

# LAGRANGIAN TENS OF PLANES, ENRIQUES SURFACES AND HOLOMORPHIC SYMPLECTIC FOURFOLDS

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*To the memory of Igor Rostislavovich Shafarevich*

ABSTRACT. Two 10-dimensional families of holomorphically symplectic fourfolds are associated to Fano models of Enriques surfaces in  $\mathbb{P}^5$ . The first one parametrizes the varieties of lines on smooth cubic hypersurfaces containing 10 mutually intersecting planes. The second one is a family of double EPW sextics introduced by K. O'Grady. The EPW sextics are associated to Lagrangian subspaces of the Plücker space of the Grassmannian  $G_2(\mathbb{P}^5)$  spanned by 10 mutually intersecting planes in  $\mathbb{P}^5$ . Also some results are obtained on the variety of general tens of mutually intersecting planes, not necessarily associated to Fano models of Enriques surfaces.

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## 1. INTRODUCTION

In this paper we study several constructions of holomorphic symplectic fourfolds arising naturally from complex Enriques surfaces. One of these constructions is of course straightforward: one considers the K3 cover  $Y$  of an Enriques surface  $S$  and then takes the Hilbert square  $Y^{[2]}$  known to have a natural structure of a holomorphic symplectic fourfold. The following two new constructions are more geometric and interesting. Both of them use a Fano model of an Enriques surface, a smooth surface of degree 10 in  $\mathbb{P}^5$ . A Fano model of a general Enriques surface comes equipped with ten elliptic pencils  $|F_i|$  of curves of degree 6 with  $F_i \cdot F_j = 4$  for  $i \neq j$ . It is known that each elliptic pencil on an Enriques surface contains 2 double fibers, each is a plane cubic curve on the Fano model. A choice of one such fiber in each pencil defines 10 planes  $\Lambda_1, \dots, \Lambda_{10}$ , the planes spanned by the cubics. Each pair of planes intersects at a point. This is our starting datum for the new constructions.

We show that, generically (in the sense of moduli of Enriques surfaces which we will make more precise in the paper), there exists a unique smooth cubic hypersurface  $\mathcal{C}$  in  $\mathbb{P}^5$  which contains these planes. The variety of lines on a smooth cubic fourfold is a known example of a holomorphic symplectic fourfold. This is our first holomorphically symplectic fourfold associated to the Fano model of an Enriques surface.

The second new construction is more elaborate. First, we view the 10 planes as a set of 10 points in the Grassmann variety  $\text{Gr}(2, \mathbb{P}^5)$  of planes in  $\mathbb{P}^5$ . In the Plücker embedding  $\text{Gr}(2, \mathbb{P}^5) \hookrightarrow \mathbb{P}(\wedge^2 \mathbb{C}^6) \cong \mathbb{P}^{19}$ , we consider ten vectors  $v_i \in \wedge^2 \mathbb{C}^6$  representing these ten planes. The natural symplectic form on  $\wedge^2 \mathbb{C}^6$  defined by the wedge product (plus a choice of a volume form

on  $\mathbb{C}^6$ ) allows one to interpret the condition that the ten planes intersect pairwise as the condition that the vectors  $v_1, \dots, v_{10}$  span an isotropic subspace  $A$ . Generically,  $\dim A = 10$ , that is,  $A$  is a Lagrangian subspace. Now we invoke a recent construction of K. O’Grady of a holomorphic symplectic fourfold arising from a Lagrangian subspace of  $\bigwedge^3 \mathbb{C}^6$ . He considers the locus of vectors  $v \in \mathbb{C}^6$  such that  $(v \wedge \bigwedge^2 \mathbb{C}^6) \cap A \neq \{0\}$ . The image of this set in  $\mathbb{P}^5$  is a hypersurface of degree 6, the so-called EPW sextic introduced by D. Eisenbud, S. Popescu and C. Walter in [24]. O’Grady shows that the hypersurface admits a double cover ramified along its singular locus, a surface of degree 40, and this double cover has a structure of a holomorphically symplectic fourfold. All of this is true under certain assumption of genericity of a choice of  $A$ . In our case, the Lagrangian subspace is certainly very special. In particular, the associated EPW sextic hypersurface is singular at additional 10 planes, the ten planes  $\Lambda_i$ . O’Grady’s double cover  $X$  of this EPW sextic is a singular symplectic variety, and we provide a description of its minimal resolution  $\tilde{X}$ , which is a holomorphically symplectic fourfold. The degree 40 surface part of the singular locus of the EPW sextic is conjecturally birational to an Enriques surface  $S$  (not isomorphic to the original one, in general). The ten planes give rise to 10 divisors on  $\tilde{X}$ , which are  $\mathbb{P}^1$ -bundles over some K3-surfaces (in general not isomorphic to the K3 double cover of the Enriques surface  $S$ ). We show that the generic holomorphically symplectic fourfold  $\tilde{X}$  obtained in this way is not birationally isomorphic to the variety of lines on the cubic fourfold  $\mathcal{C}$ , though the starting data are the same.

Some of the proofs rely, in part, on theoretical considerations involving theory of Enriques surfaces and the work of O’Grady on EPW sextics, but also on computer algebra computations, used to show that the dimension, Hilbert polynomial and certain conditions on the singularities or configuration of relevant varieties are as expected. Those conditions are verified by a Macaulay2 computation [28] for a special choice of initial data, which implies that they are valid generically. Another construction of a 10-dimensional family of holomorphically symplectic manifolds as resolutions of singularities of EPW sextics along 10 pairwise incident planes is due to A. Ferretti [26]. He obtains his family via deformation of the degenerate double EPW sextic associated to the Cayley quartic symmetroid surface. As it is known that the Cayley quartic symmetroid surface is birationally isomorphic to the K3-cover of a non-generic Enriques surface, containing a smooth rational curve, Ferretti’s construction is also somehow related to Enriques surfaces. Though we were unable to find a geometric way of identification of the two 10-dimensional families, we checked that they are isomorphic by using the Global Torelli Theorem.

In the course of our study we came upon the following interesting problem in projective geometry first considered by Ugo Morin in 1932 [44], [45]. A median is a linear subspace in  $\mathbb{P}^{2n+1}$  of dimension  $n$ . What are maximal sets

of medians in which each two medians intersect? Morin was able to classify such infinite families and he writes in the introduction to his paper [44] that he did not undertake the classification of finite maximal sets of pairwise intersecting planes in  $\mathbb{P}^5$  because of encountering considerable difficulties.<sup>1</sup> A count of constants suggests that the number of planes in such a family should be 10 and the expected dimension of the variety parametrizing such families should be 45. We show that the expected situation is the one happening in reality for the tens of planes arising from Enriques surfaces as above. But there are examples showing that in general such estimates are wrong. The preprint version of this paper that was circulated in 2010 contained descriptions of Morin infinite complete families and other examples, among which one example of a maximal family of 13 mutually incident planes in  $\mathbb{P}^5$  (Example 6.15). We also reported on this work in several conferences, see e. g. [18]. Since then there has been a substantial progress about the Morin Problem, see [55], [26], [36], [25]. (Igor:added a reference)

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## 2. FANO MODELS OF ENRIQUES SURFACES

We take for the ground field the field of complex numbers although many geometric constructions in the paper are valid for any algebraically closed field of characteristic  $\neq 2$  or even characteristic free.

**2.1. Markings and supermarkings.** We refer for many general facts about Enriques surfaces to [13] or [21]. Let  $S$  be an Enriques surface and  $\text{Pic}(S)$  its Picard group. It is known that  $\text{Pic}(S)$  coincides with the Néron-Severi group and its torsion subgroup is generated by the canonical class  $K_S$ .

The group of divisor classes modulo the numerical equivalence

$$\text{Num}(S) = \text{Pic}(S)/(K_S)$$

is equipped with the symmetric bilinear form defined by the intersection pairing on the surface. It is a unimodular even quadratic lattice of rank 10 and signature  $(1, 9)$ . As such it must be isomorphic to the orthogonal sum

$$E = E_8 \oplus U,$$

where  $E_8$  is the unique negative definite unimodular even lattice of rank 8 and  $U$  is a hyperbolic plane over  $\mathbb{Z}$ . One can realize  $E$  as a primitive

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<sup>1</sup>La classificazione dei sistemi completi formati con un numero *finito* di piano non è stato effettuato, poichè presentava notevoli complicazioni.

sublattice of the standard odd unimodular hyperbolic lattice

$$\boxed{1.1} \quad (2.1) \quad l^{1,10} = \mathbb{Z}\mathbf{e}_0 + \mathbb{Z}\mathbf{e}_1 + \cdots + \mathbb{Z}\mathbf{e}_{10}$$

where  $\mathbf{e}_0^2 = 1, \mathbf{e}_i^2 = -1, i > 0, \mathbf{e}_i \cdot \mathbf{e}_j = 0, i \neq j$ . The vector

$$\mathbf{k}_{10} = -3\mathbf{e}_0 + \mathbf{e}_1 + \cdots + \mathbf{e}_{10}$$

satisfies  $\mathbf{k}_{10}^2 = -1$  and

$$\mathbf{E} = \mathbf{k}_{10}^\perp \subset l^{1,10}.$$

The vectors

$$\boxed{\text{fi}} \quad (2.2) \quad \mathbf{f}_i = \mathbf{e}_i - \mathbf{k}_{10} = 3\mathbf{e}_0 - \sum_{j=1, j \neq i}^{10} \mathbf{e}_j, \quad i = 1, \dots, 10,$$

form an *isotropic 10-sequence* in  $\mathbf{k}_{10}^\perp$ , i.e. an ordered set of 10 isotropic vectors satisfying  $\mathbf{f}_i \cdot \mathbf{f}_j = 1, i \neq j$ .

Adding up the vectors  $\mathbf{f}_i$ , we obtain

$$\boxed{\text{defDelta}} \quad (2.3) \quad \mathbf{f}_1 + \cdots + \mathbf{f}_{10} = 3\mathbf{\Delta},$$

where

$$\mathbf{\Delta} = 10\mathbf{e}_0 - 3\mathbf{e}_1 - \cdots - 3\mathbf{e}_{10} \in \mathbf{E}.$$

Note that one can reconstruct the standard basis in  $l^{1,10}$  from the isotropic 10-sequence using that

$$\mathbf{e}_0 = \mathbf{\Delta} + 3\mathbf{k}_{10}, \quad \mathbf{e}_i = \mathbf{f}_i + \mathbf{k}_{10}, \quad i = 1, \dots, 10.$$

A basis in  $\mathbf{E}$  can be defined by the vectors

$$\boxed{\text{stbasis}} \quad (2.4) \quad \alpha_0 = \mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3, \quad \alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}, \quad i = 1, \dots, 9,$$

with the intersection matrix  $(\alpha_i \cdot \alpha_j)$  described by the following Dynkin diagram

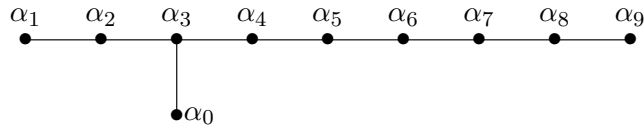


FIGURE 1. Enriques lattice

enriqueslattice

Let  $W$  be the group of orthogonal transformations of  $l^{1,10}$  which fixes  $\mathbf{k}_{10}$  and leaves the connected components of the cone  $\{x \in l^{1,10} \otimes \mathbb{R} : x^2 > 0\}$  invariant. Restricting isometries from  $W$  to isometries of  $\mathbf{k}_{10}^\perp$  defines a homomorphism from  $W$  to a subgroup  $O(\mathbf{E})'$  of index 2 of the orthogonal group  $O(\mathbf{E})$  of the lattice  $\mathbf{E}$  that consists of isometries that leave the connected components of the cone  $\{x \in \mathbf{E} \otimes \mathbb{R} : x^2 > 0\}$  invariant. Since  $\mathbf{E}$  is a unimodular sublattice of a unimodular lattice, this homomorphism is surjective. We have

$$(2.5) \quad O(\mathbf{E}) = O(\mathbf{E})' \times \{\pm \text{id}_{\mathbf{E}}\},$$

where  $\text{id}_{\mathbf{E}}$  is the identity isometry of  $\mathbf{E}$ .

The orthogonal summand  $U$  of  $E$  is spanned by  $\alpha_9 = e_9 - e_{10}$  and

$$f = 3\alpha_0 + 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8.$$

Each vector  $\alpha_i$  defines a reflection isometry

$$s_{\alpha_i} : x \mapsto x + (x, \alpha_i)\alpha_i$$

of  $\mathbb{P}^{1,10}$  which leaves  $\mathbf{k}_{10}$  invariant.

The group  $W$  is isomorphic to the Coxeter group  $W_{2,3,7}$  with the set of generators  $s_{\alpha_i}$ . This follows from the general facts about reflection groups (see, for example, [DolgachevBAMS, 4.3]). The Dynkin diagram from above becomes the Coxeter graph of the reflection group  $W_{2,3,7}$ .

Let us consider  $\mathbf{D} = E/2E$  as a vector space over  $\mathbb{F}_2$  of dimension 10. We equip  $\mathbf{D}$  with a quadratic form defined by

$$q(v + 2E) = \frac{1}{2}v^2 \pmod{2}.$$

The quadratic form is even in the sense that its Arf invariant is equal to 0, or, equivalently,  $\#q^{-1}(0) = 2^4(2^5 + 1) = 528$ , the other possibility being  $\#q^{-1}(0) = 2^4(2^5 - 1) = 496$ . In an appropriate basis the quadratic form is given by the standard quadratic form  $q = x_1x_2 + \cdots + x_9x_{10}$ . We denote its orthogonal group by  $O^+(10, \mathbb{F}_2)$ .

We have a natural surjective homomorphism of orthogonal groups

$$\boxed{\text{hom1}} \quad (2.6) \quad W \rightarrow O(E) \rightarrow O(\mathbf{D}, q) \cong O^+(10, \mathbb{F}_2).$$

Its kernel is denoted by  $W(2)$ .

Let

$$\overline{W} = \mathbf{D} \rtimes O(\mathbf{D}, q) \cong \mathbb{F}_2^{10} \rtimes O^+(10, \mathbb{F}_2) := \text{AO}^+(10, \mathbb{F}_2),$$

where the group  $O(\mathbf{D}, q)$  acts naturally on  $\mathbf{D}$ . An easy lemma from the theory of abelian groups tells us that

$$\boxed{\text{display1}} \quad (2.7) \quad \text{Aut}(E \oplus \mathbb{F}_2) \cong \text{Hom}(E, \mathbb{F}_2) \rtimes \text{Aut}(E) \cong \mathbf{D} \rtimes \text{Aut}(E),$$

where we identify  $\mathbf{D}$  with  $\text{Hom}(E, \mathbb{F}_2)$  by using the symmetric bilinear form induced by the intersection form on  $E$ .

The following lemma follows from (2.7).

**Lemma 2.1.** *Let  $O(\text{Pic}(S))$  be the group of automorphisms of  $\text{Pic}(S)$  preserving the intersection form. Then a choice of an isomorphism  $\phi : \text{Num}(S) \cong E$  and a splitting  $s : \text{Num}(S) \rightarrow \text{Pic}(S)$  defines an isomorphism of groups*

$$O(\text{Pic}(S)) \cong \mathbf{D} \rtimes O(E),$$

where  $O(E)$  acts on  $\mathbf{D}$  via its natural action on  $E/2E$ .

Let  $R$  be a smooth rational curve on  $X$ . We will often call it a  $(-2)$ -curve (since, by the adjunction formula,  $R^2 = -2$ ). Obviously the linear system  $|R|$  consists only of  $R$ , so we can identify  $R$  with its divisor class. Also observe that  $|R + K_S| = \emptyset$  because the linear system  $|2R| = |2(R + K_S)| =$

$\{2R\}$ . Thus we can also identify  $R$  with its numerical class. Each  $(-2)$ -curve defines a “reflection” automorphism of  $\text{Pic}(S)$

$$s_R : x \mapsto x + (x \cdot R)R.$$

The group generated by such reflections is a subgroup of  $\text{O}(\text{Pic}(S))$  and, under natural homomorphism  $\text{O}(\text{Pic}(S)) \rightarrow \text{O}(\text{Num}(S)) \cong \text{O}(\mathbf{E})$ , can be identified with a reflection subgroup of  $\text{O}(\mathbf{E})$ . We denote this subgroup by  $W_S^{\text{nod}}$ .

