mathcal{E}$ on $\mathbb{P}^n$ defined by an exact sequence*

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1)^{m-n-1} \to \mathcal{O}_{\mathbb{P}^n}^{m-1} \to \mathcal{E} \to 0$$

is a stable bundle of rank n with the Chern polynomial $c_t = (1-t)^{n-m+1}$.

Proof. It is enough to show that $\mathcal{E}$ is slope-stable. This was proven in [BS]. □

It follows from [DK1], Corollary 3.3, that Steiner bundles (twisted by $\mathcal{O}_{\mathbb{P}^n}(1)$) form an open subset $\mathcal{S}_{n,m}$ in an irreducible component of the moduli space $\mathcal{M}_{\mathbb{P}^n}(n; (1+t)^{m-1})$. If $n = 2$,

$$\dim \mathcal{S}_{2,m} = m(m-4).$$

The logarithmic bundles $\Omega^1(\mathcal{A})$ of generic arrangements on $\mathbb{P}^2$ depend on nm parameters. One proves that the map from the variety of general arrangements of m hyperplanes to the moduli space of vector bundles on $\mathbb{P}^n$ is a birational morphism for $m \geq n+2$. This was proved first in [DK1] for $m \geq 2n+3$ and improved later in [Va]. Thus for $n = 2$, only in the case $m = 6$ we get the equality of the dimensions.

Now let us consider the problem of stability of Steiner sheaves $\mathcal{F}$ on $\mathbb{P}^n = \mathbb{P}(V)$, not necessarily locally free. We assume that

$$\text{rank } \mathcal{F} = n,$$

hence $\mathcal{F}$ is given by an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1) \otimes U \to \mathcal{O}_{\mathbb{P}^n} \otimes W \to \mathcal{F} \to 0,$$

where $U \cong H^0(\mathbb{P}^n, \mathcal{F} \otimes \Omega_{\mathbb{P}^n}(1))$, $W \cong H^0(\mathbb{P}^n, \mathcal{F})$ and the sheaf $\mathcal{F}$ is determined by a tensor $t : V \to \text{Hom}(U, W)$. We fix vector spaces U and W of dimensions $m-1-n$ and $m-1$, respectively and consider the triples $(\mathcal{F}, a, b)$, where $\mathcal{F}$ is a Steiner sheaf and a, b are isomorphisms from above. Each such triple (a *Steiner triple*) is represented by a tensor t defining a point in $\mathbb{P}(U^* \otimes V^* \otimes W)$. The condition of non-degeneracy is defined by a non-vanishing of the hyperdeterminant. Recall from [GKZ] that the dual variety of $\mathbb{P}_k^{n_1} \otimes \cdots \otimes \mathbb{P}_k^{n_s}$, embedded by Segre, is a hypersurface if and only if $n_i \leq \sum_{j \neq i} n_j$ for any i. A tensor $t \in V_1 \otimes \cdots \otimes V_s$, where $\mathbb{P}^{n_i} = \mathbb{P}(V_i)$, defines a hyperplane section of the Segre variety. So, it is singular if only if the hyperdeterminant (which is an element of $\otimes_{i=1}^s V_i^*$) vanishes at t. In our case $n_1 + 1 = \dim U = m - 1 - n, n_2 + 1 = \dim V = n + 1, n_3 + 1 = \dim W = m - 1$, so $n_1 = n_2 + n_3 - 2n, n_2 = n_1 + n_3 + 2(m - n - 2), n_3 = n_1 + n_2$. Thus the hyperdeterminant exists if $m \geq n + 2$.

Let

$$X_{m,n} = \mathbb{P}(U^* \otimes V^* \otimes W) // \text{SL}(U) \times \text{SL}(W).$$

We can also view $X_{m,n}$ as the GIT-quotient of the Grassmannian of $m-1-n$-subspaces in $V^* \otimes W$:

$$X_{m,n} = G(m-1-n, V^* \otimes W) // \text{SL}(W).$$

The following result describes the set of semi-stable points in the Grassmannian $G(m-1-n, V^* \otimes W)$ with respect to the action of $\text{SL}(W)$ ([Ka], [Ca]).

Proposition 4.3. *A subspace $E \in G(m-1-n, V^* \otimes W)$ is semi-stable (resp. stable) if and only if for each proper linear subspace $W' \subset W$ we have*

$$\frac{\dim E \cap (W' \otimes V^*)}{\dim W'} \leq \frac{\dim E}{\dim W} \ (\textit{resp.} \ <)$$

Corollary 4.1. *Let $(\mathcal{F}, a, b)$ be a Steiner triple with the defining tensor $t \in U^* \otimes V^* \otimes W$. Assume that $\mathcal{F}$ is slope stable (resp. slope semi-stable). Then the tensor t, considered as a point in $G(m-1-n, V^* \otimes W)$ is stable (resp. semi-stable).*

Proof. Let $E \subset V^* \cap W$ considered as the image of U under the map $t : U \to V^* \otimes W$ defined by t. Let $U' = t^{-1}(E \cap W') \subset U$. It gives an exact sequence of sheaves

$$0 \to \mathcal{O}_{\mathbb{P}^n} \otimes U' \to \mathcal{O}_{\mathbb{P}^n}(1) \otimes W' \to \mathcal{F}' \to 0.$$

It is clear that $\mathcal{F}'(-1)$ is a subsheaf of the Steiner sheaf $\mathcal{F}$ with

$$\mu(\mathcal{F}'(-1)) = \frac{\dim U'}{\dim W' - \dim U'}.$$

Since $\mathcal{F}$ is slope stable (resp. slope semi-stable), we have

$$\frac{\dim U'}{\dim W' - \dim U'} \leq \mu(\mathcal{F}) = \frac{\dim U}{\dim W - \dim U} \text{ (resp. } <).$$

It is easy to see that this is equivalent to the condition of semi-stability (stability) from the previous proposition. ☐

Remark 3. The validity of the converse of the assertion in the previous corollary is unknown. It is true in the case when $m = n + 3$ and n is odd (see [Ca]).

Corollary 4.2. *Let $\mathcal{A}$ be an arrangement of m hyperplanes in $\mathbb{P}^n$ and $\mathcal{L}$ be its lattice. For any $x \in \mathcal{L}$ let $s(x)$ denote the number of hyperplanes containing x and let $r(x) = rank(x)$. Assume that there exists $x \in \mathcal{L}$ such that*

$$s(x) > \frac{m-1}{n}(r(x) - 1) + 1.$$

Then the Steiner log-sheaf $\tilde{\Omega}^1(\mathcal{A})$ is unstable (i.e. not semi-stable). If the equality holds, $\tilde{\Omega}^1(\mathcal{A})$ is not stable.