Let  $\text{Aut}(S)$  be the group of automorphisms of  $S$  and

$$\mathfrak{r} : \text{Aut}(S) \rightarrow \text{O}(\text{Pic}(S)), g \rightarrow g^*,$$

its natural representation in the orthogonal group of  $\text{O}(\text{Pic}(S))$ . Its kernel is a finite group (see [\[21, p.20\]](#), [DolgachevBrief](#)). We denote by

$$\text{Aut}(S)^* \subset \text{O}(\text{Pic}(S))$$

the image of the homomorphism  $\mathfrak{r}$  and by

$$\text{Aut}(S)^{**} \subset \text{O}(\text{Num}(S))$$

its image in  $\text{O}(\text{Num}(S))$ . The following facts are well-known (see, for example, [\[17, Theorem 5.12\]](#), [DolgachevBMS](#)).

- $\text{Aut}(S)^{**}$  is contained in  $\text{O}(\text{Num}(S))' \cong \text{O}(\mathbf{E})'$ ;
- $\text{Aut}(S)^{**} \cap W_S^{\text{nod}} = \{1\}$ ;
- $\text{Aut}(S)^{**} \cdot W_S^{\text{nod}} = W_S^{\text{nod}} \rtimes \text{Aut}(S)^{**}$ , where  $\text{Aut}(S)^{**}$  acts on  $W_S^{\text{nod}}$  by conjugation  $s_R \mapsto g \circ s_R \circ g^{-1} = s_{g(R)}$ ;
- $W_S^{\text{nod}} \rtimes \text{Aut}(S)^{**}$  is a subgroup of finite index in  $\text{O}(\text{Num}(S))$ .

It is known that, for an *unnodal Enriques surface* (i.e. not containing  $(-2)$ -curves), the groups  $\text{Ker}(\mathfrak{r})$  and  $W_S^{\text{nod}}$  are trivial, so the natural homomorphisms  $\text{Aut}(S) \rightarrow \text{Aut}(S)^{**}$  and  $\text{Aut}(S) \rightarrow \text{Aut}(S)^*$  are bijective. Moreover, the group  $\text{Aut}(S)^{**}$  contains the 2-congruence subgroup

$$\text{O}(\text{Num}(S))'(2) := \text{Ker}(\text{O}(\text{Num}(S))' \rightarrow \text{O}(\text{Num}(S)/2\text{Num}(S)))$$

(see [\[2\]](#), [DolgachevBrief](#), [\[21\]](#)).

**defmsm**

**Definition 2.2.** An isomorphism of lattices  $\phi : \mathbf{E} \rightarrow \text{Num}(S)$  is called a *marking* of  $S$ . Two markings  $\phi : \mathbf{E} \rightarrow \text{Num}(S)$  and  $\phi' : \mathbf{E} \rightarrow \text{Num}(S)$  are called equivalent if there exists  $\sigma \in \text{O}(\mathbf{E})$  such that  $\phi' = \phi \circ \sigma$  and  $\phi \circ \sigma \circ \phi^{-1} \in (W_S^{\text{nod}} \rtimes \text{Aut}(S)^{**}) \times \{\pm \text{id}_{\text{Num}(S)}\}$ . A *supermarking* of  $S$  is an isomorphism  $\tilde{\phi} : \mathbf{E} \oplus \mathbb{F}_2 \rightarrow \text{Pic}(S)$  such that its composition with the projection  $\text{Pic}(S) \rightarrow \text{Num}(S)$  is a marking  $\mathbf{E} \rightarrow \text{Num}(S)$ . Two supermarkings  $\tilde{\phi}$  and  $\tilde{\phi}'$  are equivalent if there exists  $\tilde{\sigma} \in \text{O}(\mathbf{E} \oplus \mathbb{F}_2)$  such that  $\tilde{\phi}' = \tilde{\phi} \circ \tilde{\sigma}$  and  $\tilde{\phi} \circ \tilde{\sigma} \circ \tilde{\phi}^{-1} \in (W_S^{\text{nod}} \rtimes \text{Aut}(S)^{**}) \times \{\pm \text{id}_{\text{Pic}(S)}\}$ .

If  $S$  is an unnodal Enriques surface, then  $W_S^{\text{nod}} = \{1\}$  and, for any  $g \in \text{Aut}(S)^{**}$  we have  $\phi^{-1} \circ g \circ \phi \in W(2)$ . This shows that elements of  $W(2)$  act identically on equivalence classes of markings and the set of equivalence classes of markings is a torsor under the group  $W/W(2) \cong \text{O}^+(10, \mathbb{F}_2)$ . The

set of equivalence classes of supermarkings on  $S$  is a torsor under the group  $\text{AO}^+(10, \mathbb{F}_2)$ .

**2.2. Isotropic sequences.** Let  $\text{Nef}(S)$  denote the nef cone of  $S$ . It is the subset of  $\text{Num}_{\mathbb{R}}(S) = \text{Num}(S) \otimes \mathbb{R}$  of numerical classes of  $\mathbb{R}$ -divisors  $D$  such that  $D \cdot C \geq 0$  for any effective divisor  $C$ . Since any class of an irreducible divisor with negative self-intersection is the class of a smooth rational curve  $R$ , the nef cone can also be defined as the set of effective  $\mathbb{R}$ -divisors  $D$  that have non-negative intersection with any smooth rational curve. It is known that  $\text{Nef}(S)$  is a fundamental domain for the action of  $W_S^{\text{nod}}$  on  $\text{Num}_{\mathbb{R}}(S)^+ = \{x \in \text{Num}_{\mathbb{R}}(S) : x^2 \geq 0, x \text{ is effective}\}$  [13, Chapter 4, §2]. This means that, for any effective numerical class  $x \in \text{Num}_{\mathbb{R}}(S)$  with non-negative self-intersection, there exists a unique  $w \in W_S^{\text{nod}}$  such that  $w(x) \in \text{Nef}(S)$ .

**Definition 2.3.** A nef isotropic  $c$ -sequence in  $\text{Num}(S)$  is an ordered set of  $c > 1$  isotropic vectors  $f_1, \dots, f_c \in \text{Nef}(S) \cap \text{Num}(S)$  such that  $f_i \cdot f_j = 1, i \neq j$ . A nef isotropic  $c$ -sequence in  $\text{Pic}(S)$  is an ordered set of effective divisor classes  $F_1, \dots, F_c$  such that their numerical classes  $f_i = [F_i]$  form a nef isotropic  $c$ -sequence in  $\text{Num}(S)$ .

The proof of the following proposition can be found in [13]<sup>CD</sup>, Lemma 3.3.1.

P5.1.5

**Proposition 2.4.** *Let  $(f_1, \dots, f_k)$  be an isotropic  $k$ -sequence in  $\text{Num}(S)$  of effective isotropic classes. There exists a unique  $w \in W_S^{\text{nod}}$  such that, after reindexing, the sequence  $(f'_1, \dots, f'_k) := (w(f_1), \dots, w(f_k))$  contains a nef isotropic subsequence  $f'_{i_1}, f'_{i_2}, \dots, f'_{i_c}$  with  $1 = i_1 < i_2 < \dots < i_c < i_{c+1} = k + 1$  such that, for any  $i_s < i < i_{s+1}$ ,*

$$f'_i = f'_{i_s} + R_{i_s,1} + \dots + R_{i_s,i-i_s} \in W_S^{\text{nod}} \cdot f_{i_s},$$

where  $R_{i_s,1} + \dots + R_{i_s,i-i_s}$  is a nodal cycle of type  $A_{i-i_s}$  (i.e. the exceptional curve of a minimal resolution of a double rational point of type  $A_{i-i_s}$ ).

Since  $f'_{i_1}, \dots, f'_{i_c}$  are nef, we have  $f'_{i_s} \cdot R_{ij} \geq 0$ . Since  $f'_{i_s} \cdot f'_{i_{s'}} = 1, s \neq s'$ , this implies that

$$f'_{i_s} \cdot R_{i_{s'},a} = 0, R_{i_s,a} \cdot R_{i_{s'},b} = 0, s \neq s', f'_{i_s} \cdot R_{i_s,1} = 1, f'_{i_s} \cdot R_{i_s,a} = 0, a \neq 1.$$

An isotropic  $k$ -sequence satisfying the properties of the isotropic sequence  $(f'_1, \dots, f'_k)$  from the proposition is called *canonical*. It follows from the proof of the Proposition that the  $W_S^{\text{nod}}$ -orbit of any isotropic sequence contains a unique canonical isotropic sequence. The number  $c$  of nef members in a canonical isotropic  $k$ -sequence is called the *non-degeneracy* invariant of the sequence. A canonical isotropic  $k$ -sequence with non-degeneracy invariant  $k$  is called non-degenerate. Obviously, the length  $k$  of any isotropic  $k$ -sequence is bounded by 10. Of course, if  $S$  is unnodal, then any canonical isotropic 10-sequence is non-degenerate.

C:2.5

**Corollary 2.5.** *There is a bijective correspondence between the following sets*



- equivalence classes of markings of  $S$ ;
- $\text{Aut}(S)^{**}$ -orbits of canonical isotropic 10-sequences.

Also there is a bijective correspondence between the following sets

- equivalence classes of supermarkings of  $S$ ;
- $\text{Aut}(S)^*$ -orbits of ordered sets  $(F_1, \dots, F_{10})$  of effective divisor classes such that their numerical classes define a canonical isotropic 10-sequence.

*Proof.* A marking  $\phi : \mathbf{E} \rightarrow \text{Num}(S)$  defines an isotropic 10-sequence, the image of vectors  $(\mathbf{f}_1, \dots, \mathbf{f}_{10})$  defined in (2.2). Applying some  $w \in W_S^{\text{nod}}$ , we replace this sequence by a canonical isotropic 10-sequence. Then the marking  $w \circ \phi : \mathbf{E} \rightarrow \text{Num}(S)$  is an equivalent marking. Since  $-\text{id}_{\text{Num}(S)}$  destroys effectiveness and elements  $w \in W_S^{\text{nod}}$  destroy nefness, the only equivalences of markings which preserve the canonical isotropic sequence originate from elements of  $\text{Aut}(S)^{**}$ . This gives an injective map from the set of equivalence classes of markings to the set of  $\text{Aut}(S)^{**}$ -orbits of canonical isotropic 10-sequences. Conversely, given a canonical isotropic 10-sequence  $(f_1, \dots, f_{10})$  we choose any marking  $\phi : \mathbf{E} \rightarrow \text{Num}(S)$  and consider an isotropic 10-sequence  $(\phi^{-1}(f_1), \dots, \phi^{-1}(f_{10}))$  in  $\mathbf{E}$ . It follows from section 2.1 that there is a bijection between isotropic 10-sequences  $(\mathbf{f}_1, \dots, \mathbf{f}_{10})$  in  $\mathbf{E}$  and root bases  $(\alpha_1, \dots, \alpha_{10})$  with Dynkin diagram (II). Since the group  $W = \text{O}(\mathbf{E})'$  coincides with the reflection group  $W_{2,3,7}$  generated by reflections in  $\alpha_0, \dots, \alpha_9$ , it acts simply transitively on the root bases and hence acts simply transitively on the set of isotropic 10-sequences. Hence we can find  $\sigma \in \text{O}(\mathbf{E})'$  such that  $\phi' = \phi \circ \sigma$  defines another marking that sends  $(\mathbf{f}_1, \dots, \mathbf{f}_{10})$  to  $(f_1, \dots, f_{10})$ . So the map from the set of equivalence classes of markings to the set of  $\text{Aut}(S)^{**}$ -orbits of canonical isotropic 10-sequences is surjective.

The sublattice of  $\mathbf{E}$  spanned by  $(\mathbf{f}_1, \dots, \mathbf{f}_{10})$  is of finite index in  $\mathbf{E}$ , equal to 3. Fix a set of effective representatives  $(F_1, \dots, F_{10})$  of a canonical isotropic 10-sequence  $(f_1, \dots, f_{10})$  in  $\text{Num}(S)$ . Write the numerical class  $[D]$  of a divisor as a linear combination of the isotropic 10-sequence  $(f_1, \dots, f_{10})$  with rational coefficients from  $\frac{1}{3} \cdot \mathbb{Z}$ . Then the choice of  $(F_1, \dots, F_{10})$  gives a splitting of the canonical surjection  $\text{Pic}(X) \rightarrow \text{Num}(X)$ . This is our supermarking. The proof of the bijectivity of this correspondence is similar to the previous one.  $\square$

**2.3. Fano polarization.** Let  $(f_1, \dots, f_{10})$  be a canonical isotropic 10-sequence and  $\phi : \mathbf{E} \rightarrow \text{Num}(S)$  the corresponding marking. Replacing  $\phi$  by an equivalent marking, we may assume that  $\phi(\mathbf{f}_i) = f_i, i = 1, \dots, 10$ . It follows from the proof of Proposition 2.4 in [13] that  $f_1 + \dots + f_{10}$  is a nef numerical divisor class (in fact, this is how one finds a canonical isotropic sequence: apply  $W_S^{\text{nod}}$  to make the sum nef and then prove that it satisfies the properties of a canonical sequence). Let  $\delta = \phi(\Delta)$ . Then

$$3\delta = f_1 + \dots + f_{10}.$$

**T1** **Theorem 2.6.** *For any canonical isotropic 10-sequence  $(f_1, \dots, f_{10})$  in  $\text{Num}(S)$ , there exists a unique  $\delta \in \text{Nef}(S) \cap \text{Num}(S)$  such that*

**13** (2.8) 
$$3\delta = f_1 + \dots + f_{10}.$$

*It satisfies*

$$\delta^2 = 10, \quad \delta \cdot f \geq 3 \text{ for any nef isotropic class } f.$$

*Conversely, any nef  $\delta$  satisfying this property can be written as in (2.8) for some canonical isotropic 10-sequence defined uniquely up to permutation.*

*Proof.* We have already shown that there exists some  $\delta$  for which equality (2.8) holds. Let  $f_1, \dots, f_{10}$  be the nef elements of  $(f_1, \dots, f_{10})$ . In notation of Proposition 2.4, we see that

$$f_{i_k} + f_{i_{k+1}} + \dots + f_{i_{k+1}-1} = sR_{i_k} + (s-1)R_{i_k,1} + \dots + R_{i_k,s-1},$$

where  $s = i_{k+1} - i_k$ . Intersecting both sides with  $R_{i_k,j}$ , we obtain

$$(f_{i_k} + f_{i_{k+1}} + \dots + f_{i_{k+1}-1}) \cdot R_{i_k,j} = -2(s-j) + (s-j+1) + (s-j-1) = 0.$$

We also see that  $R_{i_k,j}$  does not intersect any  $f_i$  with  $i \notin \{i_k, i_{k+1}, \dots, i_{k+1}-1\}$ . Thus

**intersection1** (2.9) 
$$\delta \cdot R_{i_k,j} = 0$$

for all  $k, j$ . For any effective divisor  $D$ , we have  $\delta \cdot [D] \geq 0$ . Indeed, by the above, we may assume that  $D$  does not contain  $R_{i_k,j}$  as irreducible components, so  $[D]$  intersects all  $f_{i_1}, \dots, f_{i_c}$  non-negatively because they are nef. Thus  $\delta$  is a nef numerical class satisfying  $\delta \cdot f_{i_k} = 3, k = 1, \dots, c$ . Since  $f_{i_k} \cdot f_{i_t} = 1$  for  $t \neq k$ , each  $f_{i_k}$  is a primitive isotropic vector in  $\text{Num}(S)$  (i.e. it is not an integer multiple of any other vector). By Hodge's Index Theorem, for any other primitive isotropic  $f$ , we have  $f \cdot f_{i_k} > 0$ . Hence  $\delta \cdot f = \frac{1}{3} \sum_{i=1}^{10} f \cdot f_i \geq \frac{10}{3} > 3$ .

Conversely, suppose  $\delta$  is a nef numerical class such that  $\delta^2 = 10$  and  $\delta \cdot f \geq 3$  for any nef isotropic vector  $f$ . By [13, Corollary 2.5.7], there are three  $O(\mathbf{E})$ -orbits of vectors of norm 10. Only one of them satisfies the property that  $|\Delta \cdot f| \geq 3$  for all isotropic vectors  $f$  in  $\mathbf{E}$ . Thus, if we fix a marking of  $S$ , we may assume that  $\delta$  corresponds to  $\Delta$ , and hence we can write  $3\delta = f_1 + \dots + f_{10}$  for some isotropic 10-sequence  $(f_1, \dots, f_{10})$ . Applying  $w \in W_S^{\text{nod}}$  we may assume that it is a canonical sequence. Since  $\delta$  is nef and the sum  $f_1 + \dots + f_{10}$  is nef, we see that  $w = 1$ . Thus  $(f_1, \dots, f_{10})$  is already a canonical sequence.  $\square$

Let  $f$  be a nef primitive isotropic numerical class. Then  $f = [F] = [F + K_S]$ , where  $F$  is a nef effective divisor with  $F^2 = 0$ . By [13, Proposition 3.1.2],  $|2F|$  is an elliptic pencil, and  $F + K_S \sim F'$  for some other effective divisor with  $2F' \in |2F|$ . The two divisors  $F, F'$  are the two half-fibers of an elliptic fibration. Since  $|F|$  consists of the single divisor  $F$ , we will identify  $F$  with its divisor class.

Let  $\Delta \in \text{Pic}(S)$  with  $\delta = [\Delta]$  be as in Theorem [F1](#) [2.6](#). Then  $\Delta$  is a nef effective divisor with  $\Delta^2 = 10$  and  $\Delta \cdot F \geq 3$  for any nef effective divisor  $F$  of arithmetic genus 1 on  $S$ . In fact,  $\Delta \cdot F = 3$  if the numerical class  $[F]$  coincides with one of the classes  $f_i$  and  $\Delta \cdot F > 3$  otherwise. By [\[13\]](#), [Theorem 4.6.1](#)], the linear system  $|\Delta|$  defines a birational morphism onto a normal surface of degree  $\Delta^2 = 10$

$$\text{mapFano} \quad (2.10) \quad \phi_\Delta : S \rightarrow \bar{S} \subset |\Delta|^* \cong \mathbb{P}^5.$$

We call  $\bar{S}$  a *Fano model* of  $S$  and the divisor class  $\Delta$  a *Fano polarization*.

ample

**Proposition 2.7.** *A Fano polarization is ample if and only if the corresponding canonical isotropic 10-sequence is non-degenerate.*

*Proof.* It follows from [\(2.9\)](#) [intersection1](#) that  $(f_1, \dots, f_{10})$  is non-degenerate if  $\delta$  is ample. Conversely, if  $(f_1, \dots, f_{10})$  is non-degenerate, then for any smooth rational curve  $R$ , we have  $R \cdot f_i \geq 0$ , hence  $\Delta$  is ample unless  $f_i \cdot R = 0$  for all  $i$ . However,  $f_1, \dots, f_{10}$  generate  $\text{Num}(S)$  over  $\mathbb{Q}$ , so this is impossible.  $\square$

Suppose an irreducible curve  $R$  is blown down under  $\phi_\Delta$ . Then  $R \cdot \Delta = 0$  implies, by Hodge Index Theorem that  $R^2 < 0$ . By the adjunction formula,  $R^2 = -2$ . If  $R \neq R_{i_k, j}$  for any  $k, j$ , then  $R \cdot \Delta = 0$  implies that  $R \cdot f_i = 0$  for all  $i$ . As we remarked in the proof of the previous proposition this is impossible. So, we see that, if  $\Delta$  is not ample, the singular points of the Fano model  $\bar{S}$  are double rational points of types  $A_{i_{k+1}-i_k-1}$ , where  $i_{k+1} - i_k > 1$ .

The following theorem follows from the previous discussion.

supmark

**Theorem 2.8.** *Any non-degenerate supermarking on  $S$  defines an ample Fano polarization  $\Delta$  and 10 half-fibers  $F_i$  of elliptic fibrations such that*

$$\text{12} \quad (2.11) \quad 3\Delta \sim F_1 + \dots + F_{10}.$$

*The sequence  $(F_1, \dots, F_{10})$  is uniquely determined by the equivalence class of the supermarking.*

Assume that  $\Delta$  is an ample Fano polarization. Under the map [\(2.10\)](#) [mapFano](#) the curves  $F_i$  and  $F_{-i} \sim F_i + K_S$  are mapped to plane cubics lying in planes which we will denote  $\Lambda_i$  and  $\Lambda_{-i}$  respectively. In this way we get 20 planes  $\Lambda_{\pm i}$ . Since  $f_i \cdot f_j = 1, i \neq j$ , we see that  $\Lambda_i \cap \Lambda_j \neq \emptyset$  if  $i + j \neq 0$ .

twentyplanes1

**Proposition 2.9.** *Let  $\Lambda_{\pm i}$  be the 20 planes in  $\mathbb{P}^5$  defined by an ample Fano polarization of an Enriques surface  $S$ .*

- (i) *For any  $i \neq j$ , the planes  $\Lambda_i, \Lambda_j, i + j \neq 0$ , span a hyperplane, or, equivalently,  $\Lambda_i$  and  $\Lambda_j$  intersect at one point.*

*Assume additionally that  $S$  is an unnodal Enriques surface.*

- (ii) *If  $3\Delta \sim F_1 + \dots + F_{10}$ , then the intersection points  $p_{ij} = F_j \cap F_i, j \neq i$ , are all distinct and their sum taken with multiplicity two forms the base locus of a pencil of curves of arithmetic genus 1 and degree 6 in the plane  $\Lambda_i$  (an Halphen pencil of index 2, see [\[7\]](#) [Cantat](#)). For any fixed  $i$ , no six points among the points  $p_{ij}$  lie on a conic, no three of them lie on a line.*

- (iii) If  $3(\Delta + K_S) \sim F_1 + \cdots + F_{10}$ , the intersection points  $F_j \cap F_i, j \neq i$ , form the base locus of a pencil of cubic curves in the plane  $\Lambda_i$  (an Halphen pencil of index 1).

*Proof.* (i) If  $\dim \Lambda_i \cap \Lambda_j \neq 0$ , then the only possibility is that  $\Lambda_i \cap \Lambda_j$  is a line  $\ell$ . Since  $S$  is unnodal,  $\ell \not\subset \phi_\Delta(F_i) \cup \phi_\Delta(F_j)$ . Since  $\phi_\Delta(F_i)$  and  $\phi_\Delta(F_j)$  are cubic curves in  $\Lambda_i$  and  $\Lambda_j$ , the line  $\ell$  intersects the Fano model  $\bar{S} = \phi_\Delta(S)$  at finitely many points. Obviously  $B = \phi^{-1}(\ell \cap \bar{S})$  is the base locus of the linear system  $|\Delta - F_i - F_j|$ . But this contradicts the equality  $(\Delta - F_i - F_j)^2 = 0$ .

(ii) Suppose  $\Lambda_i \cap \Lambda_j \cap \Lambda_k \neq \emptyset$ . Then  $F_i$  and  $F_j$  intersect  $F_k$  at the same point. Consider the natural exact sequence coming from restriction of the sheaf  $\mathcal{O}_S(F_i - F_j)$  to  $F_k$ :

$$0 \rightarrow \mathcal{O}_S(F_i - F_j - F_k) \rightarrow \mathcal{O}_S(F_i - F_j) \rightarrow \mathcal{O}_{F_k}(F_i - F_j) \rightarrow 0.$$

Suppose  $F_i \cap F_j \cap F_k \neq \emptyset$ . Then  $h^0(\mathcal{O}_{F_k}(F_i - F_j)) = h^0(\mathcal{O}_{F_k}) = 1$ . Since  $S$  is unnodal and  $(F_i + F_j - F_k)^2 = -2$ , we have  $h^0(D) = 0$  for any divisor  $D$  numerically equivalent to  $F_i + F_j - F_k$  or  $F_k - F_i - F_j$ . For the same reason  $h^0(\mathcal{O}_S(F_i - F_j)) = 0$ . Applying Riemann-Roch and Serre's Duality, we find that  $h^1(\mathcal{O}_S(F_i - F_j - F_k)) = 0$ , and the exact sequence of cohomology gives  $\mathcal{O}_{F_k}(F_i - F_j) = 0$ , a contradiction.

Thus, the 9 points  $p_{ij} = F_i \cap F_j$  on  $F_i$  are indeed all distinct. Since  $F_i \cdot F_j = 1$ , they are nonsingular points of  $F_i$  and their sum is an effective Cartier divisor  $\eta$  such that

$$\mathcal{O}_{F_i}(\eta) \cong \mathcal{O}_{F_i}(F_1 + \cdots + F_{10} - F_i) \cong \mathcal{O}_{F_i}(3\Delta - F_i).$$

Since  $2F_i$  moves in a pencil,  $\mathcal{O}_{F_i}(2F_i) \cong \mathcal{O}_{F_i}$ . On the other hand, the exact sequence

$$0 \rightarrow \mathcal{O}_S \rightarrow \mathcal{O}_S(F_i) \rightarrow \mathcal{O}_{F_i}(F_i) \rightarrow 0$$

together with the fact the  $h^0(\mathcal{O}_S(F_i)) = 1$  implies that  $h^0(\mathcal{O}_{F_i}(F_i)) = 0$ , hence, in the plane embedding  $F_i \subset \Lambda_i$ ,  $\mathcal{O}_{F_i}(\eta)$  is not isomorphic to  $\mathcal{O}_{F_i}(3)$  but  $\mathcal{O}_{F_i}(2\eta) \cong \mathcal{O}_{F_i}(6)$ . A standard argument shows that in the linear system of sextics cutting out  $2\eta$  there exists a sextic curve with 9 double points at the points  $p_{ij}$ . Together with the cubic  $F_i$ , taken with multiplicity 2, they form an Halphen pencil of index 2. We leave to the reader to prove the other assertions by considering the exact sequences

$$0 \rightarrow \mathcal{O}_S(D_6 - 2\Delta - F_i) \rightarrow \mathcal{O}_S(D_6 - 2\Delta) \rightarrow \mathcal{O}_{F_i}(D_6 - 2\Delta) \rightarrow 0$$

and

$$0 \rightarrow \mathcal{O}_S(D_3 - \Delta - F_i) \rightarrow \mathcal{O}_S(D_3 - \Delta) \rightarrow \mathcal{O}_{F_i}(D_3 - \Delta) \rightarrow 0,$$

where  $D_6$  (resp.  $D_3$ ) is the sum of 6 (resp. 3) curves  $F_j, j \neq i$ .

(iii) Apply the same argument with the only change that this time we have

$$\mathcal{O}_{F_i}(\eta) \cong \mathcal{O}_{F_i}(F_1 + \cdots + F_{10} + K_S - F_i) \cong \mathcal{O}_{F_i}(3\Delta - F_i + K_S)$$

and we get that  $\mathcal{O}_{F_i}(\eta) \cong \mathcal{O}_{F_i}(3)$  in the plane embedding. This shows that there exists a cubic curve cutting out the divisor  $\eta$  on  $F_i$ . Together with  $F_i$  they form a pencil of cubic curves intersecting at nine points  $p_{ij}$ .  $\square$

### 3. THE MODULI SPACES OF MARKED AND SUPERMARKED ENRIQUES SURFACES

**3.1. Periods of Enriques surfaces.** The construction of the coarse moduli space of marked Enriques surfaces via periods is well-known. It is a special case of the construction of the moduli space of lattice polarized K3 surfaces [15].<sup>2</sup> Recall that, given an even lattice  $M$  of signature  $(1, \rho - 1)$ , a  $M$ -polarization of a K3 surface  $Y$  is a primitive embedding  $j : M \hookrightarrow \text{Pic}(Y)$  such that the image has non-empty intersection with the closure of the ample cone on  $Y$ . If the image contains an ample divisor class, the polarization is called ample (see more precise definitions in loc. cit.).

Let  $\pi : Y \rightarrow S$  be the canonical K3-cover of an Enriques surface. We denote by  $\tau$  the covering involution so that  $\pi$  can be identified with the quotient map  $Y \rightarrow Y/(\tau) \cong S$ . Let  $\pi^* : \text{Pic}(S) \rightarrow \text{Pic}(Y)$  be the inverse image homomorphism. The standard argument using the Hochschild-Serre exact sequence in étale cohomology (see, for example, [35]<sup>Jensen</sup>) shows that  $\text{Ker}(\pi^*) = \mathbb{Z}K_S$  and  $\pi^*(\text{Pic}(S) = \text{Pic}(Y)^{(\tau)})$ , the subgroup of  $\tau$ -invariant divisor classes. Thus the homomorphism  $\pi^* : \text{Pic}(S) \rightarrow \text{Pic}(Y)$  factors through the injective homomorphism  $\text{Num}(S) \rightarrow \text{Pic}(Y)$  for which we will keep the same notation  $\pi^*$ . Since, for any divisor classes  $D, D'$  on  $S$ , we have  $\pi^*(D) \cdot \pi^*(D') = 2D \cdot D'$ , the homomorphism  $\pi^*$  is a lattice embedding homomorphism of lattices

$$\pi^* : \text{Num}(S)(2) \hookrightarrow \text{Pic}(Y).$$

Here, for any lattice  $M$  and an integer  $k$ ,  $M(k)$  denotes the lattice  $M$  with the quadratic form multiplied by  $k$ . Since the sublattice  $\text{Pic}(Y)^{(\tau)}$  is primitive, in  $\text{Pic}(Y)$ , the lattice embedding  $\pi^*$  is primitive.

**P:3.2** **Proposition 3.1.** *Composing an ample marking  $j : \mathbf{E} \rightarrow \text{Num}(S)$  with  $\pi^*$  defines a bijection between the sets of ample markings of  $S$  and ample lattice  $\mathbf{E}(2)$ -markings of  $Y$ .*

*Proof.* Since the pre-image of an ample divisor is ample, we obtain that  $\pi^* \circ j$  defines an ample lattice polarization on  $Y$ .

Conversely, let  $(Y, j)$  be an ample  $\mathbf{E}(2)$  lattice polarization of a K3 surface. It is known that  $H^2(Y, \mathbb{Z})$  considered as a lattice via the cup-product is isomorphic to the K3-lattice

$$\mathbf{L}_{K3} = \mathbf{E}_8^{\oplus 2} \oplus \mathbf{U}^{\oplus 3},$$

where  $\mathbf{U}$  is an integral hyperbolic plane and  $\mathbf{E}_8$  is an even negative definite unimodular lattice of rank 8. It is known that the orthogonal group

<sup>2</sup>There is some additional technical requirement to the embedding, for which we refer to loc. cit. and which can be ignored in our case.

of  $\mathbf{L}_{K3}$  acts transitively on primitive sublattices isomorphic to  $\mathbf{E}(2)$  [46, <sup>Namikawa</sup>Theorem (1.4)], so we may choose  $\phi$  in such a way that the embedding  $\mathbf{E}(2) = \mathbf{E}_8(2) \oplus \mathbf{U}(2) \rightarrow \mathbf{L}_{K3}$  is defined by  $(x, y) \mapsto (x, x, y, y, 0)$ . Let  $\iota$  be the involution of  $\mathbf{L}_{K3}$  defined by switching the two factors of  $\mathbf{E}_8$  and the first two factors of  $\mathbf{U}$ . Then  $\mathbf{E}(2)$  is identified with the sublattice of invariant elements  $\mathbf{L}_{K3}^\iota$ . Let  $\sigma = \phi \circ \iota \circ \phi$  be the corresponding involution of  $H^2(Y, \mathbb{Z})$ . The restriction of  $\sigma$  to  $\text{Pic}(Y)$  acts identically on the sublattice  $j(\mathbf{E}(2))$  of  $\text{Pic}(Y)$  that contains an ample class. By the Global Torelli Theorem of Pyatetsky-Shapiro and Shafarevich, there exists an involution  $g$  of  $Y$  such that  $\sigma = g^*$ . The involution  $g^*$  of  $H^2(Y, \mathbb{Z})$  has trace equal to  $10 - 12 = -2$ , hence, applying the *Lefschetz fixed-point formula*,

$$\boxed{\text{lefschetz}} \quad (3.1) \quad \text{Lef}(g) := \sum (-1)^i \text{Tr}(g^* | H^i(Y, \mathbb{Z})) = \chi(X^g)$$

we obtain that the Euler-Poincaré characteristic of the locus of fixed points  $Y^g$  of  $g$  is equal to zero. The involution  $\iota$  acts as the minus identity on

$$\boxed{\text{latticeT}} \quad (3.2) \quad \mathbf{T} = j(\mathbf{E}(2))^\perp \cong \mathbf{E}(2) \oplus \mathbf{U}$$

embedded into  $\mathbf{L}_{K3}$  by  $(x, y, z) \mapsto (-x, x, y, -y, z)$ . Since  $\mathbf{T}_{\mathbb{C}}$  contains  $H^{2,0}(Y)$ , we obtain that  $g$  has no isolated fixed points. Applying <sup>lefschetz</sup>(3.1), we obtain that the set  $Y^g$  of fixed points of  $g$  is either empty, and hence  $X = Y/(g)$  is an Enriques surface, or  $Y^g$  is equal to the union  $F$  of disjoint elliptic curves. It is easy to see from the classification of algebraic surfaces that  $X$  must be a non-minimal rational surface, hence it contains a  $(-1)$ -curve. Its pre-image on  $Y$  is  $g$ -invariant  $(-2)$ -curve. Since  $\mathbf{E}(2) = H^2(Y, \mathbb{Z})^g$  has no vectors of square norm  $-2$ , we get a contradiction. Now, we identify  $\mathbf{E}(2)$  with  $\text{Num}(S)(2)$  and hence obtain a marking  $\phi : \mathbf{E} \rightarrow \text{Num}(S)$  inducing the original  $\mathbf{E}(2)$ -lattice polarization of  $Y$ .  $\square$

Next we invoke the theory of periods for lattice polarized K3 surfaces from <sup>DolgachevMirrors</sup>[15]. For any even lattice  $M$  of signature  $(1, r-1)$  primitively embeddable into the K3-lattice  $\mathbf{L}_{K3}$ , one can construct the coarse moduli space  $\mathcal{M}_{K3, M}$  (resp.  $\mathcal{M}_{K3, M}^a$ ) of isomorphism classes of lattice  $M$ -polarized (resp. amply polarized) K3 surfaces  $X$ , i.e. isomorphism classes of pairs  $(Y, j)$ , where  $j : M \hookrightarrow \text{Pic}(Y)$  is a primitive lattice embedding such that the image contains a nef and big (resp. ample) divisor class.