Proof. Assume such $x = L_I$ with $r(x) = r$ exists. Without loss of generality we may assume that the hyperplanes containing L_I are the hyperplanes $L_i = V(f_i), i = 1, \ldots, s$ and $f_1, \ldots, f_r$ are linearly independent. This implies that, for any $i = r+1, \ldots, s$, we can write $f_i = \sum_{j=1}^r a_{ij} f_j$. The corresponding relations span a subspace U' of U of dimension $s - r$. By definition of the defining tensor of $\mathcal{A}$, it maps U' to the subspace $V^* \otimes W'$ of $V^* \otimes W \subset V^* \otimes k^m$ generated by

$$(a_{r+11} f_1, \ldots, a_{r+1r} f_r, -f_{r+1}, 0, \ldots, 0), \ldots,$$
$$(a_{s1} f_1, \ldots, a_{sr} f_r, 0, \ldots, 0, -f_s, 0, \ldots, 0).$$

Thus, in the notation of Proposition 4.3, we have $\dim W' = s - 1$ and $\dim U' = s - r = \dim E \cap W \otimes V^*$ and

$$\frac{\dim E \cap (W' \otimes V^*)}{\dim W'} - \frac{\dim E}{\dim W}$$
$$= \frac{s-r}{s-1} - \frac{m-1-n}{m-1} = \frac{sn - n - (m-1)(r-1)}{(m-1)(s-1)}.$$

By assumption, the last number is positive, hence t is unstable. By Corollary 4.2, the sheaf $\tilde{\Omega}^1(\mathcal{A})$ is unstable. ☐

Proposition 4.4. *The sheaf $\tilde{\Omega}^1(\mathcal{A})$ is slope stable (resp. slope semi-stable) if and only if the sheaf $\Omega^1(\mathcal{A})$ is slope-stable (resp. slope semi-stable).*

Proof. More generally, let

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{K} \to 0$$

be an exact sequence of sheaves with rank $\mathcal{K} = 0$. Since $c_1(\mathcal{K}) = 0$ and rank $\mathcal{F} =$ rank $\mathcal{G}$, we have

$$\mu(\mathcal{F}) = \mu(\mathcal{G}).$$

Let $\mathcal{F}'$ be a subsheaf of $\mathcal{F}$ with $\mu(\mathcal{F}') > \mu(\mathcal{F})$, then $\mathcal{F}'$ is a subsheaf of $\mathcal{G}$ with $\mu(\mathcal{F}') > \mathcal{G}$. Thus $\mathcal{G}$ is unstable if $\mathcal{F}$ is. Conversely, if $\mathcal{G}'$ is a subsheaf of $\mathcal{G}$ with $\mu(\mathcal{G}') > \mu(\mathcal{G})$, we take $\mathcal{F}'$ to be the kernel of the projection to $\mathcal{K}$. Since $c_1(\mathcal{K}) = 0$, we have $\mu(\mathcal{F}') = \mu(\mathcal{G}') > \mu(G) = \mu(\mathcal{F})$. Hence $\mathcal{F}$ is unstable if $\mathcal{G}$ is. This shows that slope semi-stability of $\mathcal{F}$ is equivalent to slope semi-stability of $\mathcal{G}$. A similar proof, with replacing strict inequalities with non strict inequalities proves that slope stability of $\mathcal{F}$ is equivalent to slope stability of $\mathcal{G}$. We apply this to our situation using exact sequence (3.3). ☐

Definition 4.1. An arrangement of hyperplanes $\mathcal{A}$ is called *stable* (resp. *semi-stable*, resp. *unstable*) if the sheaf $\tilde{\Omega}^1(\mathcal{A})$, or, equivalently, the sheaf $\Omega^1(\mathcal{A})$ is stable (resp. semi-stable, resp. unstable).

Example 4.2. Let $\mathcal{A}$ be a free arrangement. In this case the module of differentials $\Omega^1_{S/k}(\log f)$ is free, hence isomorphic to a direct sum of modules of type $S(a_i)$. This shows that

$$\Omega^1(\mathcal{A}) \cong \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^n}(a_i). \tag{4.11}$$

Its slope is equal to $(a_1 + \cdots + a_n)/n$. Let us assume that $a_1 \leq \cdots \leq a_n$. Then the inequality $a_n \geq (a_1 + \cdots + a_n)/n$ shows that $\mu(\mathcal{O}_{\mathbb{P}^n}(a_n)) \geq \mu(\Omega^1(\mathcal{A}))$ with equality only in the case $a_1 = \cdots = a_n$. Hence $\Omega^1(\mathcal{A})$ is unstable unless $a_1 = \cdots = a_n$ in which case it is semi-stable.

Example 4.3. Take $n = 2$. The only interesting r is $r = 2$, i.e. x is a point in $\mathbb{P}^2$. We get that $s(x) > \frac{m-1}{2} + 1$ implies unstability. For example, if $m = 6$, we need 4 lines passing through x. One should compare it with an inductive sufficient condition for slope stability and slope semi-stability of the bundle $\Omega^1(\mathcal{A})$ from [Sch, Theorem 4.5]. Note that the condition $s(x) \leq 3$ for any x with $\text{rank}(x) = 2$ is not sufficient for semi-stability. The reflection arrangement of type A_3 (its dual set of points in $\check{\mathbb{P}}^2$ is the set of vertices of a complete quadrilateral) is free. By (4.6), $c_t(\Omega^1(\mathcal{A}))) = 1 + 3t + 2t^2 = (1 + t)(1 + 2t)$, hence $a_1 = 1, a_2 = 2$ in (4.11). This shows that $\Omega^1(\mathcal{A})$ is unstable. This also can be proved without appealing to the freeness of the arrangement. It is known ([OSS, p. 168]) that a vector bundle $\mathcal{E}$ on $\mathbb{P}^2$ is unstable if

$$8\Delta(\mathcal{E}) = 4c_2(\mathcal{E}) - c_1(\mathcal{E}) < 0.$$

By (4.6), this is equivalent to the inequality

$$4\sum_{x\in\mathcal{P}}(s(x)-1) - (m-1)(m+3) < 0. \tag{4.12}$$

In the case of A_3-arrangement, the left-hand-side is equal to $44 - 45 < 0$, so the sheaf $\Omega^1(\mathcal{A})$ is unstable.

Recall that for any arrangement $\mathcal{A}$ in $\mathbb{P}^n = \mathbb{P}(V)$ there is the *associated arrangement* $\mathcal{A}^{\mathrm{as}}$ (defined only up to projective equivalence) in $\mathbb{P}^{m-n-2} = \mathbb{P}(U)$ (see [DK1]). The corresponding sheaf $\tilde{\Omega}^1(\mathcal{A}^{\mathrm{as}})$ is the Steiner sheaf defined by the same tensor $t \in U^* \otimes V^* \otimes W$ with the role of U and V exchanged.

For any arrangement one defines the subset $D(\mathcal{A})$ of the set of subsets of $\{1,\ldots,m\}$ of cardinality $n+1$ which consists of subsets $(i_0,\ldots,i_n)$ such that $V(f_{i_0})\cap\ldots\cap V(f_{i_n}) \neq \emptyset$. In terms of the matrix of coordinates of the functions f_i, this is just the set of vanishing minors of maximal order. It follows from [DO], Lemma 1, p. 37, that the map $I \mapsto \{1,\ldots,m\}\setminus I$ is a bijection between the sets $D(\mathcal{A})$ and $D(\mathcal{A}^{\mathrm{as}})$. In particular, $\mathcal{A}$ is generic if and only if $\mathcal{A}^{\mathrm{as}}$ is generic.

Conjecture. $\tilde{\Omega}^1(\mathcal{A})$ is stable if and only if $\tilde{\Omega}^1(\mathcal{A})^{\mathrm{as}}$ is stable.