Let  $N$  be the orthogonal complement of  $M$  in  $\mathbf{L}_{K3}$ . Let

$$\boxed{\text{period1}} \quad (3.3) \quad \mathcal{D}_N = \{[\omega] \in \mathbb{P}(N \otimes \mathbb{C}) : \omega \cdot \omega = 0, \omega \cdot \bar{\omega} > 0\},$$

which is the disjoint union of two copies of the  $(20-r)$ -dimensional bounded symmetric domains of orthogonal type. Let  $(Y, j)$  be a pair consisting of a K3 surface  $Y$  and an ample  $M$ -lattice polarization  $j : M \rightarrow \text{Pic}(Y)$ . We choose a marking  $\phi : \mathbf{L}_{K3} \rightarrow H^2(Y, \mathbb{Z})$  such that its restriction to  $M \subset \mathbf{L}_{K3}$  coincides with the composition  $M \xrightarrow{j} \text{Pic}(Y) \subset H^2(Y, \mathbb{Z})$ . We assign to a triple  $(Y, j, \phi)$  the point  $p_{(Y, j, \phi)}$  in  $\mathcal{D}_N$  corresponding to the line  $\phi_{\mathbb{C}}^{-1}(H^{2,0})$ , where  $\phi_{\mathbb{C}}$  is the extension of  $\phi$  to a map  $H^2(Y, \mathbb{C}) \rightarrow N_{\mathbb{C}}$ . We use here the

fact that  $j(M)$  consists of algebraic cycles, hence  $H^{2,0} \subset j(M)^\perp$ . The point  $p_{(Y,j,\phi)}$  is called the *period* of a marked lattice  $M$ -polarized K3 surface  $Y$ .

Let  $A_M = M^\vee/M$  be the discriminant group of an even lattice  $M$  equipped with the quadratic map

$$q_{A_M} : A_M \rightarrow \mathbb{Q}/2\mathbb{Z}, \quad x^* \mapsto x^{*2} \bmod 2\mathbb{Z},$$

where we extend the quadratic form of  $M$  to  $M_\mathbb{Q}$  and then restrict it to the dual lattice  $M^\vee \subset M_\mathbb{Q}$ . Let  $O(A_M)$  be the group of automorphisms of  $A_M$  preserving the quadratic form  $q_{A_M}$ . We have a natural homomorphism

$$\boxed{\text{rN}} \quad (3.4) \quad r_N : O(N) \rightarrow O(A_N)$$

As explained, for example in <sup>Nikulin</sup>[49, §I], an isometry of  $M$  from the kernel of this map can be lifted to an isometry of a unimodular lattice  $L$  that contains  $N$  as a primitive sublattice and the lifted isometry acts identically on the orthogonal complement  $N^\perp$  of  $N$  in  $L$ . The discriminant groups of  $N$  and its orthogonal complement  $N^\perp$  in  $L$  are isomorphic but  $q_{A_{N^\perp}} = -q_{A_N}$  (see loc. cit.).

Let

$$\boxed{\text{monodromy}} \quad (3.5) \quad \Gamma_N = \text{Ker}(r_N : O(N) \rightarrow O(A_N)).$$

It follows from the Global Torelli Theorem of Pyatetsky-Shapiro and Shafarevich that an element  $\sigma$  of  $\Gamma_N$  applied to the period point  $p_{(Y,j,\phi)}$  by composing  $\phi$  with  $\sigma$  lifted to  $\mathbf{L}_{K3}$  can be represented in the form  $p_{(Y,j,g^*\circ\phi)}$  for some automorphism  $g$  of  $Y$ . In this way we can assign to an isomorphism class of lattice polarized K3 surfaces  $Y$  a well-defined point  $\Gamma_N \cdot p_{(Y,j,\phi)}$  in the orbit space  $\Gamma_N \backslash \mathcal{D}$ . We can also extend this correspondence to families of marked lattice polarized surfaces to show that

$$\mathcal{M}_{K3,M} := \Gamma_N \backslash \mathcal{D}_N$$

is the coarse *moduli space* of  $M$ -polarized K3 surfaces.

For any vector  $\delta \in N$  let  $H_N(\delta)$  denote the intersection of  $\mathcal{D} \subset \mathbb{P}(N_\mathbb{C})$  with the hyperplane  $\mathbb{P}(\delta^\perp)$ . Let

$$\mathcal{H}_N(-2) = \bigcup_{\delta \in \mathbf{T}_{-2}} H_N(\delta),$$

where for any lattice  $M$  and any integer  $k$  we denote by  $M_k$  the set of vectors of square  $k$ . Let

$$\mathcal{D}_N^o = \mathcal{D}_N \setminus \mathcal{H}_N(-2).$$

Then

$$\mathcal{M}_{K3,M}^a := \Gamma_N \backslash \mathcal{D}_N^o$$

becomes the coarse moduli space of ample lattice  $M$ -polarized K3 surfaces.

We apply this to the case when  $M$  is the lattice  $\mathbf{E}(2)$  primitively embedded in  $\mathbf{L}_{K3}$ . We have an isomorphism

$$A_{\mathbf{E}(2)} = \mathbf{E}(2)^\vee/\mathbf{E} \cong \mathbf{D} = \mathbf{E}/2\mathbf{E}, \quad x \mapsto \frac{1}{2}x \bmod 2\mathbf{E}$$

and the transfer of the quadratic form  $q_{\mathbf{E}(2)}$  to a quadratic form on  $\mathbf{D}$  is a nondegenerate quadratic form of even type. In particular,

$$\mathrm{O}(A_{\mathbf{E}(2)}) \cong \mathrm{O}^+(10, \mathbb{F}_2).$$

By [Nikulin](#) [49, Proposition 1.14.2] the homomorphism  $r_{\mathbf{T}}$  is surjective and we denote its kernel by  $\Gamma_{\mathrm{Enr}}$ .

Let  $f : \mathcal{X} \rightarrow T$  be a smooth, projective family of Enriques surfaces. Let  $\mathcal{P}_{\mathcal{X}/T} = R^1 f_* \mathbb{G}_m$  be the relative Picard sheaf for the faithfully flat topology represented by the relative Picard scheme  $\mathrm{Pic}_{\mathcal{X}/T}$ . Let  $\mathcal{P}_{\mathcal{X}/T}^\tau$  be the quotient of  $\mathcal{P}_{\mathcal{X}/T}$  by the numerical equivalence. It is equipped with the induced symmetric bilinear form defined by the intersection form of divisor classes.

We define a *family of marked Enriques surfaces* as a flat smooth family  $f : \mathcal{X} \rightarrow T$  of Enriques surfaces together with an isomorphism of abelian sheaves  $\mathbf{E}_T \rightarrow \mathcal{P}_{\mathcal{X}/T}^\tau$  preserving the intersection product

$$\mathcal{P}_{\mathcal{X}/T}^\tau \times \mathcal{P}_{\mathcal{X}/T}^\tau \rightarrow \mathbb{Z}_T.$$

Here for any abelian group  $A$ , we denote by  $A_T$  the constant sheaf with fiber  $A$ . Two families of marked surfaces are said to be isomorphic if they differ by a composition with an isomorphism of the families. Let  $\mathcal{F}_{\mathrm{Enr}}^m : \mathrm{Sch}/T \rightarrow (\mathrm{Sets})$  be the functor defined by these families.

The following proposition follows from the definition and the construction of the coarse moduli space of lattice polarized K3 surfaces.

**P3.2** **Proposition 3.2.** *The functor  $\mathcal{F}_{\mathrm{Enr}}^m$  admits a coarse moduli space  $\mathcal{M}_{\mathrm{Enr}}^m$  in the category of analytic spaces isomorphic to the orbit space  $\mathcal{M}_{K3, \mathbf{E}(2)}^a = \Gamma_{\mathrm{Enr}} \backslash \mathcal{D}_{\mathbf{E}(2)^\perp}^o$ .*

The group  $\Gamma_{\mathrm{Enr}} \subset \mathrm{O}(\mathbf{E}(2)^\perp) = \mathrm{O}(\mathbf{E}(2) \oplus \mathbf{U})$  contains the isometry of  $-\mathrm{id}_{\mathbf{U}} \oplus \mathrm{id}_{\mathbf{E}(2)}$  that interchanges the two connected components of  $\mathcal{D}_{\mathbf{E}(2)^\perp}^o$  [Dolgachev Mirrors](#) [15, Proposition 5.6]. This shows that the moduli space  $\mathcal{M}_{\mathrm{Enr}}^m$  is irreducible. Since  $\Gamma_{\mathrm{Enr}}$  is an arithmetic subgroup, the quotient is an irreducible quasi-projective variety.

Let

$$\mathcal{H}_{\mathbf{E}(2)^\perp}(-4) = \bigcup_{\delta \in (\mathbf{E}(2)^\perp)_{-4, \mathrm{ev}}} H_\delta,$$

where  $(\mathbf{E}(2)^\perp)_{-4, \mathrm{ev}}$  is the set of vectors in  $\mathbf{E}(2)^\perp$  of square norm  $-4$  and of even type (see [Gritsenko Kondoz](#) [29], [20]). Let

$$\mathcal{D}_{\mathbf{E}(2)^\perp}^{oo} := \mathcal{D}_{\mathbf{E}(2)^\perp}^o \setminus (\mathcal{H}_{\mathbf{E}(2)^\perp}(-2) \cup \mathcal{H}_{\mathbf{E}(2)^\perp}(-4)).$$

The quotient

$$\mathcal{M}_{\mathrm{Enr}}^{m, \mathrm{un}} := \Gamma_{\mathrm{Enr}} \backslash \mathcal{D}_{\mathbf{E}(2)^\perp}^{oo}$$

is the coarse moduli space for marked unnodal Enriques surfaces (see loc.cit.). Its complement in  $\mathcal{M}_{\mathrm{Enr}}^m$  is the *nodal divisor*

$$\mathcal{M}_{\mathrm{Enr}}^{m, \mathrm{nod}} := \Gamma_{\mathrm{Enr}} \backslash (\mathcal{D}_{\mathbf{E}(2)^\perp}^o \cap \mathcal{H}_{\mathbf{E}(2)^\perp}(-4))$$



equal to the coarse moduli space of marked nodal Enriques surfaces.

The group  $O(E(2)^\perp)$  acts on  $\mathcal{D}_{E(2)^\perp}^o$  and defines an action on  $\mathcal{M}_{\text{Enr}}^m$  by changing the markings. The quotient

$$\mathcal{M}_{\text{Enr}} := O(E(2)^\perp) \backslash \mathcal{D}_{E(2)^\perp}^o$$

is an irreducible quasi-projective variety. By a well-know result of S. Kondō [KondōRationality \[39\]](#), it is a rational variety.

Since the homomorphisms  $\rho_{E(2)^\perp}$  and  $\rho_{E(2)}$  are surjective, for any isometry of  $E(2)$  there exists an isometry  $\sigma$  of  $E(2)^\perp$  such that the pair  $(\alpha, \sigma)$  can be lifted to an isometry of  $\mathbb{L}_{K3}$ . This shows that there is a natural bijection between the points of  $\mathcal{M}_{\text{Enr}}$  and the isomorphism classes of Enriques surfaces. Only for this reason,  $\mathcal{M}_{\text{Enr}}$  is called *moduli space of Enriques surfaces*. Since any marked nodal Enriques surface  $(S, j)$  contains  $\text{Ker}(\rho_{E(2)})$  in  $j^{-1} \circ \text{Aut}(S)^{**} \circ j$ , we see that the action of  $O(E(2)^\perp)$  on  $\mathcal{D}_{E(2)^\perp}^{oo}$  factors through the action of the finite group

$$O^+(10, \mathbb{F}_2) = O(E(2)')/O(E(2))'(2) = W_{2,3,7}/W_{2,3,7}(2).$$

Thus

$$\mathcal{M}_{\text{Enr}}^{\text{un}} := O^+(10, \mathbb{F}_2) \backslash \mathcal{M}_{\text{Enr}}^{m, \text{un}}$$

should be considered as the *moduli space of unnodal Enriques surfaces*. The complement

$$\mathcal{M}_{\text{Enr}}^{\text{nod}} := \mathcal{M}_{\text{Enr}} \setminus \mathcal{M}_{\text{Enr}}^{\text{un}}$$

is an irreducible codimension one subvariety parameterizing nodal Enriques surfaces. According to [DolgachevKondo2 \[20\]](#), it is a rational variety.

Recall from [\[16\]](#) that a *Coble surface* (of K3 type) is a rational surface obtained as the quotient of a K3 surface by an involution whose set of fixed points is the disjoint union of  $k$  smooth rational curves. We can interpret the boundary

$$\mathcal{M}_{K3, E(2)} \setminus \mathcal{M}_{K3, E(2)}^a = \Gamma_{\text{Enr}} \backslash \mathcal{H}_{E(2)^\perp}(-2)$$

as the coarse moduli space  $\mathcal{M}_{\text{Coble}}^m$  of marked Coble surfaces (see [DolgachevKondo2 \[20\]](#)). In fact, it is proven in Proposition 3.2 of loc. cit. that

$$\mathcal{M}_{\text{Coble}}^m \cong \mathcal{M}_{K3, M}$$

for  $M = E(2) \oplus \langle -2 \rangle$ , where for any integer  $m$  we denote by  $\langle m \rangle$  the lattice  $\mathbb{Z}e$  with  $e^2 = m$ .

**3.2. The Hilbert scheme of Fano models.** Let  $\text{Hilb}_{2n}$  be the Hilbert scheme of Enriques surfaces embedded in  $\mathbb{P}^n$  by a complete linear system  $|D|$  with  $D^2 = 2n$ . For any such surface  $S \subset \mathbb{P}^n$ , the Hilbert polynomial  $P_S(t) = \chi(S, \mathcal{O}_S(t)) = 2nt^2 + 1$ . Let  $\mathcal{N}_S$  be the normal bundle of  $S$  in  $\mathbb{P}^n$ .

We have a natural exact sequence

$$\boxed{\text{normal}} \quad (3.6) \quad 0 \rightarrow \Theta_S \rightarrow \Theta_{\mathbb{P}^n} \otimes \mathcal{O}_S \rightarrow \mathcal{N}_S \rightarrow 0,$$

where  $\Theta_S, \Theta_{\mathbb{P}^n}$  denote the tangent sheaves of  $S, \mathbb{P}^n$ .