5. Unstable hyperplanes

Let $\mathrm{Ar}_{n,m}$ be the variety of arrangements of $m \geq n+2$ hyperplanes in $\mathbb{P}^n$. This is just an open Zariski subset of $(\check{\mathbb{P}}^n)^m/S_m$ or, equivalently, a locally closed subset of the projective space of polynomials of degree m which consists of products of m distinct linear polynomials. We denote by $\mathrm{Ar}^{\mathrm{ss}}_{n,m}$ (resp. $\mathrm{Ar}^{\mathrm{s}}_{n,m}$) the subset of semi-stable (resp. stable) arrangements. Let $\mathcal{S}_{n,m}$ be a connected component of the Maruyama moduli space $\mathcal{M}_{\mathbb{P}^n}(n,(1-t)^{n-m+1})$ which contains Steiner vector bundles defined by exact sequence (4.1). We have a map

$$\log : \mathrm{Ar}^{\mathrm{ss}}_{n,m} \to \mathcal{S}_{n,m}, \quad \mathcal{A} \mapsto \tilde{\Omega}^1(\mathcal{A}). \tag{5.1}$$

We have already mentioned that this map is injective on the subset of generic arrangements which do not osculate a normal rational curve of degree n (i.e. the corresponding points in the dual projective space do not lie on such a curve)([DK1], [Va]). The generic arrangements osculating a normal rational curve are blown down to the locus of Schwarzenberger bundles.

The main idea of Valles's proof is to reconstruct the hyperplanes from the arrangement as *unstable* hyperplanes of the sheaf $\tilde{\Omega}^1(\mathcal{A})$.

Definition 5.1. Let $\mathcal{F}$ be a Steiner sheaf of rank n on $\mathbb{P}^n$. A hyperplane L is called an *unstable* hyperplane of $\mathcal{F}$ if

$$H^{n-1}(L, \mathcal{F}(-n)|L) \neq \{0\}.$$

We denote by $W(\mathcal{F})$ the set of unstable hyperplanes of $\mathcal{F}$.

Here $\mathcal{F}|L$ is the scheme-theoretical restriction, i.e.

$$\mathcal{F}|L = i^*\mathcal{F} = F \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_L,$$

where $i : L \hookrightarrow \mathbb{P}^n$ is the inclusion map.

Proposition 5.1. *Let L be a hyperplane from a hyperplane arrangement $\mathcal{A}$. Then L is an unstable hyperplane of the sheaf $\tilde{\Omega}^1(\mathcal{A})$.*

Proof. Without loss of generality we may assume that $L = L_1$. We use the residue exact sequence (3.2). Tensoring it with $\mathcal{O}_L$ we obtain an exact sequence

$$(5.2) \qquad 0 \to \mathcal{T}or_1^{\mathbb{P}^n}(\mathcal{O}_L, \mathcal{O}_L) \xrightarrow{\alpha} \Omega^1_{\mathbb{P}^n}|L \to \tilde{\Omega}^1(\mathcal{A})|L \to \mathcal{O}_L \oplus \bigoplus_{i=2}^{m} \mathcal{O}_{L_t \cap L} \to 0.$$

Consider the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1) \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_L \to 0$$

corresponding to the inclusion of the ideal sheaf of L in $\mathcal{O}_{\mathbb{P}^n}$. Tensoring it with $\mathcal{O}_L$, we get an exact sequence

$$0 \to \mathcal{T}or_1^{\mathbb{P}^n}(\mathcal{O}_L, \mathcal{O}_L) \to \mathcal{O}_L(-1) \to \mathcal{O}_L \to \mathcal{O}_L \to 0.$$

This shows that $\mathcal{T}or_1^{\mathbb{P}^n}(\mathcal{O}_L, \mathcal{O}_L) \cong \mathcal{O}_L(-1)$. Using (2.12), it is easy to identify the cokernel of the map α with Ω^1_L. Thus we get an exact sequence

$$0 \to \Omega^1_L \to \tilde{\Omega}^1(\mathcal{A})|L \to \epsilon_{1*}\left(\mathcal{O}_L \oplus \bigoplus_{t=2}^{m} \mathcal{O}_{L_t \cap L}\right) \to 0.$$

Twisting by $\mathcal{O}_L(-n)$ and applying cohomology, we get a surjection

$$H^{n-1}(L, \tilde{\Omega}^1(\mathcal{A})(-n)|L) \to H^{n-1}\left(L, \mathcal{O}_L(-n) \oplus \bigoplus_{i=2}^{m} \mathcal{O}_{L_i \cap L}(-n)\right) = H^{n-1}(L, \mathcal{O}_L(-n)) = k.$$

This proves the assertion. □

Lemma 5.1. *Let $\mathcal{A}'$ be the arrangement obtained from an arrangement $\mathcal{A}$ of $m \geq n+3$ hyperplanes by deleting a hyperplane L. There exists an exact sequence*

$$0 \to \tilde{\Omega}^1(\mathcal{A}') \to \tilde{\Omega}^1(\mathcal{A}) \to \mathcal{O}_L \to 0.$$

Proof. The assertion probably follows from the residue exact sequence without the assumption on m, but this requires the verification that $\mathrm{res}^{-1}(\mathcal{O}_L)$ is isomorphic to $\tilde{\Omega}^1(\mathcal{A}')$, so we prefer to give a simpler proof. We use that $\tilde{\Omega}^1(\mathcal{A})$ and $\tilde{\Omega}^1(\mathcal{A}')$ are Steiner sheaves. We have a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^n}(-1) & \longrightarrow & \mathcal{O}_{\mathbb{P}^n} & \longrightarrow & \mathcal{O}_L & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^n}(-1)^{m-n-1} & \longrightarrow & \mathcal{O}_{\mathbb{P}^n}^{m-1} & \longrightarrow & \tilde{\Omega}^1(\mathcal{A}) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^n}(-1)^{m-n-2} & \longrightarrow & \mathcal{O}_{\mathbb{P}^n}^{m-2} & \longrightarrow & \tilde{\Omega}^1(\mathcal{A}') & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Here the top horizontal sequence is the exact sequence of the definition of the sheaf $\mathcal{O}_L$. The first two vertical exact sequences are obtained from composing the defining tensor $t_{\mathcal{A}} : V \to \mathrm{Hom}(U, W)$ of $\mathcal{A}$ with the restriction map $\mathrm{Hom}(U, W) \to \mathrm{Hom}(U', W')$, where $t_{\mathcal{A}'} : V \to \mathrm{Hom}(U', W')$ is the defining tensor of $\mathcal{A}'$. The right vertical sequence is the needed exact sequence. □

Proposition 5.2. *Let $\mathcal{A}'$ be the arrangement obtained from an arrangement $\mathcal{A}$ by deleting a hyperplane L. Then $W(\tilde{\Omega}^1(\mathcal{A})) \subset W(\tilde{\Omega}^1(\mathcal{A}')) \cup \{L\}$.*

Proof. It is enough to show that any $L' \in W(\tilde{\Omega}^1(\mathcal{A})) \setminus \{L\}$ belongs to $W(\tilde{\Omega}^1(\mathcal{A}'))$. Tensoring the exact sequence from the previous Lemma by $\mathcal{O}_{L'}(-n)$ we get an exact sequence

$$0 \to \tilde{\Omega}^1(\mathcal{A}')(-n)|L' \to \tilde{\Omega}^1(\mathcal{A})(-n)|L' \to \mathcal{O}_{L'\cap L}(-n) \to 0.$$

Taking cohomology, we get a surjection

$$H^{n-1}(L', \tilde{\Omega}^1(\mathcal{A}')(-n)|L') \to H^{n-1}(L', \tilde{\Omega}^1(\mathcal{A})(-n)|L').$$

This shows that $L' \in W(\tilde{\Omega}^1(\mathcal{A}'))$ if $L' \in W(\tilde{\Omega}^1(\mathcal{A}))$. □

In the case of general arrangements this result is Proposition 2.1 from [Va] and Theorem 3.13 from [AO] (where the inclusion is taken in scheme-theoretical sense, see below).