It is known that

$$h^i(\Theta_S) = h^i(\Omega_Y^1) - h^i(S, \Omega_S^1) = \begin{cases} 10 & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

<sup>CD</sup>[T3, Chapter 1, §4]. Applying the exact sequence <sup>normal</sup>(3.6) and the exact sequence

$$0 \rightarrow \mathcal{O}_S \rightarrow \mathcal{O}_S(1)^{\oplus(n+1)} \rightarrow \Theta_{\mathbb{P}^n} \otimes \mathcal{O}_S \rightarrow 0$$

obtained from the known resolution of the tangent sheaf of a projective space, we obtain

$$\boxed{\text{dimnormal}} \quad (3.7) \quad \dim H^i(S, \mathcal{N}_S) = \begin{cases} 10 + n^2 + 2n & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus we see that the Hilbert scheme is smooth at the point  $[S]$  and of dimension  $10 + n^2 + n$ . Taking  $n = 5$ , we obtain

hilbert **Proposition 3.3.** *The Hilbert scheme  $\text{Hilb}_{\text{Fano}}$  of Fano models of Enriques surfaces is a smooth variety. The dimension of each its connected (=irreducible) component is equal to 45.*

Let  $\mathcal{F}_{\text{Enr}, 2n}$  be the moduli functor of equivalence classes of families  $f : \mathcal{X} \rightarrow T$  of Enriques surfaces together with a relatively ample invertible sheaf  $\mathcal{L}$  on  $\mathcal{X}$  such that  $\mathcal{L} \otimes \mathcal{O}_{\mathcal{X}_t}$  is a very ample of degree  $2n$  on each fiber  $\mathcal{X}_t$  of  $f$ . We can also consider  $\mathcal{L}$  as a section of the relative Picard sheaf  $\mathcal{P}_{\mathcal{X}/T}$ . Locally, we can trivialize this sheaf by considering a family of supermarked Enriques surface given by an isomorphism of sheaves of abelian group  $\phi : (\mathbf{E} \oplus \mathbb{Z}/2\mathbb{Z})_T \rightarrow \mathcal{P}_{\mathcal{X}/T}$ . We say that  $\phi^{-1}(\mathcal{L})$  is the *type* of the family. For any connected  $T$ , different trivializations define the types belonging to the same orbit of the group  $\mathbf{D} \times \mathbf{O}(\mathbf{E})'$  acting on  $\mathbf{E} \oplus \mathbb{Z}/2\mathbb{Z}$ . Since there is only a finite set of orbits of  $\mathbf{O}(\mathbf{E})$  on vectors of positive norm square, we see that the Hilbert scheme has only finitely many connected components.

The group  $\text{PGL}(n+1)$  acts on  $\text{Hilb}_{\text{Fano}}$  via its action on  $\mathbb{P}^n$ . According to a result of Matsusaka and Mumford <sup>MatsusakaMumford</sup>[43], the action is proper (i.e. the orbits are closed). It also has finite stabilizer groups. It follows that the geometric quotient  $\mathcal{H}_{2n}/\text{PGL}(n+1)$  exists as a separated algebraic space <sup>KeelMori</sup>[37, Theorem 1.1]. We will consider it as an analytic space.

**Proposition 3.4.** *The quotient  $\mathcal{M}_{\text{Enr}, 2n} := \text{Hilb}_{\text{Fano}}/\text{PGL}(n+1)$  is a coarse moduli space for the functor  $\mathcal{F}_{\text{Enr}, 2n}$ .*

This is analogous to the similar result for the moduli functor of polarized K3 surfaces and we refer for the proof to <sup>Huybrechts</sup>[32, Chapter 5, Theorem 2.4].

Similarly we define the moduli functor  $\mathcal{F}_{\text{Enr}, 2n}^\tau$  of families of numerically polarized Enriques surfaces by replacing  $\mathcal{L}$  by its class  $[\mathcal{L}]$  in  $\text{Num}(\mathcal{X})$ , considered as a section of the sheaf  $\mathcal{P}_{\mathcal{X}/T}^\tau$ . A local trivialization  $\phi : \mathbf{E}_T \rightarrow \mathcal{P}_{\mathcal{X}/T}^\tau$  defines a *type* of a numerically polarized family which is an  $\mathbf{O}(\mathbf{E})$ -orbit of a vector  $h \in \mathbf{E}$ .

The functor  $\mathcal{F}_{\text{Enr},2n}$  admits a natural involution that sends a family  $(\mathcal{X} \rightarrow T, \mathcal{L})$  to  $(\mathcal{X} \rightarrow T, \mathcal{L} \otimes \omega_{\mathcal{X}/T})$ . The quotient by this involution is the functor  $\mathcal{F}_{\text{Enr},2n}^\tau$ . The coarse moduli space  $\mathcal{M}_{\text{Enr},2n}$  is an étale double cover

$$\boxed{\text{doublecover}} \quad (3.8) \quad \mathcal{M}_{\text{Enr},2n} \rightarrow \mathcal{M}_{\text{Enr},2n}^\tau$$

of the coarse moduli space  $\mathcal{M}_{\text{Enr},2n}^\tau$  of  $\mathcal{F}_{\text{Enr},2n}^\tau$

Fixing a type  $h$  of a numerical polarization we decompose the functor and its coarse moduli space into the connected sum of components  $\mathcal{F}_{\text{Enr},h}$ , where  $h$  denotes the orbit of  $\text{O}(\mathbf{E})$  of vectors  $v$  with  $v^2 = 2n$  such that there exists an Enriques surface for which the image of  $h$  under some marking  $\mathbf{E} \rightarrow \text{Num}(S)$  is a very ample polarization.

The double cover  $\boxed{\text{doublecover}}$  (3.8)

$$\boxed{\text{gritsenko}} \quad (3.9) \quad \mathcal{M}_{\text{Enr},h} \rightarrow \mathcal{M}_{\text{Enr},h}^\tau$$

is discussed in  $\boxed{\text{GritsenkoHulek}}$  [29], where  $\mathcal{M}_{\text{Enr},h}^\tau$  is constructed as a quotient of an appropriate open subset  $\mathcal{M}_{\text{Enr},h}^{m,a}$  of  $\mathcal{M}_{\text{Enr}}^m$  by the finite group

$$\Gamma_h = r_{\mathbf{E}(2)^\perp}^{-1}(r_{\mathbf{E}(2)}(\text{O}(\mathbf{E})'_h)) \subset \Gamma_{\text{Enr}}.$$

Since the varieties  $\mathcal{M}_{\text{Enr},h}^{m,a}$  are arithmetic quotients of open subsets in a homogeneous domain,  $\mathcal{M}_{\text{Enr},2n}^\tau$  consists of finitely many connected components parameterized by types of numerical polarizations.

Now we specialize by taking  $h$  to be the orbit of the vector  $\mathbf{\Delta} \in \mathbf{E}$ , i.e. we consider families of Fano polarized Enriques surface. It is known that there is only one orbit of vectors of square norm 10 in  $\mathbf{E}$  that can realize a very ample polarization. So, we obtain that there is only one connected component of  $\mathcal{M}_{\text{Enr},10}^\tau$  which we denote by  $\mathcal{M}_{\text{Enr},\text{Fano}}^\tau$ .

$\boxed{\text{verra}}$  **Proposition 3.5.** *The Hilbert scheme  $\mathcal{H}_{10}$  of Fano polarized Enriques surfaces in  $\mathbb{P}^5$  is a smooth irreducible quasi-projective variety of dimension 45. The orbit space  $\mathcal{H}_{10}/\text{PGL}(6)$  exists as a separated algebraic space and admits an étale double cover map onto the coarse moduli space of  $\mathcal{M}_{\text{Enr},\text{Fano}}^\tau$  of Enriques surfaces with numerical Fano polarization.*

*Proof.* The only not proven assertion here is the irreducibility of the Hilbert scheme  $\mathcal{H}_{10}$ . This non-trivial result follows from  $\boxed{\text{verra}}$  [58]. Since it is not stated explicitly there, we have to briefly sketch the proof (see the details in  $\boxed{\text{Ciliberto}}$  [8]).  $\boxed{\text{Igor:Added a reference}}$  It is known that the moduli space  $\mathcal{M}_{\text{Enr}}$  is rationally dominated by a 10-dimensional affine space  $\mathbb{A}^{10}$  parameterizing *Enriques sextics*, i.e. surfaces  $V_q$  in  $\mathbb{P}^3$  given by equation

$$F_q = \sum_{1 \leq i < j < k \leq 4} x_i^2 x_j^2 x_k^2 + x_1 x_2 x_3 x_4 q(x_1, x_2, x_3, x_4) = 0,$$

where  $q$  is a nondegenerate quadratic form. The normalization  $S_q$  of  $V_q$  is an Enriques surface. The pre-image of each edge  $\ell_{ij} = V(x_i, x_j)$  of the coordinate tetrahedron  $x_1 x_2 x_3 x_4 = 0$ , is a half-fiber  $F_{ij}$  of an elliptic pencil on  $S_q$ . Verra chooses the edges  $\ell_{12}$  and  $\ell_{34}$  and considers the family

$\mathcal{F}$  of quintic elliptic curves  $E$  in  $\mathbb{P}^3$  that do not pass through the vertices of the coordinate tetrahedron  $x_1x_2x_3x_4 = 0$ , intersect exactly in one point the edges  $\ell_{12}, \ell_{34}$  and intersect exactly at two points the remaining edges. He shows that the linear system  $|C| = |F_{12} + F_{34} + E|$  is a Fano polarization on  $S_q$  [Verra, Proposition 3.1]. In Proposition 1.1. he shows that  $\mathcal{F}$  is an irreducible rational variety of dimension 10 that dominates  $\mathcal{M}_{\text{Enr}, \text{Fano}}$ .  $\square$

If we write  $3h$  as a sum of a nondegenerate canonical isotropic sequence  $(f_1, \dots, f_{10})$ , then we see that the group  $O(E)_h$  is isomorphic to the image of the permutation group  $\mathfrak{S}_{10} \subset O(E)'$  in  $O(10, \mathbb{F}_2)$ . Since  $\mathfrak{S}_{10}$  is almost simple group it is easy to see that it embeds in  $O^+(10, \mathbb{F}_2)$  (in fact the 2-level subgroup  $O(E)'(2)$  has no elements of finite order larger than 2 [DolgachevBrief, Theorem 8]). Thus

s10quotient

$$(3.10) \quad \mathcal{M}_{\text{Enr}, \text{Fano}}^\tau \cong \mathfrak{S}_{10} \backslash \mathcal{M}_{\text{Enr}, \Delta}^{\text{m}, a}.$$

It is a finite cover of  $\mathcal{M}_{\text{Enr}}$  of degree equal to  $[O^+(10, \mathbb{F}_2) : \mathfrak{S}_{10}] = 2^{13} \cdot 3 \cdot 17 \cdot 31$ .

**3.3. Moduli space of supermarked Enriques surfaces.** Following Beauville [4] we define the moduli stack  $\mathcal{K}_{K3, M, h}$  of lattice  $M$ -polarized K3 surfaces such that  $\mathcal{K}_{K3, M, h}$  is the groupoid of families of K3 surfaces as above together with a primitive embedding  $j_T : M_T \hookrightarrow \mathcal{P}_{\mathcal{X}/T}$  that sends a vector  $h \in M$  to a section of  $\mathcal{P}_{\mathcal{X}/T}$  that defines an ample polarization of each fiber.

The following proposition is proven in [Beauville, Proposition 2.6].

beauville

**Proposition 3.6.** *The stack  $\mathcal{K}_{K3, M, h}$  is smooth and irreducible of dimension  $20 - \text{rank} M$ .*

For any  $h \in E$  with  $h^2 > 0$  a family  $(\mathcal{X} \rightarrow T, \phi : E \rightarrow \mathcal{P}_{\mathcal{X}/T}^\tau)$  of marked Enriques surfaces defines a family of Enriques surfaces with numerical polarization of type  $h$ , by taking a section of  $\mathcal{P}_{\mathcal{X}/T}^\tau$  equal to  $\phi(h)$ .

We take  $h$  to be the vector  $\Delta \in E$  and consider it as a vector in  $E(2)$  with square norm equal to 20. It follows from our discussion of the moduli spaces of Enriques surfaces via the moduli spaces of lattice polarized K3 surfaces, that the stack  $\mathcal{K}_{K3, E(2), \Delta}$  is equivalent to the stack  $\mathcal{E}_{\text{Enr}}^{\text{m}}$  defined by families of marked Enriques surfaces with numerical Fano polarization.

Applying Corollary [2.5], we see that the groupoid  $\mathcal{E}_{\text{Enr}}^{\text{m}}(T)$  can be considered as a section  $s$  of the Cartesian product  $(\mathcal{P}_{\mathcal{X}/T}^\tau)^{10}$ . The pre-image  $\tilde{s}$  of  $s$  under the map  $(\mathcal{P}_{\mathcal{X}/T})^{10} \rightarrow (\mathcal{P}_{\mathcal{X}/T}^\tau)^{10}$ , possibly after a finite étale base change, splits into  $2^{10}$  sections  $(F_1, \dots, F_{10})$ , each defining a supermarking.

We define the stack  $\mathcal{E}_{\text{Enr}}^{\text{sm}, a}$  of supermarked Enriques surfaces by considering families  $\mathcal{X} \rightarrow T$  of Enriques surfaces with a section of  $(\mathcal{P}_{\mathcal{X}/T})^{10}$  defining an ample supermarking. It comes endowed with forgetful morphism

$$\sigma : \mathcal{E}_{\text{Enr}}^{\text{sm}, a} \rightarrow \mathcal{E}_{\text{Enr}}^{\text{m}, a},$$

which is a Galois étale cover of stacks with Galois group  $G = (\mathbb{Z}/2\mathbb{Z})^{10}$ . We refer to [48] for the definitions where it is also proved (Theorem 1.1) that the fundamental group of the stack is naturally isomorphic to the fundamental group of its coarse moduli space. This implies that there exists an étale Galois analytic map

$$\Phi : \mathcal{M}_{\text{Enr}}^{\text{sm}} \rightarrow \mathcal{M}_{\text{Enr}}^{\text{m}}.$$

with the Galois group isomorphic to  $(\mathbb{Z}/2\mathbb{Z})^{10}$ . We take the cover  $\mathcal{M}_{\text{Enr}}^{\text{sm}}$  as our definition of the *coarse moduli space of supermarked Enriques surfaces*. We leave to the experts in the theory of stacks to prove that it coincides with the coarse moduli space of the stack  $\mathcal{E}_{\text{Enr}}^{\text{sm}}$ .

irreducibility

**Theorem 3.7.** *The moduli space  $\mathcal{M}_{\text{Enr}}^{\text{sm}}$  is irreducible.*

*Proof.* Recall from the previous section that we have defined the moduli spaces  $\mathcal{M}_{\text{Enr},\text{Fano}}$ ,  $\mathcal{M}_{\text{Enr},\text{Fano}}^{\tau}$  and the open set  $\mathcal{M}_{\text{Enr}}^{\text{m},a} := \mathcal{M}_{\text{Enr},\Delta}^{\text{m},a}$  of  $\mathcal{M}_{\text{Enr}}^{\text{m}}$ . Let  $\mathcal{M}_{\text{Enr}}^{\text{sm},a} = \Phi^{-1}(\mathcal{M}_{\text{Enr}}^{\text{m},a})$ , then we have the following triangular commutative diagram

$$\begin{array}{ccc} & \mathcal{M}_{\text{Enr}}^{\text{sm},a} & \\ \swarrow & & \searrow \\ \mathcal{M}_{\text{Enr},\text{Fano}} & \xrightarrow{2:1} & \mathcal{M}_{\text{Enr},\text{Fano}}^{\tau} \end{array},$$

in which the downward arrows are étale covers. The r. h. s. one is the quotient under the group of the supermarking changes  $\mathbf{G}$  of order  $10! \cdot 2^{10}$  that acts by permutations of the elliptic half-fibers  $F_i$  and by switching  $F_i \leftrightarrow F_{-i} \in |F_i + K_S|$ ; the covering group  $\tilde{\mathbf{G}}$  of the l. h. s. cover is of order  $10! \cdot 2^9$  and is generated by the permutations of the  $F_i$  and by switching an even number of them.

We want to prove that  $\mathcal{M}_{\text{Enr}}^{\text{sm}}$  is irreducible. To this end, let us choose one of its irreducible components  $M$  and prove that  $M = \mathcal{M}_{\text{Enr}}^{\text{sm},a}$ . We know that  $\mathcal{M}_{\text{Enr}}^{\text{m}}$  is an arithmetic quotient of a Hermitian symmetric domain, and hence is irreducible. Also, according to Proposition 3.5,  $\mathcal{M}_{\text{Enr},\text{Fano}}$  is irreducible. Hence the restrictions to  $M$  of the cover maps from the above diagram provide a similar triangular diagram

$$\begin{array}{ccc} & M & \\ \swarrow / \tilde{G} & & \searrow / G \\ \mathcal{M}_{\text{Enr},\text{Fano}} & \xrightarrow{2:1} & \mathcal{M}_{\text{Enr},\text{Fano}}^{\tau} \end{array},$$

in which the downward maps are irreducible étale covers with monodromy groups  $\tilde{G} \subset \tilde{\mathbf{G}}, G \subset \mathbf{G}$ .

The factorization

**factor** (3.11) 
$$\mathcal{M}_{\text{Enr}}^{\text{sm},a} \rightarrow \mathcal{M}_{\text{Enr}}^{\text{m},a} \rightarrow \mathcal{M}_{\text{Enr},\text{Fano}}$$

of the quotient map over  $G$  in the last diagram provides the exact triple of covering groups

$$(3.12) \quad 0 \rightarrow K \rightarrow G \xrightarrow{\kappa} \mathfrak{S}_{10} \rightarrow 1,$$

where  $K \subset \mathbf{D} \cong (\mathbb{Z}/2\mathbb{Z})^{10}$ . The irreducibility of  $\mathcal{M}_{\text{Enr}}^{\text{sm},a}$  is equivalent to the equality  $K = \mathbf{D}$ , which we are now going to prove.

We observe that  $\tilde{G}$  is a subgroup of index 2 in  $G$  and that the restriction  $\tilde{\kappa}$  of  $\kappa$  to  $\tilde{G}$  is surjective by the covering homotopy argument. Hence we have the exact triple

$$0 \rightarrow K' \rightarrow \tilde{G} \xrightarrow{\tilde{\kappa}} \mathfrak{S}_{10} \rightarrow 1,$$

where  $K' = K \cap \mathbf{D}'$ ,  $\mathbf{D}' = \{(m_1, \dots, m_{10}) \in (\mathbb{Z}/2\mathbb{Z})^{10} \mid \sum_i m_i = 0\}$ .

The fact that  $K' \subsetneq K$  means that  $K$ , identified with its image in  $(\mathbb{Z}/2\mathbb{Z})^{10}$ , contains an element  $m = (m_1, \dots, m_{10})$  with odd number of nonzero components. Let  $\text{supp}(m)$  denote the set of subscripts  $i$  for which  $m_i \neq 0$ . Assume we have chosen a  $m$  with smallest possible support. For any permutation  $\sigma \in \mathfrak{S}_{10}$  and any  $g \in \kappa^{-1}(\sigma)$ , we have  $\text{supp}(gmg^{-1}) = \sigma(\text{supp}(m))$ . Thus conjugating  $m$  with an appropriate element of  $G$ , we can assume that  $\text{supp}(m) = \{1, \dots, k\}$ , where  $k \in \{1, 3, 5, 7, 9\}$ . We will now see that  $k = 1$ . Indeed, if  $3 \leq k \leq 9$ , then conjugating  $m$  with some lifts of the transpositions  $(k-1, k+1)$ ,  $(k, k+1)$ , we obtain two elements  $m', m''$  of  $K$  with respective supports  $\{1, \dots, k-2, k, k+1\}$ ,  $\{1, \dots, k-2, k-1, k+1\}$ , so that  $\text{supp}(m + m' + m'') = \{1, \dots, k-2\}$  is smaller than  $\text{supp}(m)$ , which is a contradiction. Hence  $k = 1$  and  $K$  contains the monodromy transformation switching just one pair of divisor classes  $F_1 \leftrightarrow F_{-1}$ . But then  $K$  contains the transformations  $F_i \leftrightarrow F_{-i}$  for all  $i = 1, \dots, 10$ , which implies that  $K = \mathbf{D}$ .  $\square$

*Remark 3.8.* It is natural to expect that the complex analytic space  $\mathcal{M}_{\text{Enr}}^{\text{sm},a}$  is isomorphic to the quotient of an open subset of the period domain for a normal subgroup of the monodromy group  $\Gamma_{\text{Enr}}$  with quotient group isomorphic to  $\mathbf{D} \cong \mathbb{F}_2^{10}$ . According to R. Borcherds,  $\Gamma_{\text{Enr}}$  contains such a subgroup. Let us explain his construction.

Let  $N = \mathbf{E}(2)^\perp$ . We have

$$N/2N^\vee \cong (\mathbf{E}_8(2) \oplus \mathbf{U}(2) \oplus \mathbf{U}) / (2\mathbf{E}_8(2)^\vee \oplus 2\mathbf{U}(2)^\vee \oplus 2\mathbf{U}^\vee) \cong \mathbf{U}/2\mathbf{U} \cong \mathbb{F}_2^2.$$

Let  $f, g$  be the standard isotropic generators of  $\mathbf{U}$  and  $\bar{f}, \bar{g}$  be their cosets in  $\mathbf{U}/2\mathbf{U}$ . The subgroup  $\Gamma_{\text{Enr}}$  of  $\mathbf{O}(N)$  in its natural action on  $(N/2N)^\vee$  leaves the vector  $\eta = \bar{f} + \bar{g}$  invariant (since it is the only vector of square 2 mod 4 in  $\mathbb{F}_2^2$  with quadratic form inherited from  $\mathbf{U}$ ). Let  $A$  be the quotient  $\mathbb{F}_2^2/\mathbb{F}_2\eta \cong \mathbb{F}_2$ . Define a map from

$$\alpha : \Gamma_{\text{Enr}} \rightarrow \text{Hom}(N^\vee/N, A) \cong \mathbb{F}_2^{10}$$

as follows. The image of  $g \in \Gamma_{\text{Enr}}$  is equal to the linear function  $l(w + N) = g(w) - w \bmod \mathbb{F}_2\eta$ . One can show that the images of reflections in vectors of square 2 are nonzero, and generate  $\text{Hom}(N^\vee/N, A) \cong \mathbb{F}_2^{10}$ . One can hope

that the quotient space of an open subset of  $\mathcal{D}_N$  by the group  $\text{Ker}(\alpha)$  is the moduli space of the stack  $\mathcal{E}_{\text{Enr}}^{\text{sm}}$ .

#### 4. REYE CONGRUENCES

**4.1. General nodal Enriques surfaces.** An Enriques surface is called *general nodal* if the numerical classes of smooth rational curves on it are congruent modulo  $2\text{Num}(S)$  (see [DolgachevBrief \[21\]](#)). One can show that the isomorphism classes of general nodal surfaces are parameterized by a constructive subset of  $\mathcal{M}_{\text{Enr}}$  which is a dense subset of  $\mathcal{M}_{\text{Enr}}^{\text{nod}}$ . We denote it by  $\mathcal{M}_{\text{Enr}}^{\text{gnod}}$ . Its pre-image under the map  $\mathcal{M}_{\text{Enr}}^m \rightarrow \mathcal{M}_{\text{Enr}}$  is the locus of points whose periods in  $\mathcal{D}_{\mathbf{T}}$  do not belong to the intersection of any two irreducible components of the divisor  $\mathcal{H}_{-4}$ . One has the following characterization of general nodal surfaces (the proof can be found in a forthcoming new two-volumes edition of [CP \[13\]](#)).

gennodal

**Theorem 4.1.** *Let  $S$  be a nodal Enriques surface. The following conditions are equivalent.*

- (a)  $S$  is general nodal.
- (b) Any genus one fibration on  $S$  has irreducible double fibers and has at most one reducible non-multiple fiber that consists of two irreducible components.
- (c) Let  $\mathcal{R}_S$  be the set of smooth rational curves on  $S$ . For any Fano polarization  $h$ , the set  $\Pi_h = \{R \in \mathcal{R}_S : R \cdot h \leq 4\}$  consists of one element.
- (d) For any  $d \leq 4$ ,  $S$  admits a Fano polarization  $h$  such that  $\Pi_h = \{R\}$ , where  $R \cdot h = d$ .
- (e) A genus one pencil that admits a smooth rational curve as a 2-section does not contain reducible fibres.
- (f) The Nikulin  $R$ -invariant is isomorphic to the root lattice  $A_1$  and Nikulin's  $r$ -invariant consists of one element (see the definitions in [DolgachevBrief \[21, §5\]](#)).

It follows from property (c) that any canonical isotropic 10-sequence on a general nodal surface has non-degeneracy invariant equal to 9 or 10. Thus a Fano model of a general nodal surface is either smooth or has one ordinary double point. The moduli space of marked general nodal surfaces consists of 5 irreducible components distinguished by the minimal possible degree  $h \cdot R \in \{0, 1, 2, 3, 4\}$  of a  $(-2)$ -curve  $R$  on it (the proof uses the computation of the group of automorphisms of a general nodal surface [DolgachevBrief \[21, Theorem 7\]](#) and can be found in volume 2 of the new edition of [CP \[13\]](#)).

**4.2. Reye polarizations.** A Fano polarization is called a *Reye polarization* if the image of the map  $\phi_{\Delta} : S \rightarrow \mathbb{P}^5$  lies in a nonsingular quadric. A nonsingular quadric  $Q$  in  $\mathbb{P}^5$  can be identified with the Plücker embedded Grassmannian variety  $\text{Gr}(2, 4)$  of planes in  $\mathbb{C}^4$  or, equivalently, lines in  $\mathbb{P}^3$ . It is known that  $Q$  has two irreducible 3-dimensional families of planes and the cohomology classes of members of these families freely generate  $H^4(Q, \mathbb{Z})$ .

In the Grassmannian interpretation, one family consists of planes  $\sigma_x$  of lines through a fixed point  $x$ , and another family consists of planes  $\sigma_\pi$  of lines contained in a fixed plane  $\pi$ . It follows from this that two different planes belong to the same family if and only if they intersect at one point, or equivalently, span a hyperplane in  $\mathbb{P}^5$ .

An irreducible surface  $X$  in  $\text{Gr}(2, 4)$  is called a *congruence of lines*. Its cohomology class  $[X]$  in  $H^4(\text{Gr}(2, 4), \mathbb{Z})$  is equal to  $m[\sigma_x] + n[\sigma_\pi]$ , where  $m$  is called the *order* of  $X$  and  $n$  is called the *class* of  $X$ . The degree of  $X$  in  $\mathbb{P}^5$  is equal to  $m + n$ .

**Proposition 4.2.** *An Enriques surface  $S$  admitting a Reye polarization  $\Delta$  is nodal.*

*Proof.* We may assume that the Reye polarization is ample (otherwise  $S$  is nodal). Let  $Q$  be the nonsingular quadric containing the image of  $S$  under a map given by the Reye polarization. If we write  $3\Delta \sim F_1 + \dots + F_{10}$ , then it follows from Proposition [2.9](#) <sup>twentyplanes1</sup> that the images of twenty curves  $F_i$  and  $F_{-i} \in |F_i + K_S|$  under the map given by the Reye polarization span planes  $\Lambda_i \subset Q$  and the planes  $\Lambda_i$  and  $\Lambda_j$  intersect at one point if  $i + j \neq 0$ . Since  $\Lambda_i$  and  $\Lambda_{-i}$  intersects  $\Lambda_j, j \neq i, -i$ , at one point, we see that they belong to the same family and hence intersect at one point. So, all twenty planes intersect each other at one point (the images of the curves  $F_i$  and  $F_{-i}$  still do not intersect but the planes which they span do intersect). Since  $\Lambda_i, \Lambda_{-i}$  span a hyperplane,  $|\Delta - F_i - F_{-i}| = |\Delta - 2F_i + K_S| \neq \emptyset$ . Since  $(\Delta - F_i - F_{-i})^2 = -2$ , we see that  $S$  must be a nodal surface.  $\square$

The converse is also true <sup>DR</sup> [[14](#), Theorem 1]:

**Theorem 4.3.** *Let  $\Delta$  be a Fano polarization on an Enriques surface  $S$ . Suppose there exists a half-fiber  $F$  of an elliptic fibration such that  $|\Delta - 2F + K_S| \neq \emptyset$ . Then there exists a rank 2 vector bundle  $E$  on  $S$  with  $c_1(E) = [\Delta], c_2 = 10$  and  $h^0(E) = 4$  such that the evaluation map  $S \rightarrow \text{Ker}(\text{ev}_x : H^0(S, E) \rightarrow E_x)$  defines an embedding of  $S$  into the Grassmann quadric  $\text{Gr}(2, H^0(E)) \subset \mathbb{P}^5$ . The image is a smooth congruence of lines of order 7 and class 3.*

**Corollary 4.4.** *A general nodal Enriques surface  $S$  admits an ample Reye polarization.*

*Proof.* We have already noticed in Theorem [4.1](#) <sup>gennodal</sup> that smooth rational curves on  $S$  form one class modulo  $2\text{Num}(S)$ . It follows from Theorem [4.1](#) <sup>gennodal</sup> that  $S$  contains a degree 4 curve with respect to a numerical Fano polarization  $h$  defined by a non-degenerate isotropic sequence  $(f_1, \dots, f_{10})$ . Its numerical class is equal to  $\delta - 2f_1$  <sup>CossecDolgachev</sup> [[12](#), Remark 4.7]. Thus, we can find a representative  $\Delta$  of  $h$  in  $\text{Pic}(S)$  such that  $|\Delta - 2F_1 + K_S| \neq \emptyset$ , and we can apply the previous theorem.  $\square$



A smooth congruence with  $(m, n) = \binom{7}{\text{CR}} \binom{3}{\text{CAG}}$  is called a *Reye congruence*. It is constructed as follows (see [T1] or [T9, 1.1.7]). Let  $W = \mathbb{P}(L)$  be a 3-dimensional linear system (a web) of quadrics in  $\mathbb{P}^3 = \mathbb{P}(V_4)$ . Via polarization, it defines a 3-dimensional linear system  $W^p$  of symmetric divisors of type  $(1, 1)$  on  $\mathbb{P}^3 \times \mathbb{P}^3$ . We say that  $W$  is a *regular web* of quadrics if the base locus  $\text{PB}(W)$  of  $W^p$  is smooth. By the adjunction formula,  $\text{PB}(W)$  is a K3 surface. The *Reye congruence of lines*  $\text{Rey}(W)$  associated with  $W$  is the variety of lines in  $\mathbb{P}^3$  that are contained in a pencil of quadrics from  $W$ . One can show that the regularity assumption implies the following good properties of  $W$ :

- (i) The base locus of  $W$  is empty.
- (ii) The involution automorphism  $\tau \in \text{Aut}(\mathbb{P}^3 \times \mathbb{P}^3)$  that switches the factors has no fixed points on  $\text{PB}(W)$ , the orbits of  $\tau$  span Reye lines and hence the quotient space  $\text{PB}(W)/(\tau)$  is isomorphic to  $\text{Rey}(W)$ .
- (iii) The image  $\text{St}(W)$  of  $\text{PB}(W)$  under the projection to any factor is a quartic surface with at most nodes as singularities. It is the *Steinerian surface* (or the *Jacobian surface*) of  $W$ , the locus of singular points of singular quadrics in  $W$ .
- (iv) The *discriminant surface*  $D_W$  of  $W$  parametrizing singular quadrics in  $W$  is an irreducible quartic surface with 10 nodes corresponding to rank 2 quadrics in  $W$ .
- (v) The surface  $\tilde{D}_W = \{(Q, x) \in W \times \mathbb{P}^3 : x \in \text{Sing}(Q)\}$  is smooth and the two projections  $\pi_i$  are minimal resolution of singularities of the surfaces  $D_W$  and  $\text{St}(W)$ .

We summarize these constructions by the following diagram

$$\boxed{\text{incidence}} \quad (4.1) \quad \begin{array}{ccccc} & \tilde{D}_W & & \text{PB}(W) & \\ & \swarrow \pi_1 & & \swarrow p_1=p_2 & \searrow /(\tau) \\ D_W & & \text{St}(W) & & \text{Rey}(W). \end{array}$$

Considered  $\text{PB}(W)$  as a surface in  $\mathbb{P}^{15}$  via the Segre embedding of  $\mathbb{P}^3 \times \mathbb{P}^3$ , the quotient map

$$\boxed{\text{mapreye}} \quad (4.2) \quad \pi : \text{PB}(W) \rightarrow \text{Rey}(W) \subset G(2, V_4) \subset \mathbb{P}^5 = \mathbb{P}\left(\bigwedge^2 V_4\right)$$

is the projection from the image of the diagonal of  $\mathbb{P}^3 \times \mathbb{P}^3$  equal to the second Veronese variety of  $\mathbb{P}^3$  spanning  $\mathbb{P}^9$ .