Corollary 5.1. *Assume $\mathcal{A} = \mathcal{A}' + L$, where $\mathcal{A}'$ is an arrangement such that $W(\tilde{\Omega}^1(\mathcal{A}'))$ consists of $m-1$ unstable hyperplanes. Then*

$$W(\tilde{\Omega}^1(\mathcal{A})) = W(\tilde{\Omega}^1(\mathcal{A}')) \cup \{L\}.$$

Proof. $W(\tilde{\Omega}^1(\mathcal{A}'))$ consists of hyperplanes from $\mathcal{A}'$. Thus $W(\tilde{\Omega}^1(\mathcal{A}')) \cup \{L\} \subset W(\tilde{\Omega}^1(\mathcal{A}))$. By Proposition 5.2, we have the opposite inclusion. □

The set $W(\mathcal{F})$ of unstable hyperplanes of a Steiner sheaf $\mathcal{F}$ has a natural structure of a closed subscheme of the dual projective space $\check{\mathbb{P}}^n$ (see [AO]). In fact, one can construct a closed subscheme of $\tilde{\mathcal{S}}_{n,m} \subset \mathcal{S}_{n,m} \times \check{\mathbb{P}}^n$ such that the projection

$$p : \tilde{\mathcal{S}}_{n,m} \to \mathcal{S}_{n,m}$$

has fibres isomorphic to the varieties $W(\mathcal{F})$ under the projection to the second factor. The image of p_1 is a proper closed subvariety. Let

$$p' : \widetilde{\mathrm{Ar}}^{\mathrm{ss}}_{n,m} \to \mathrm{Ar}^{\mathrm{ss}}_{n,m}$$

be the pull-back of the map p with respect to the map $\log : \mathrm{Ar}^{\mathrm{ss}}_{n,m} \to \mathcal{S}_{n,m}$. We know that over an open subset of generic arrangements which do not osculate a normal rational curve, the map p' is an unramified cover of degree m. Over the locus of generic arrangements osculating a normal rational curve the fibres are isomorphic to $\mathbb{P}^1_k$. It follows that there exists an open Zariski subset $U \subset Ar^{\mathrm{ss}}_{n,m}$ containing generic arrangements not osculating a normal rational curve such that, for any $\mathcal{F} \in U$, the scheme $W(\mathcal{F})$ is a reduced 0-dimensional and consists of m points.

Definition 5.2. An arrangement $\mathcal{A}$ of m hyperplanes is called a *Torelli arrangement* if $W(\tilde{\Omega}^1(\mathcal{A}))$ consists of m hyperplanes of $\mathcal{A}$.

Theorem 5.1. *Let U be the subset of $Ar^{ss}_{n,m}$ which consists of Torelli arrangements. Then U is an open subset of $Ar^{ss}_{n,m}$ and the map $\log : U \to \mathcal{S}_{n,m}$ is injective.*

Examples of Torelli arrangements are generic arrangements of $m \geq n+2$ which do not osculate a normal rational curve in $\mathbb{P}^n$ [Va]. It follows from Proposition 5.1 that any arrangement which contains a Torelli arrangement is a Torelli arrangement. In particular any arrangement which contains a generic arrangement $\mathcal{A}'$ with at least $n+2$ hyperplanes not osculating a normal rational curve is a Torelli arrangement.

Conjecture. A semi-stable arrangement of $m \geq n+2$ hyperplanes in $\mathbb{P}^n$ is a Torelli arrangement unless the corresponding points in $\check{\mathbb{P}}^n$ lie on a stable normal rational curve of degree n.

Recall that a stable normal rational curve in $\mathbb{P}^n$ is a connected reduced curve of arithmetic genus 0 and degree n in $\mathbb{P}^n$. It is the union of smooth rational curves $C_1, \ldots, C_s$ of degrees $d_1, \ldots, d_s$ satisfying the following conditions

(i) $n = d_1 + \cdots + d_s$;

(ii) each curve C_i spans a subspace $\langle C_i \rangle = \mathbb{P}(V_i)$ of $\mathbb{P}^n = \mathbb{P}(V)$ of dimension d_i;

(iii) $V = V_1 + \cdots + V_s$.

6. Line arrangements

Here we assume $n = 2$. Recall that a line L is called a *jumping line* of a rank 2 vector bundle $\mathcal{E}$ on $\mathbb{P}^2$ if the splitting type of the restriction of $\mathcal{E}$ to L is different from the splitting type of the restriction of $\mathcal{E}$ to a general line in the plane. This means that

$$\mathcal{E}|L \not\cong \begin{cases} \mathcal{O}_L(a) \oplus \mathcal{O}_L(a) & \text{if } c_1(\mathcal{E}) = 2a, \\ \mathcal{O}_L(a) \oplus \mathcal{O}_L(a-1) & \text{if } c_1(\mathcal{E}) = 2a-1. \end{cases}$$

Equivalently, $H^1(\mathcal{E}(-a-1)|L) \neq 0$ if $c_1(\mathcal{E}) = 2a$ and $H^1(\mathcal{E}(-a)|L) \neq 0$ if $c_1(\mathcal{E}) = 2a-1$. It is easy to see that $H^1(\mathcal{E}(-2)|L) = 0$ implies $H^1(\mathcal{E}(-2-s)|L) = 0$ for any $s \geq 0$. In [DK1] an unstable line of $\Omega^1(\mathcal{A})$ for a generic arrangement $\mathcal{A}$ was called a *super-jumping line*. Note that the notions of an unstable line of $\Omega^1(\mathcal{A})$ and a jumping line of $\Omega^1(\mathcal{A})$ coincide only if $m = 5$ or 6. The exact sequence (3.3) shows that any unstable line of $\tilde{\Omega}^1(\mathcal{A})$ not passing through its singular locus is a jumping line of $\Omega^1(\mathcal{A})$.

Let $\mathcal{M}_{\mathbb{P}^2}(2; c_1, c_2)$ be the moduli space of semi-stable sheaves of rank 2 on $\mathbb{P}^2$ with fixed Chern classes c_1, c_2. If there exists a stable vector bundle with these Chern classes (e.g. if $(c_1, c_2) = 1$) then it is an irreducible variety of dimension $4c_2 - c_1^2 - 3$ ([Ma], [Ba], [Hu]). Consider its boundary $\partial\mathcal{M}_{\mathbb{P}^2}(2; c_1, c_2)$ formed by sheaves which are not locally free. For any sheaf $\mathcal{F}$ from the boundary, the double dual sheaf $\mathcal{F}^{**}$ is a semi-stable vector bundle with the same c_1 and $c_2(\mathcal{F}^{**}) = c_2 - \delta$ for some $\delta \geq 0$. Let $\mathcal{M}_{\mathbb{P}^2}(2; c_1, c_2)^\delta$ be the subset of $\mathcal{M}_{\mathbb{P}^2}(2; c_1, c_2)$ which parametrizes isomorphism classes of such sheaves (or, more precisely, the corresponding S-equivalence classes if the sheaves are not stable but semi-stable). Since all bundles with $c_1^2 - 4c_2 > 0$ are known to be unstable (see [OSS, p. 168]),

$$\mathcal{M}_{\mathbb{P}^2}(2; c_1, c_2)^\delta = \emptyset, \quad \delta > 4c_2 - c_1^2.$$