It follows from property (ii) that  $\text{Rey}(W)$  is an Enriques surface and  $\text{PB}(W)$  is its K3 cover. Let  $W^p = \mathbb{P}(\tilde{L})$ , where  $\tilde{L}$  is the image of  $L \subset S^2 V_4^\vee$  under the polarization map  $S^2 V_4^\vee \rightarrow V_4^\vee \otimes V_4^\vee$ . It is a 4-dimensional subspace in  $V_4^\vee \otimes V_4^\vee = H^0(\mathbb{P}^3 \times \mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1, 1))$ . Let

$$\boxed{\text{decom}} \quad (4.3) \quad V_4^\vee \otimes V_4^\vee = S^2 V_4^\vee \oplus \bigwedge^2 V_4^\vee.$$

The linear system  $|\wedge^2 V_4^\vee|$  restricted to  $\text{PB}(W)$  defines morphism [\(4.2\)](#) <sup>mapre</sup>.

In fact,  $\text{Rey}(W)$  is a Fano model of the Enriques surface  $\text{PB}(W)/(\tau)$ . Let us see geometrically the corresponding Fano polarization  $\Delta$  and a sequence of 10 half-fibers  $(F_1, \dots, F_{10})$  satisfying [\(2.8\)](#).

One can consider  $\tilde{D}_W$  as the graph of the rational map

$$\boxed{\text{st}} \quad (4.4) \quad D_W \dashrightarrow \text{St}(W), \quad Q \mapsto \text{Sing}(Q),$$

(classically known as the *Steinerian map*). The projection  $\pi_2$  is not an isomorphism in general. Its fiber over a point  $x$  is isomorphic, under the first projection  $\pi_1$ , to the pencil of quadrics in  $W$  with singular point at  $x$ . The projection is an isomorphism if and only if the determinantal surface  $D_W$  does not contain lines. A regular web with this property is called an *excellent web*. The assumption that  $S$  is a general nodal surface implies that the web  $W$  is an excellent web. For an excellent web  $W$ , the projections  $\pi_2$  and  $p_1 = p_2$  are isomorphisms and all three surfaces  $\tilde{D}_W, \text{St}(W), \text{PB}(W)$  can be identified with the K3-cover  $Y$  of  $\text{Rey}(W)$ . The surface  $Y$  contains two natural divisor classes

$$\eta_H = c_1(\pi_1^* \mathcal{O}_{D_W}(1)), \quad \eta_S = c_1(p_2^* \mathcal{O}_{\text{St}(W)}(1)).$$

Let  $\Theta_i, i = 1, \dots, 10$ , be the divisor classes of fibers of  $p_1$  over the ten nodes. These are  $(-2)$ -curves on  $\text{PB}(W)$ . We have a fundamental relation

$$\boxed{\text{fr}} \quad (4.5) \quad |2\eta_S| = |3\eta_H - \Theta_1 - \dots - \Theta_{10}|$$

expressing the fact that  $D_W$  is given by the symmetric determinant (see [\[19\]](#) <sup>CAG</sup>, Proposition 4.2.5). Here the linear system on the right-hand side can be naturally identified with the linear system of polar cubics of the determinantal surface  $D_W$ . It defines the Steinerian rational map [\(4.4\)](#) <sup>st</sup>. The linear system on the left-hand side can be identified with the linear system of quadrics in  $\mathbb{P}^3 = \mathbb{P}(V_4)$ . Under the identification of these linear systems via [\(4.5\)](#) <sup>fr</sup>, the intersection of a polar hypersurface with  $D_W$  is mapped under the Steinerian map [\(4.4\)](#) <sup>st</sup> to the intersection of the corresponding quadric with  $\text{St}(W)$ .

Let

$$E_i = \eta_S - \Theta_i, \quad i = 1, \dots, 10$$

They define the elliptic pencils  $|E_i|$  on  $\text{PB}(W)$  cut out by the pre-images on  $\text{PB}(W)$  of plane sections of  $\text{St}(W)$  through the line  $\pi_2(\Theta_i)$ . The pencils  $|E_i|$  are  $\tau$ -invariant. In fact, if we let  $\eta = \eta_S + \tau^*(\eta_S)$ , we find that  $\eta$  is equal to  $\pi^* \mathcal{O}_{\text{Rey}(W)}(1)$  and hence  $\eta^2 = 20$  and  $\eta_S \cdot \tau^*(\eta_S) = 6$ . This gives

$$E_i \cdot \tau^*(E_i) = (\eta_S - \Theta_i) \cdot (\eta_S - \Theta_j) = 0,$$

hence  $|E_i|$  and  $|\tau^*(E_i)|$  define the same elliptic pencil. We have (see [\[11, 2.4\]](#)):

$$\begin{aligned}
(4.6) \quad \eta_H^2 &= \eta_S^2 = 4, \\
\eta_H \cdot \eta_S &= 6, \\
\eta_H \cdot \Theta_i &= 0, \quad \eta_S \cdot \Theta_i = 1, \\
\Theta_i \cdot \Theta_j &= 0, \quad i \neq j, \quad \Theta_i^2 = -2, \\
\eta_S \cdot E_i &= 3, \quad \eta_H \cdot E_i = 6, \\
E_i \cdot E_j &= 2, \quad i \neq j, \quad E_i^2 = 0, \\
E_i \cdot \Theta_j &= 1, \quad i \neq j, \quad E_i \cdot \Theta_i = 3.
\end{aligned}$$

We leave to the reader to use the previous relations to check the assertions of the following theorem.

**T4.1** **Theorem 4.5.** *Assume  $W$  is an excellent web and denote by  $Y$  a minimal nonsingular model of  $D_W$  isomorphic to  $\tilde{D}_W$ , or  $PB(W)$ , or  $St(W)$ . Let  $\eta = \eta_S + \tau^*(\eta_S)$ . Then  $\eta = \pi^*(\Delta)$ , where  $\Delta$  is a hyperplane section of  $Rey(W)$  in its Plücker embedding. The divisors  $E_i$  are nef divisor classes on  $Y$  and  $|E_i|$  is an elliptic pencil on  $Y$  with  $|E_i| = |\tau^*(E_i)|$ . We have  $E_i + \tau(E_i) = \pi^*(2F_i)$  for some elliptic pencil  $|2F_i|$  on  $Rey(W)$  and  $f_i = [F_i], i = 1, \dots, 10$ , form a non-degenerate canonical isotropic 10-sequence such that*

$$3[\Delta] = f_1 + \dots + f_{10}.$$

*The pre-images of the half-fibers of  $|2F_i|$  are the cubic curves on  $St(W)$  that are residual to the line  $\Theta_i$  in the plane components of the quadrics from  $W$  with the singular line  $\Theta_i$ . The curves  $R_i = \pi(\Theta_i)$  on  $Rey(W)$  are curves of degree 4 such that*

$$R_i \cdot F_j = 1, \quad j \neq i, \quad R_i \cdot F_i = 3.$$

Consider the discriminant quartic surface  $D_W$ . It is classically called a *quartic symmetroid*. Among all normal quartic surfaces with 10 nodes, quartic symmetroids are distinguished by the following *Cayley property* (see [\[11, Corollary 2.4.6\]](#)):

- The projection from any node defines a double cover of  $\mathbb{P}^2$  branched along the union of two cubic curves everywhere tangent to the same conic.

Recall that for any point  $x$  lying on a reduced irreducible hypersurface  $V$  in  $\mathbb{P}^n$ , the *enveloping cone*  $EC_x(V)$  of  $V$  at the point  $x$  is the closure of the union of lines passing through the point  $x$  that are tangent to  $V$  at some other point. The closure of the locus of such points is the branch divisor of the projection of  $V$  to  $\mathbb{P}^{n-1}$  with center at  $x$ . It is cut out in  $V$  by the first polar  $P_x(V)$  of  $V$  with pole at  $x$  (see [\[19, 1.1\]](#)). We can rephrase the Cayley property by saying that the enveloping cone of the quartic at its node splits into the union of two cubic cones. If we choose coordinates such that the

node is  $(1, 0, 0, 0)$ , then we can write the equation of the quartic in the form

$$F = x_0^2 F_2(x_1, x_2, x_3) + 2x_0 F_3(x_1, x_2, x_3) + F_4(x_1, x_2, x_3) = 0.$$

The enveloping cone has the equation

envelop

$$(4.7) \quad F_3(x_1, x_2, x_3)^2 - F_2(x_1, x_2, x_3)F_4(x_1, x_2, x_3) = 0.$$

The quadric  $V(F_2)$  is the tangent cone of  $V$  and its projection is a conic in  $\mathbb{P}^2$  which is everywhere tangent to the branch divisor, the tangency locus being the intersection  $V(F_2) \cap V(F_3)$ . The additional singular points are projected to singular points of the branch divisor.

*Remark 4.6.* The Cayley property satisfied by the surfaces  $D_W$  shows that quartic surfaces used by M. Artin and D. Mumford in their celebrated paper [1] are exactly quartic symmetroids, often called Cayley quartic symmetroids. Note also that a choice of an irreducible component in each enveloping cone is equivalent to a supermarking of  $\text{Rey}(W)$ . According to [34, Theorem 4.3], this choice is equivalent to a choice of a small resolution of the double cover of  $\mathbb{P}^3$  branched along the quartic symmetroid. [IngallsKuznetsov]

**4.3. The bitangent surface.** Let  $X$  be a normal quartic surface in  $\mathbb{P}^3$ . The closure of lines that are tangent to  $X$  at two points is a congruence of lines in  $\text{Gr}(2, 4)$  which we denote by  $\text{Bit}(X)$ . A general plane section of  $X$  is a smooth plane quartic curve, so it admits 28 bitangent lines. This shows that the class of  $\text{Bit}(X)$  is equal to 28. A less obvious fact is that the order of  $\text{Bit}(X)$  is equal to 12 [57, p. 281]. Thus its bidegree is equal to  $(12, 28)$  and it is a surface of degree 40 in the Grassmannian  $\text{Gr}_1(\mathbb{P}^3)$ . [Salmon]

We take  $X$  to be the discriminant surface  $D_W \subset W \subset \mathbb{P}(S^2 V_4^\vee)$ . We assume that  $W$  is an excellent web of quadrics. Any line in  $\mathbb{P}(L)$  is a pencil of quadrics contained in  $W$ . Since no line is contained in  $D_W$ , any pencil in  $W$  contains a nonsingular quadric. Also it has no quadrics of rank 1. [HodgePedoe]

The known classification of pencils of quadrics in  $\mathbb{P}^3$  [33, pp. 305-307] gives the following.

wall

**Lemma 4.7.** *Let  $\mathcal{P}$  be a pencil of quadrics in  $\mathbb{P}^3$ . Assume that it contains a nonsingular quadric, it does not contain quadrics of rank 1 and the number  $r$  of its singular members is at most 2. Let  $D_{\mathcal{P}} = \mathcal{P} \cap D_W$  considered as an effective divisor on  $\mathcal{P}$  and  $m$  be the number of reducible quadrics in  $\mathcal{P}$ . Then one of the following cases occurs (we use the numbering of cases given in loc. cit.).*

- (iii)  $r = 2, m = 2$ ,  $D_{\mathcal{P}} = 2a + 2b$  and  $Bs(\mathcal{P})$  consists of four different lines;
- (vii)  $r = 2, m = 1$ ,  $D_{\mathcal{P}} = 3a + b$  and  $Bs(\mathcal{P})$  consists of two tangent conics;
- (viii)  $r = 2, m = 1$ ,  $D_{\mathcal{P}} = 2a + 2b$  and  $Bs(\mathcal{P})$  is a smooth conic and a pair of lines intersecting at a point not lying on the conic;
- (x)  $r = 2, m = 0$ ,  $D_{\mathcal{P}} = 2a + 2b$  and  $Bs(\mathcal{P})$  is the union of a rational normal cubic and a line intersecting it at two different points;

- (xi)  $r = 1, m = 1, D_{\mathcal{P}} = 4a$  and  $Bs(\mathcal{P})$  is the union of 3 lines, one is taken with multiplicity 2;
- (xii)  $r = 2, m = 0, D_{\mathcal{P}} = 3a + b$  and  $Bs(\mathcal{P})$  is an irreducible cuspidal quartic curve;
- (xiii)  $r = 1, m = 1, D_{\mathcal{P}} = 4a$  and  $Bs(\mathcal{P})$  is the union of a smooth conic and a pair of lines intersecting at a point on the conic;
- (xiv)  $r = 1, m = 0, D_{\mathcal{P}} = 4a$  and  $Bs(\mathcal{P})$  is the union of a rational normal cubic and its tangent line.

A Reye line  $\ell$  is contained in the base locus of a pencil  $\mathcal{P}_{\ell}$  of quadrics in  $W$ . The remaining part of the base locus is a curve  $\gamma$  of degree 3 lying on a nonsingular member of the pencil as a curve of bidegree  $(1, 2)$ . For a general Reye line, the curve  $\gamma$  is a rational normal cubic intersecting the line at two different points. Thus it defines a pencil of type (x) whose closure is equal to the bitangent surface. This defines a map

$$\boxed{\text{reyemap}} \quad (4.8) \quad \nu : \text{Rey}(W) \rightarrow \text{Bit}(D_W) \subset \text{Gr}(2, L), \quad \ell \mapsto \mathcal{P}_{\ell}.$$

Note that, since  $W$  has no base points, each line in  $\mathbb{P}^3$  is contained in the linear system in  $W$  of dimension  $\leq 1$ , so the map is a regular map. Since  $\nu$  is bijective on the set of lines of type (x) and  $\text{Rey}(W)$  is smooth, the map  $\nu$  is the normalization map.

*Remark 4.8.* The lines of type (xi) intersect  $D_W$  with multiplicity 4 at one point but do not belong to  $\text{Bit}(D_W)$ . In fact one of the lines in the base locus of such a pencil is the singular line of a reducible quadric in  $W$  and it is known <sup>CR</sup> that it cannot be a Reye line if  $W$  is excellent (this property is taken in [11] as one of the properties defining a regular web of quadrics and it is shown that the definition coincides with our definition of an excellent web). The lines of types (vii), (xii) define pencils of quadrics with no lines in their base locus. They belong to the surface of flex lines. Lines of all other types belong to the bitangent surface. Lines of type (viii) passes through a node of  $D_W$ , it is a general line of the enveloping cone. Lines of type (xiii) are common lines of the enveloping cone and the tangent cone.

The following <sup>wall</sup> theorem follows from the description of Reye lines based on Lemma 4.7.

$\boxed{\text{bit}}$  **Theorem 4.9.** *Assume that  $W$  is an excellent web of quadrics in  $\mathbb{P}^3$ . The bi-degree of the congruence  $\text{Bit}(D_W)$  is equal to  $(12, 28)$ , in particular, it is a surface of degree 40 in the Plücker space  $\mathbb{P}^5$ . Its singular locus consists of 10 pairs of smooth plane cubic curves  $B_i, B'_i$  representing the generators of the enveloping cone of  $D_W$  at the 10 nodes. The union  $B_i \cup B'_i$  is equal to the intersection of  $\text{Bit}(D_W)$  with the plane of lines through the node  $q_i$ . Two cubics  $B_i$  and  $B'_i$  intersect at 9 points corresponding to the lines  $\ell_{ij} = \langle q_i, q_j \rangle \in \text{Bit}(D_W)$ . Two cubics from different pairs  $(B_i, B'_i)$  and  $(B_j, B'_j)$  intersect at one point  $\ell_{ij}$ . The normalization of  $\text{Bit}(D_W)$  is naturally isomorphic to the Reye congruence  $\text{Rey}(W)$  and the restriction of the*

normalization map  $\nu : \text{Rey}(W) \rightarrow \text{Bit}(D_W)$  to  $\nu^{-1}(B_i)$  is an unramified double cover  $\nu^{-1}(B_i) \rightarrow B_i$  (the same is true for  $B'_i$ ). The pre-image of  $\ell_{ij}$  consists of 4 points.

*Remark 4.10.* One can see the normalization map [\(4.8\)](#) <sup>reymap</sup> also as follows (see [\[26\]](#) <sup>F3</sup>). Let  $D_W^{[2]}$  be the Hilbert square of  $D_W$ . Consider a birational involution  $\tau$  of  $D_W^{[2]}$  that assigns to a pair of points the residual pair of points on the line joining the pair. The involution  $\tau$  lifts to a biregular involution  $\tilde{\tau}$  of the Hilbert square  $Y^{[2]}$  of a minimal nonsingular model  $Y$  of  $D_W$ . Its set of fixed points is isomorphic to the Reye congruence  $\text{Rey}(W)$ . The restriction of the map  $Y^{[2]} \rightarrow D_W^{[2]}$  is the normalization of the image of the map isomorphic to  $\text{Bit}(D_W)$ .