Note that

$$\partial\mathcal{M}_{\mathbb{P}^2}(2; c_1, c_2) = \cup_{\delta > 0}\mathcal{M}_{\mathbb{P}^2}(2; c_1, c_2)^\delta.$$

Let

$$0 \to \mathcal{F} \to \mathcal{F}^{**} \to \mathcal{T} \to 0,$$

be the canonical exact sequence corresponding to the natural inclusion $\mathcal{F} \subset \mathcal{F}^{**}$. The sheaf $\mathcal{T}$ is concentrated at the set of singular points of $\mathcal{F}$. Let δ_x be the length of the $\mathcal{O}_{\mathbb{P}^2,x}$-module $\mathcal{T}_x$. Let

$$Z(\mathcal{F}) = \sum_{x \in \mathbb{P}^2} \delta_x x \in \mathrm{Sym}^\delta(\mathbb{P}^2)$$

be the corresponding point of the symmetric product of the plane. The set-theoretical union

$$\mathcal{M}_{\mathbb{P}^2}(2; c_1, c_2)^U = \coprod_{\delta \geq 0} \mathcal{M}_{\mathbb{P}^2}(2; c_1, c_2 - \delta)^0 \times \mathrm{Sym}^\delta(\mathbb{P}^2)$$

has a structure of a projective algebraic variety and is called the Uhlenbeck compactification of the moduli space of semi-stable vector bundles $\mathcal{M}_{\mathbb{P}^2}(c_1, c_2)^0$ (see [Li]). The natural map

$$\mathcal{M}_{\mathbb{P}^2}(2; c_1, c_2) \to \mathcal{M}_{\mathbb{P}^2}(2; c_1, c_2)^U, \quad \mathcal{F} \mapsto (\mathcal{F}^{**}, Z(\mathcal{F}))$$

is a morphism of algebraic varieties. Its fibre over a point $Z = \sum \delta_x x$ is isomorphic to the product of punctual quotient schemes $\mathrm{Quot}(2\delta_x)$ parametrizing quotient sheaves of $\mathcal{O}_{\mathbb{P}^2}^2$ concentrated at x and of length δ_x. It is an irreducible variety of dimension $2\delta_x - 1$. There is an open subset of $\mathcal{M}_{\mathbb{P}^2}(2; c_1, c_2)^U$ corresponding to points $Z = \sum_x \delta_x x$ such that $\delta_x \leq 1$. The pre-image of this set in $\mathcal{M}_{\mathbb{P}^2}(2; c_1, c_2)^\delta$ is an open subset of dimension equal to $\dim \mathcal{M}_{\mathbb{P}^2}(2; c_1, c_2 - \delta)$. It projection to $\mathcal{M}_{\mathbb{P}^2}(2; c_1, c_2 - \delta)^0$ has fibres of dimension 3δ.

Now let us specialize to our situation. Consider exact sequence (3.3)

$$0 \to \tilde{\Omega}^1(\mathcal{A}) \to \Omega^1(\mathcal{A}) \to \mathcal{T} \to 0,$$

where $\mathcal{T} = \mathcal{E}xt^2_{\mathbb{P}^2}(\mathfrak{c}_\mathcal{A}/\mathcal{J}_\mathcal{A}, \mathcal{O}_{\mathbb{P}^2})$. The stalks of $\mathfrak{c}_\mathcal{A}$ and $\mathcal{J}_\mathcal{A}$ are easy to compute using the Jung-Milnor formula from the proof of Corollary 2.2. We have

$$\mathrm{length}(\mathfrak{c}_\mathcal{A}/\mathcal{J}_\mathcal{A})_x = \tbinom{s(x)-1}{2}.$$

Since $\mathcal{E}xt^2_{\mathbb{P}^2}(k, \mathcal{O}_{\mathbb{P}^2}) \cong k$, this gives

$$\text{(6.1)} \qquad \mathrm{length}\ \mathcal{T}_x = \tbinom{s(x)-1}{2}.$$

We know from (4.6) that

$$h^0(\mathcal{T}) = \sum_{x \in \mathcal{P}} \mathrm{length}\ \mathcal{T}_x = \tbinom{m}{2} - \sum_{x \in \mathcal{P}} (s(x) - 1).$$

This gives a well-known combinatorial formula

$$\text{(6.2)} \qquad \tbinom{m}{2} - \sum_{x \in \mathcal{P}} (s(x) - 1) = \sum_{x \in \mathcal{P}} \tbinom{s(x)-1}{2}.$$

We set

$$\delta_x(\mathcal{A}) := \tbinom{s(x)-1}{2}, \quad \delta(\mathcal{A}) := \sum_{x \in \mathcal{P}} \delta_x(\mathcal{A}).$$

Note that $\delta(\mathcal{A}) = 0$ if and only if $\mathcal{A}$ is a generic arrangement. It follows from (4.6), that the numbers d and δ determine the Chern polynomial of $\Omega^1(\mathcal{A})$. Recall that the moduli space of Steiner sheaves $\mathcal{S}_{2,m}$ is equal to the moduli space $\mathcal{M}_{\mathbb{P}^2}(2; c_1, c_2)$, where $c_1 = m-3, c_2 = \tbinom{m-2}{2}$. Let $\mathcal{S}_{2,m}^\delta = \mathcal{M}_{\mathbb{P}^2}(2; c_1, c_2)^\delta$. Let $\mathrm{Ar}_{2,m}^{\mathrm{ss}}(\delta)$ be the set of semi-stable arrangements with fixed $\delta(\mathcal{A}) = \delta$. The restriction of the map (5.1) to $\mathrm{Ar}_{2,m}^{\mathrm{ss}}(\delta)$ defines a map

$$\log_\delta : \mathrm{Ar}_{2,m}^{\mathrm{ss}}(\delta) \to \mathcal{S}_{2,m}^\delta.$$

One can rewrite the condition of unstability from (4.12) in the form

$$\text{(6.3)} \qquad \mathrm{Ar}_{2,m}^{\mathrm{ss}}(\delta) = \emptyset, \quad \delta > \frac{(m-3)(m-1)}{5}.$$

We also know from above that

$$\mathrm{codim}_{\mathcal{S}_{2,m}}(\mathcal{S}_{2,m}^{\delta}) = \delta.$$

Also taking the double dual defines a morphism

$$u_\delta : \mathcal{S}_{2,m}^{\delta} \to \mathcal{M}_{\mathbb{P}^2}(2; m-3, \tbinom{m-2}{2} - \delta).$$

The composition

$$u_\delta \circ \log_\delta : \mathrm{Ar}_{2,m}^{\mathrm{ss}}(\delta) \to \mathcal{M}_{\mathbb{P}^2}(2; m-3, \tbinom{m-2}{2} - \delta)$$

is just the map $\mathcal{A} \mapsto \Omega^1(\mathcal{A})$. It is easy to see that $\mathrm{Ar}_{2,m}^{\mathrm{ss}}(1)$ is irreducible and of codimension 1 in $\mathrm{Ar}_{2,m}^{\mathrm{ss}}$. However, $\mathrm{Ar}_{2,m}^{\mathrm{ss}}(2)$ consists of two irreducible components, each of codimension 2. I do not know neither the number of irreducible component of $\mathrm{Ar}_{2,m}^{\mathrm{ss}}(\delta)$ not their codimension for arbitrary m and δ.

Remark 4. It follows from Schenk's inductive criterion of semi-stability [Sch] that all arrangements with $\delta(\mathcal{A}) = 1$ are stable for $m \geq 6$.

Example 6.1. Let $m = 4$. Here only generic arrangements are stable. The moduli space $\mathcal{M}_{\mathbb{P}^2}(2; 1, 1) \cong \mathcal{M}_{\mathbb{P}^2}(2; -1, 1)$ consists of one point, representing the sheaf $\Omega^1_{\mathbb{P}^2}(2)$. Thus

$$\tilde{\Omega}^1(\mathcal{A}) = \Omega^1(\mathcal{A}) \cong \Omega^1_{\mathbb{P}^2}(2) \cong \Theta_{\mathbb{P}^2}(-1).$$

The exact sequence

$$0 \to \mathcal{O}_L(-1) \to \Omega^1_{\mathbb{P}^2}|L \to \Omega_L \to 0$$

shows that

$$H^1(L, \Omega^1(\mathcal{A})(-2)|L) \cong H^1(L, \Omega^1_{\mathbb{P}^2}|L) \cong H^1(L, \Omega^1_L) \cong k.$$

Thus any line is an unstable line of $\Omega^1(\mathcal{A})$.

Example 6.2. Let $m = 5$. The moduli space $\mathcal{S}_{2,5} = \mathcal{M}_{\mathbb{P}^2}(2; 2, 3) \cong \mathcal{M}_{\mathbb{P}^2}(2; 0, 2)$ is a 5-dimensional variety. Its open subset $\mathcal{S}_{2,5}^0$ representing vector bundles is isomorphic to an open subset U of $\mathbb{P}^5$. If we identify the latter with the space of curves of degree 2 in the dual plane, then U is equal to the set of nonsingular conics and the isomorphism is defined by assigning to a vector bundle $\mathcal{E}$ its set of jumping lines (see [Ba]). The variety $\mathcal{M}_{\mathbb{P}^2}(2; 2, 2) \cong \mathcal{M}_{\mathbb{P}^2}(2; 0, 1)$ is 2-dimensional. A sheaf $\mathcal{F}$ from $\mathcal{M}_{\mathbb{P}^2}(2; 2, 2)$ is determined by an extension

$$0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{F} \to \mathcal{I}_A(2) \to 0,$$

where $\mathcal{I}_A$ is the ideal sheaf of a 0-dimensional closed subscheme in the plane with $h^0(\mathcal{O}_A) = 2$. It shows that $h^0(\mathcal{F}(-1)) \neq 0$, hence $\mathcal{F}$ contains a subsheaf

$\mathcal{O}_{\mathbb{P}^2}(1)$ of slope 1. Since $\mu(\mathcal{F}) = 1$, this shows that $\mathcal{M}_{\mathbb{P}^2}(2;2,2)$ represents the S-equivalence classes of semi-stable but not stable sheaves. Each such class consists of vector bundles represented uniquely (up to isomorphism) by an extension

$$0 \to \mathcal{O}_{\mathbb{P}^2}(1) \to \mathcal{E} \to \mathcal{I}_x(1) \to 0 \tag{6.4}$$

for some point x. The only non-locally free semi-stable sheaf in this class is the sheaf $\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{I}_x(1)$, where x is a point.

The variety $\mathcal{M}_{\mathbb{P}^2}(2;2,1) \cong \mathcal{M}_{\mathbb{P}^2}(2;0,0)$ is a one-point set. It represents the S-equivalence class of the sheaf $\mathcal{O}_{\mathbb{P}^2}(1)^2$.

Thus for a generic arrangement $\mathcal{A}$ of 5 lines we have $\tilde{\Omega}^1(\mathcal{A}) \cong \Omega^1(\mathcal{A})$ is the Schwarzenberger vector bundle with the curve of jumping lines equal to the unique nonsingular conic in the dual plane containing the five lines of the arrangement. The map $\mathrm{Ar}^{\mathrm{ss}}_{2,5}(0) \to \mathcal{M}_{\mathbb{P}^2}(2;2,3)^0 = U$ is a surjective map with 5-dimensional fibres.

The set $\mathrm{Ar}^{\mathrm{ss}}_{2,5}(1)$ consists of arrangement with one triple point. All these arrangements are semi-stable but not stable. The sheaf $\Omega^1(\mathcal{A})$ belongs to $\mathcal{M}_{\mathbb{P}^2}(2;2,2)$ and is S-equivalent to the sheaf $\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{I}_x(1)$, where x is a point. Observe that the two lines, say L_1, L_2 of $\mathcal{A}$ not passing through the triple point are jumping lines of $\tilde{\Omega}^1(\mathcal{A})$ and hence of $\Omega^1(\mathcal{A})$. The set of unstable lines of a sheaf given by an extension (6.4) is equal to the set of lines passing through x. This shows that $x = L_1 \cap L_2$.

Thus all arrangements with the same point of intersection of two lines L_0 and L_1 not passing through the triple point have bundle $\Omega^1(\mathcal{A})$ given by extension (6.4), where $x = L_0 \cap L_1$. The sheaf $\tilde{\Omega}^1(\mathcal{A})$ determines $\Omega^1(\mathcal{A})$ as its double dual, and determines the triple point y, as its singular point. So it determines a reducible conic in the dual plane, union of the line dual to the triple point and the line dual to the point $L_0 \cap L_1$. All arrangements defining the same conic have the same S-equivalence class of the sheaf $\tilde{\Omega}^1(\mathcal{A})$. It is represented by the sheaf $\mathcal{I}_x(1) \oplus \mathcal{I}_y(1)$. Since $\mathrm{Ext}^1_{\mathbb{P}^2}(\mathcal{I}_x(1), \mathcal{I}_y(1)) \cong k$ if $x \neq y$, we obtain that there is a unique nontrivial extension class of an extension

$$0 \to \mathcal{I}_x(1) \to \mathcal{F} \to \mathcal{I}_y(1) \to 0,$$

where $x \neq y$. Since $\Omega^1(\mathcal{A}) = \tilde{\Omega}^1(\mathcal{A})^{**} \ncong \mathcal{O}_{\mathbb{P}^2}(1)^2$, we conclude that that $\tilde{\Omega}^1(\mathcal{A})$ is given by a unique non-trivial extension

$$0 \to \mathcal{I}_x(1) \to \tilde{\Omega}^1(\mathcal{A}) \to \mathcal{I}_y(1) \to 0,$$

where x is the triple point and y is the intersection point of two lines not passing through x. Tensoring by $\mathcal{O}_L(-2)$ and using that, for any point $z \notin L$, we have an exact sequence

$$0 \to \mathcal{T}or_1^{\mathbb{P}^2}(\mathcal{O}_z, \mathcal{O}_L) \to \mathcal{I}_z \otimes_{\mathcal{O}_{\mathbb{P}^2}} \mathcal{O}_L \to \mathcal{O}_L(-1) \to 0, \tag{6.5}$$

we see that $W(\tilde{\Omega}^1(\mathcal{A}))$ consists of lines through x or y. It is the union of two lines in the dual plane.

Finally $\mathrm{Ar}^{\mathrm{ss}}_{2,5}(2)$ consists of arrangements with 2 triple points. The dual set of points lies on the union of two lines, three points on each line, one is the intersection point. The sheaf $\Omega^1(\mathcal{A})$ is S-equivalent to the sheaf $\mathcal{O}_{\mathbb{P}^2}(1)^2$ (in fact, it is isomorphic to this sheaf). It has no jumping lines. The sheaf $\tilde{\Omega}^1(\mathcal{A})$ is S-equivalent to the sheaf $\mathcal{I}_x(1) \oplus \mathcal{I}_y(1)$, where x, y are the triple points. As in the previous case we obtain that $\tilde{\Omega}^1(\mathcal{A})$ is given by a unique non-trivial extension

$$0 \to \mathcal{I}_x(1) \to \tilde{\Omega}^1(\mathcal{A}) \to \mathcal{I}_y(1) \to 0,$$

where x, y are the triple points of $\mathcal{A}$. The variety $W(\tilde{\Omega}^1(\mathcal{A}))$ is the union of two lines, dual to the points x, y. So, we see that all semi-stable arrangements of 5 lines are not Torelli arrangements. Of course they always lie on a conic.

Example 6.3. Let $m = 6$. In the case when $\mathcal{A}$ is a generic arrangements the vector bundle $\Omega^1(\mathcal{A})$ was extensively studied in [DK2]. Here we are interested in non-generic arrangements. Since $\mu(\tilde{\Omega}^1(\mathcal{A})) = 3/2$, all semi-stable arrangements are stable. Also we have $\dim \mathrm{Ar}_{2,6} = \dim \mathcal{S}_{2,6} = 12$, so the map

$$\log : \mathrm{Ar}^s_{2,6} \to \mathcal{S}_{2,6} = \mathcal{M}_{\mathbb{P}^2}(2; 3, 6) \cong \mathcal{M}_{\mathbb{P}^2}(2; -1, 4)$$

is a birational morphism which is an isomorphism on the set of Torelli arrangements.

Let $\mathcal{A} \in \mathrm{Ar}^s_{2,6}(1)$. The bundle $\Omega^1(\mathcal{A})$ belongs to the 8-dimensional variety $\mathcal{M}_{\mathbb{P}^2}(2; 3, 5) \cong \mathcal{M}_{\mathbb{P}^2}(2; -1, 3)$. The three lines from $\mathcal{A}$ which do not pass through the unique triple point $x \in \mathcal{A}$ are the jumping lines of $\Omega^1(\mathcal{A})$. It is known that a vector bundle $\mathcal{E}$ from $\mathcal{M}_{\mathbb{P}^2}(2; 3, 5)$ with 3 non-concurrent jumping lines L_1, L_2, L_3 is unique up to an automorphism of $\mathbb{P}^2$([Hu]). Its set of jumping lines is the set $\{L_1, L_2, L_3\}$ and it is given by an extension

$$0 \to \mathcal{O}_{\mathbb{P}^2}(1) \to \mathcal{E} \to \mathcal{I}_Z(2) \to 0, \tag{6.6}$$

where Z is a 0-dimensional reduced closed subscheme of $\mathbb{P}^2$ which consists of three points $p_{ij} = L_i \cap L_j$. Twisting by $\mathcal{O}_{\mathbb{P}^2}(-1)$ we see that

$$h^0(\mathcal{E}(-1)) = 1.$$

This shows that the extension is determined uniquely by the isomorphism class of $\mathcal{E}$. The set of non-isomorphic extensions as in (6.6) is naturally isomorphic to $E = \mathbb{P}(H^0(\mathcal{O}_Z)) \cong \mathbb{P}^2$. The open subspace of E which consists of sections non-vanishing at any point of Z corresponds to stable sheaves. They are all vector bundles. The isomorphism class of $\mathcal{E}$ is uniquely determined by Z and the class of the extension. Since the map $u \circ \log_1 : \mathrm{Ar}^s_{2,6}(1) \to \mathcal{M}_{\mathbb{P}^2}(2; 3, 5)$ is PGL(3)-equivariant, we obtain that any vector bundle from $\mathcal{M}_{\mathbb{P}^2}(2; 3, 5)$ is isomorphic to $\Omega^1(\mathcal{A})$ for some arrangement $\mathcal{A}$ with $\delta(\mathcal{A}) = 1$. It determines three lines of $\mathcal{A}$ not passing through the triple point.

Since any coherent sheaf $\mathcal{T}$ supported at one point x with $h^0(\mathcal{T}) = 1$ is isomorphic to the sheaf $\mathcal{O}_x$, the sheaf $\tilde{\Omega}^1(\mathcal{A})$ for such an arrangement is given by an extension (3.3)

$$0 \to \tilde{\Omega}^1(\mathcal{A}) \to \Omega^1(\mathcal{A}) \xrightarrow{\alpha} \mathcal{O}_x \to 0, \tag{6.7}$$

where x is the triple point of $\mathcal{A}$. The restriction of α to the subsheaf $\mathcal{O}_{\mathbb{P}^2}(1)$ from (6.6) is not zero. Indeed, otherwise we get that $\tilde{\Omega}^1(\mathcal{A})$ is given by an extension

$$0 \to \mathcal{O}_{\mathbb{P}^2}(1) \to \tilde{\Omega}^1(\mathcal{A}) \to \mathcal{I}_{Z\cup x}(2) \to 0. \tag{6.8}$$

Tensoring by $\mathcal{O}_{\mathbb{P}^2}(-1)$ we obtain that $h^0(\tilde{\Omega}^1(\mathcal{A})(-1)) = 1$. The residue exact sequence (3.2) shows that $h^0(\tilde{\Omega}^1(\mathcal{A})(-1)) = 0$. In fact, stable sheaves defined by extensions of type (6.8) define Hulsbergen vector bundles $\mathcal{E}$ with $h^0(\mathcal{E}(-1)) = 1$. They are not isomorphic to $\Omega^1(\mathcal{A})$ for any generic arrangement $\mathcal{A}$. Since α is not zero on $\mathcal{O}_{\mathbb{P}^2}(1)$ we see that $\tilde{\Omega}^1(\mathcal{A})$ is given by an extension

$$0 \to \mathcal{I}_x(1) \to \tilde{\Omega}^1(\mathcal{A}) \to \mathcal{I}_Z(2) \to 0, \tag{6.9}$$

where x is the triple point of $\mathcal{A}$, and Z is the set of intersection points of the lines not passing through x. A standard calculation shows that

$$\mathbb{P}(\mathrm{Ext}^1_{\mathbb{P}^2}(\mathcal{I}_Z(2), \mathcal{I}_x(1))) \cong \mathbb{P}^3.$$

Any arrangement of 6 lines with one triple point is a Torelli arrangement. Indeed, suppose L is an unstable line which is not a component of $\mathcal{A}$. By tensoring with $\mathcal{O}_L(-2)$, we easily see that L must contain the triple point. Since $W(\tilde{\Omega}^1(\mathcal{A}))$ cannot be a finite set of more than 6 points, $W(\tilde{\Omega}^1(\mathcal{A}))$ contains the pencil of lines through x. Let L_1 be a line from $\mathcal{A}$ from this pencil. Since the lines $L_2, \ldots, L_6$ form a generic arrangement osculating a nonsingular conic, we see that $W(\tilde{\Omega}^1(\mathcal{A} \setminus \{L_1\}))$ is the dual conic C. By Proposition 5.1, $W(\tilde{\Omega}^1(\mathcal{A})) \subset C \cup \{L_1\}$. This shows that $W(\tilde{\Omega}^1(\mathcal{A}))$ cannot contain a line. Counting parameters we see that any arrangement with one triple point is uniquely determined by the sheaf $\tilde{\Omega}^1(\mathcal{A})$ which is given by a unique extension (6.9). So the boundary $\mathrm{Ar}^1_{2,6}$ is birationally isomorphic to a $\mathbb{P}^2 \times \mathbb{P}^1$ fibration over $\mathcal{M}_{\mathbb{P}^2}(2;-1,3)'$, where $\mathcal{M}_{\mathbb{P}^2}(2;-1,3)'$ is the open subset of $\mathcal{M}_{\mathbb{P}^2}(2;-1,3)$ representing vector bundles with 3 non-concurrent jumping lines.

Let $\mathcal{A} \in \mathrm{Ar}^s_{2,6}(2)$ be an arrangement with two triple points x, y. There are two irreducible components of $\mathrm{Ar}^s_{2,6}(2)$, each one is of codimension 2 in $\mathrm{Ar}_{2,6}$. The first one F_1 consists of arrangements such that the line $\langle x, y\rangle$ is a component of $\mathcal{A}$. The second one F_2 consists of arrangements with each line passing through x or y. The vector bundle $\Omega^1(\mathcal{A})$ belongs to $\mathcal{M}_{\mathbb{P}^2}(2;3,4) \cong \mathcal{M}_{\mathbb{P}^2}(2;-1,2)$. The variety $\mathcal{M}_{\mathbb{P}^2}(2;-1,2)^0$ is explicitly described in [Hu]. It is isomorphic to the 4-dimensional variety of reducible but not multiple conics. The conic is the conic in $\check{\mathbb{P}}^2$ of jumping lines of the second kind of a bundle $\mathcal{E}$ from $\mathcal{M}_{\mathbb{P}^2}(2;3,4)$. Its singular point is the unique jumping line of $\mathcal{E}$. Each $\mathcal{E}$ is isomorphic to $\Omega^1(\mathcal{A})$ for some arrangement $\mathcal{A}$. If $\mathcal{A} \in F_1$ (resp. $\mathcal{A} \in F_2$), then the unique jumping line of $\Omega^1(\mathcal{A})$ is the line from $\mathcal{A}$ which does not pass through the triple points of $\mathcal{A}$ (resp. the line $\langle x, y\rangle$) (see [Sch]). We have an extension

$$0 \to \mathcal{O}_{\mathbb{P}^2}(1) \to \Omega^1(\mathcal{A}) \to \mathcal{I}_Z(2) \to 0, \tag{6.10}$$

where Z is a closed 0-dimensional subscheme of $\mathbb{P}^2$ with $h^0(\mathcal{O}_Z) = 2$ contained in the jumping line. All extension classes with fixed Z are parametrized by $\mathbb{P}^1$ and define isomorphic vector bundles. The two points of Z represent the curve of jumping lines of the second kind. So, we see that $\Omega^1(\mathcal{A})$ determines very little of $\mathcal{A}$.

As in the previous case, one can show that $\tilde{\Omega}^1(\mathcal{A})$ is defined by an extension

$$0 \to \mathcal{I}_{x,y}(1) \to \tilde{\Omega}^1(\mathcal{A}) \to \mathcal{I}_Z(2) \to 0. \tag{6.11}$$

All such extensions with fixed Z and x, y are parametrized by $\mathbb{P}^s$, where $s = 3 - \#(Z \cap \{x, y\})$. Each isomorphism class of sheaves is determined by a $\mathbb{P}^1$ of extensions.

Any arrangements from F_1 is a Torelli arrangement. The proof is similar to the case of arrangements with $\delta(\mathcal{A}) = 1$. We choose the conic osculating the lines from $\mathcal{A}$ different from the line $\langle x, y\rangle$. The sheaf $\tilde{\Omega}^1(\mathcal{A})$ is given by (6.11), where Z does not lie on the line $\langle x, y\rangle$.

For any arrangements $\mathcal{A}$ from F_2 with triple points x, y the sheaf $\Omega^1(\mathcal{A})$ has the unique jumping line $\langle x, y\rangle$. This shows that the image of the map $\log : F_2 \to \mathcal{M}_{\mathbb{P}^2}(2; -1, 2)$ is of dimension ≤ 2. Since $\tilde{\Omega}^1(\mathcal{A})$ is determined by $\Omega^1(\mathcal{A})$ and the surjective map $\Omega^1(\mathcal{A}) \to \mathcal{O}_{x,y}$ we see that the sheaves $\tilde{\Omega}^1(\mathcal{A})$ with fixed x, y depend on at most 4 parameters. Thus the arrangement $\mathcal{A}$ is not a Torelli arrangement.

Let $\mathcal{A} \in \mathrm{Ar}^s_{2,6}(3)$. The variety $\mathrm{Ar}^s_{2,6}(3)$ is an irreducible variety of dimension 8, it belongs to the closure of the irreducible component F_1 of $\mathrm{Ar}^s_{2,6}(3)$. The arrangement $\mathcal{A}$ has 3 triple points. In this case $\mathcal{M}_{\mathbb{P}^2}(2; 3, 3) \cong \mathcal{M}_{\mathbb{P}^2}(2; -1, 1)$ consists of one point represented by the bundle $\Omega^1_{\mathbb{P}^2}(3)$ with no jumping lines. So

$$\Omega^1(\mathcal{A}) \cong \Omega^1_{\mathbb{P}^2}(3) \cong \Theta_{\mathbb{P}^2}.$$

A nonzero section of $\Theta_{\mathbb{P}^2}$ defines an extension

$$0 \to \mathcal{O}_{\mathbb{P}^2} \to \Theta_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(3) \to 0.$$

The sheaf $\tilde{\Omega}^1(\mathcal{A})$ is isomorphic to the kernel of a surjective morphism of sheaves $\Omega^1(\mathcal{A}) \to \mathcal{O}_x \oplus \mathcal{O}_y \oplus \mathcal{O}_z$, where x, y, z are the triple points of $\mathcal{A}$. Arguing as in the previous cases, we obtain that $\tilde{\Omega}^1(\mathcal{A})$ is given by an extension

$$0 \to \mathcal{I}_{x,y,z} \to \tilde{\Omega}^1(\mathcal{A}) \to \mathcal{O}_{\mathbb{P}^2}(3) \to 0.$$

The classes of non-trivial extensions are parametrized by $\mathbb{P}^2$. The trivial extension is unstable. It is easy to see that any unstable line of $\tilde{\Omega}^1(\mathcal{A})$ must pass through one of the points x, y, z, i.e. $W(\tilde{\Omega}^1(\mathcal{A}))$ is contained in the union of three lines. On the other hand, after deleting the line $L = \langle x, y\rangle$ from $\mathcal{A}$, we obtain by Corollary 5.1 that $W(\tilde{\Omega}^1(\mathcal{A})) \subset W(\tilde{\Omega}^1(\mathcal{A}')) \cup \{L\}$, where $\mathcal{A}' \in \mathrm{Ar}_{2,5}(1)$. It follows from the previous example that the latter consists of two pencils of lines through z and the point $p = L_i \cap L_j$, where L_i, L_j are the lines from $\mathcal{A}'$ not passing through z. Now changing the pair x, y to x, z and y, z, and applying the same argument we see that $\mathcal{A}$ is a Torelli arrangement.

Our computations show that the only non-Torelli semi-stable arrangement of 6 lines is the arrangement whose dual points in $\check{\mathbb{P}}^2$ are nonsingular points of a conic, nonsingular if the arrangement is generic, and reducible otherwise. This confirms Conjecture 5.

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