Let us add that the normalization map [\(4.8\)](#) <sup>reymap</sup> is given by a linear subsystem of  $|\mathcal{O}_{\text{Rey}(W)}(2)|$ . To see this, we may assume that  $W = \mathbb{P}(L)$  is spanned by four quadrics  $Q_i$  which we represent by symmetric matrices  $A_i$ . Take a Reye line  $\ell$  represented by 2 points  $([v], [w]) \in \text{PB}(W) \subset \mathbb{P}^3 \times \mathbb{P}^3$ . The map  $\nu$  assigns to  $\ell$  the pencil of quadrics in  $W$  containing  $\ell$ , i.e. the projectivized kernel  $\text{Ker}(r)$  of the restriction map  $r : L \rightarrow H^0(\ell, \mathcal{O}_\ell(2))$ . It is clear that  $\text{Ker}(r)$  is equal to the kernel of the evaluation map

$$L \cong \mathbb{C}^4 \rightarrow \mathbb{C}^3, \quad A \mapsto (vAv, wAw, vAw).$$

The Plücker coordinates of the pencil are equal to the maximal minors of the matrix

$$\begin{pmatrix} vA_1v & vA_2v & vA_3v & vA_4v \\ wA_1w & wA_2w & wA_3w & wA_4w \end{pmatrix}$$

It is easy to see that they are expressed by quadratic polynomials in Plücker coordinates of  $\ell$ . It shows that  $\text{Bit}(D_W)$  is isomorphic to the projection of  $\text{Rey}(W)$  embedded in  $|\mathcal{O}_{\mathbb{P}^5}(2)|^* \cong \mathbb{P}^{20}$  by the complete linear system  $|\mathcal{O}_{\mathbb{P}^5}(2)|$ .

Cayley

**4.4. Cayley models.** Consider the decomposition [\(4.3\)](#) <sup>decom</sup>. The linear system  $|S^2V_4^\vee|$  of quadrics in  $\mathbb{P}(V_4)$  defines a morphism

$$\phi : \mathbb{P}(V_4) \times \mathbb{P}(V_4) \rightarrow \mathbb{P}(S^2V_4)$$

which factors through a closed embedding

$$\mathbb{P}(V_4) \times \mathbb{P}(V_4) / (\tau) \hookrightarrow \mathbb{P}(S^2V_4).$$

We know that  $\text{PB}(W)$  is the base locus of a 3-dimensional linear system of sections of  $\mathbb{P}(V_4) \times \mathbb{P}(V_4)$  defined by the bilinear forms associated to quadratic forms from  $L$ . This implies that  $\text{PB}(W)$  is equal to the pre-image under  $\phi$  of  $\mathbb{P}(L^\perp)$ , where  $L^\perp$  is the annihilator of  $L$  in  $S^2V_4 = (S^2V_4^\vee)^\vee$ . Thus we obtain a closed embedding

$$\iota_c : \text{Rey}(W) \hookrightarrow \mathbb{P}(L^\perp) \cong \mathbb{P}^5.$$

The image  $\text{Cay}(W)$  is called the *Cayley model* of the Enriques surface  $\text{Rey}(W)$ . It is contained in the 5-dimensional subspace  $\mathbb{P}(L^\perp) \subset \mathbb{P}(S^2V_4)$ ,

the space of quadrics in the dual projective space  $\mathbb{P}(V_4^\vee)$  which are *apolar* to quadrics from  $W$ . We can view  $\text{Cay}(W)$  as the intersection

$$\text{Cay}(W) = \phi(\mathbb{P}(V_4) \times \mathbb{P}(V_4)) \cap \mathbb{P}(L^\perp).$$

Since  $\phi(\mathbb{P}(V_4) \times \mathbb{P}(V_4))$  is the locus of reducible quadrics in  $\mathbb{P}(S^2V_4)$ , we see that the Cayley model  $\text{Cay}(W)$  is equal to the locus of reducible quadrics in  $\mathbb{P}(L^\perp)$ .

**Proposition 4.11.** *Let  $|\Delta^r|$  be the linear system defining the Reye model of  $\text{Rey}(W)$ . Then the Cayley model of  $\text{Rey}(W)$  is defined by the adjoint linear system  $|\Delta^c| := |\Delta^r + K_{\text{Rey}(W)}|$ .*

*Proof.* Recall that the two embeddings of  $\text{Rey}(W)$  into  $\mathbb{P}(\bigwedge^2 V_4)$  and into  $\mathbb{P}(L^\perp)$  are defined by the projections of the image of  $\text{PB}(W)$  in  $\mathbb{P}(V_4 \otimes V_4)$  under the Segre map  $\mathbb{P}(V_4) \times \mathbb{P}(V_4) \rightarrow \mathbb{P}(V_4 \otimes V_4)$ . The center of the projection for the former map is the Segre image of the diagonal, the center of the projection of the latter map is the subspace  $\mathbb{P}(\bigwedge^2 V_4)$ . Thus the pre-images of the linear system  $|\Delta^r|$  defining the Reye embedding and the linear system  $|\Delta^c|$  defining the Cayley embedding are different linear subsystems of the same linear system on the K3-cover  $\text{PB}(W)$ . This shows that they are defined by invertible sheaves  $\mathcal{L}$  and  $\mathcal{L}'$  such that  $\pi^*(\mathcal{L}) \cong \pi^*(\mathcal{L}')$  and hence  $\mathcal{L} \cong \mathcal{L}'$  or  $\mathcal{L} \cong \mathcal{L}' \otimes \omega_S$ . The linear systems that define the two embedding are complete, and we will see in section 4.5 that the homogeneous ideals of the embedded surfaces are different. Thus  $\mathcal{L} \not\cong \mathcal{L}'$  and  $\mathcal{L} \cong \mathcal{L}' \otimes \omega_S$ , i.e.  $\Delta^c = \Delta^r + K_{\text{Rey}(W)}$ .  $\square$

Since  $\Delta^r$  is an ample Fano polarization of the Enriques surface  $\text{Rey}(W)$ , and  $\Delta^c$  is numerically equivalent to  $\Delta^r$ , we see that  $\Delta^c$  is also an ample Fano polarization. Thus the Cayley model  $\text{Cay}(W)$  must contain 20 plane cubics that span 20 planes  $\Lambda_{\pm i}^c$ , the same cubics which we find for the Reye model  $\text{Rey}(W)$ .

One more property distinguishing the surfaces  $\text{Cay}(W)$  and  $\text{Rey}(W)$  is stated in the following theorem (see [11, Theorem 4] and [10, Theorem (2.12)] and Proposition 3.14).

T4.11

**Theorem 4.12.** *Let  $\text{Tri}(X)$  be the variety of trisecant lines of a surface in  $\mathbb{P}^5$ . If  $X = \text{Cay}(W)$ , then  $\text{Tri}(X)$  is a 3-dimensional variety isomorphic to the blow-up of 20 points in  $\mathbb{P}(L^\vee)$  corresponding to plane components of reducible quadrics in  $W$ . If  $X = \text{Rey}(W)$ , then  $\text{Tri}(X)$  consists of 20 planes  $(\Lambda_{\pm i}^c)^*$ . The union of trisecants of  $\text{Cay}(W)$  is equal to the determinantal quartic hypersurface  $D_{\mathbb{P}(L^\perp)}$ .*

Let us summarize what we have found about the differences between the Cayley and Reye models of a nodal Enriques surface.

- $\text{Rey}(W)$  is contained in a quadric, and  $\text{Cay}(W)$  is not.



- The variety of trisecant lines of  $\text{Rey}(W)$  consists of 20 planes, the lines in the planes spanned by the images of the curves  $F_{\pm i}$ . The variety of trisecant lines of  $\text{Cay}(W)$  is of dimension 3.
- The 20 planes  $\Lambda_i^r, i = \pm 1, \dots, \pm 10$ , containing the 20 plane cubic curves on the Reye model pairwise intersect. On the Cayley model the planes  $\Lambda_i^c$  and  $\Lambda_j^c$  intersect only if  $i + j \neq 0$ .
- Write  $3\Delta^c = F_1 + \dots + F_{10}$  as in (2.8). Then  $|\Delta^r - 2F_i| = \emptyset$  and  $|\Delta^c - 2F_i| = |\Delta^r - F_i - F_{-i}| \neq \emptyset$ .
- $\text{Rey}(W)$  bears a unique stable rank 2 bundle  $\mathcal{E}$  with  $c_1(\mathcal{E}) = \Delta^r$  and  $c_2(\mathcal{E}) = 3$ , and any rank 2 bundle  $\mathcal{E}'$  on  $\text{Cay}(W)$  with  $c_1(\mathcal{E}') = \Delta^c$  and  $c_2(\mathcal{E}') = 3$  is unstable and fits into an extension of the form  $0 \rightarrow \mathcal{O}(\Delta^c - F_i) \rightarrow \mathcal{E}' \rightarrow \mathcal{O}(F_i) \rightarrow 0$ .

The proof of the last property can be found in [14].

#### 4.5. The equations of a Fano model.

P4.6

**Proposition 4.13.** *Let  $W = \mathbb{P}(L)$  be a regular web of quadrics in  $\mathbb{P}(V_4)$ . Let  $\text{Rey}(W) \subset \mathbb{P}(\wedge^2 V)$  and  $\text{Cay}(W) \subset \mathbb{P}(L^\perp)$  be the Reye model and the Cayley model of a nodal Enriques surface. Then the linear system of cubics in  $\mathbb{P}(\wedge^2 V)$  (resp.  $\mathbb{P}(L^\perp)$ ) with base locus  $\text{Rey}(W)$  (resp.  $\text{Cay}(W)$ ) is 9-dimensional. Moreover, the homogeneous ideal defining  $\text{Cay}(W)$  is generated by 10 cubics, whilst the homogeneous ideal of  $\text{Rey}(W)$  is generated by one quadric and four cubics.*

cayley-reye

*Proof.* Let  $S \subset \mathbb{P}^5$  be a Fano model of an Enriques surface defined by an ample divisor  $D$  with  $D^2 = 10$ . Since  $3D + K_S$  is ample, by Kodaira's Vanishing Theorem,  $h^1(3D) = h^1(2K_S + 3D) = 0$ . By Riemann-Roch, we get  $h^0(3D) = 46$ . Since  $h^0(\mathcal{O}_{\mathbb{P}^5}(3)) = 56$ , the usual exact sequence for the ideal sheaf of a subvariety of  $\mathbb{P}^5$  shows that the dimension of the linear system of cubics containing  $S$  is greater or equal than 9. It is exactly 9 if  $S$  is 3-normal, that is  $h^1(\mathcal{I}_{S, \mathbb{P}^5}(3)) = 0$ . The 3-normality of Fano models follows from Theorem (1.1) of [27]. We can give a more explicit description of the cubics containing  $S$  for Cayley and Reye models.

Consider the variety of singular quadrics in  $\mathbb{P}^3$ . It is a discriminant quartic hypersurface  $D$  in  $\mathbb{P}^9$ . The variety  $D(2)$  of quadrics of corank 2 is known to be its singular locus and hence is equal to the intersection of 10 cubic hypersurfaces defined by the partials of the discriminant quartic. The Cayley model is the intersection of  $D(2)$  with a 5-dimensional plane in  $\mathbb{P}^9$ . So it is contained in the base locus of a linear system of cubics of dimension  $\leq 9$ . Taking the minimal resolution of the ideal of  $D(2)$  (see [62, Weyman Theorem 6.3.1]) and restricting it to a transversal  $\mathbb{P}^5$ , we obtain the resolution of the Cayley model  $S = \text{Cay}(W)$ :

resolution

$$(4.9) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^5}(-5)^{\oplus 6} \rightarrow \mathcal{O}_{\mathbb{P}^5}(-4)^{\oplus 15} \rightarrow \mathcal{O}_{\mathbb{P}^5}(-3)^{\oplus 10} \rightarrow \mathcal{I}_S \rightarrow 0.$$

It shows that  $S$  is projectively normal and that the ten partials of the discriminant cubic generate the sheaf of ideals  $\mathcal{I}_S$  of  $S$  in  $\mathbb{P}^5$ .



On the other hand, if  $S \subset \text{Gr}(2, V_4)$  is a Reye model, we have the following resolution of the ideal sheaf of  $S$  in  $\text{Gr}(2, L)$ :

$$0 \rightarrow S^2\mathcal{U}(-3) \rightarrow L^\vee \otimes \mathcal{O}_{\text{Gr}(2, V_4)}(-3) \rightarrow \mathcal{O}_{\text{Gr}(2, V_4)} \rightarrow \mathcal{O}_S \rightarrow 0,$$

where  $\mathcal{U}$  is the tautological subbundle over the Grassmannian <sup>[IngallsKuznetsov 34, Lemma 5.1]</sup>. It shows that the homogeneous ideal of  $S$  is generated by a quadric and four cubics.  $\square$

**C4.7** **Corollary 4.14.** (*F. Cossec*) *Let  $X$  be a Fano model of a general Enriques surface. Then its homogenous ideal is generated by 10 cubics.*

*Proof.* By <sup>[CR 11]</sup>, a Fano model of a general Enriques surface is not contained in a quadric, so the result follows by the argument from the first paragraph of the proof of Proposition <sup>[P4.6 4.13]</sup>.  $\square$

*Remark 4.15.* The Cayley model is scheme-theoretically defined by 6 cubics, the partials of the quartic symmetroid in  $\mathbb{P}^5$ . The 6 cubics generate a non-saturated ideal of  $S^\bullet(L^\perp)^\vee$ , whose saturation is generated by the 10 partials of  $\mathcal{D} \subset \mathbb{P}^9$  restricted to  $\mathbb{P}^5$ ; see Appendix, Fact <sup>[Betti\_numbers 9.1.1]</sup>.

Coble\_surfaces

## 5. COBLE SURFACES

**5.1. Degree 10 polarizations.** A *Coble surface* is a smooth projective rational surface  $S$  such that  $|-K_S| = \emptyset$  but  $|-2K_S| \neq \emptyset$ . We refer to <sup>[P2 16]</sup> for a classification of such surfaces. A Coble surface is said to be *terminal of K3 type* (Igor:added terminal, i.e. cannot blow-up anymore) if  $|-2K_S|$  contains a smooth divisor  $C_0$ , which is necessarily isolated and consists of the disjoint union of  $k$  smooth rational curves with self-intersection  $-4$ . The number  $k$  is equal to  $-K_S^2$ . The double cover of  $S$  defined by the line bundle  $\mathcal{O}_S(-K_S)$  is a K3 surface  $X$  and the locus of fixed points of the covering involution is the union of  $k$  smooth rational curves, the pre-images of the irreducible components of  $C_0$ . As we observed in section 3.1, the isomorphism classes of Coble surfaces are parameterized by the boundary of the moduli space of Enriques surfaces.

One proves that a Coble surface (of K3 type) always admits a birational morphism  $\pi : S \rightarrow \mathbb{P}^2$ . The image of the anti-bicanonical divisor  $C_0$  is a divisor in  $|-2K_{\mathbb{P}^2}|$ , i.e. a plane curve of degree 6.

From now on we assume that  $k = 1$ , that implies that  $K_S^2 = -1$  and the surface is obtained by blowing up the set  $\Sigma = \{p_1, \dots, p_{10}\}$  of ten points in  $\mathbb{P}^2$ . Since, for any irreducible curve  $C$  which is not a component of  $C_0$ , we have  $C \cdot K_S = -\frac{1}{2}C \cdot C_0 \leq 0$ , the adjunction formula implies that any smooth rational curve  $C$  on  $S$  has  $C^2 \in \{-1, -2, -4\}$ . This easily implies that any connected curve  $\mathcal{E}$  blown down to a point in  $\mathbb{P}^2$  is an *exceptional configuration*, i.e.  $\mathcal{E} = E_1 + \dots + E_k$ , where  $E_i \cong \mathbb{P}^1$ ,  $E_1^2 = -1$ ,  $E_i^2 = -2$ ,  $i \neq 1$  and  $E_i \cdot E_{i+1} = 1$ , and all other intersection numbers are zero. Thus the pre-images of the points  $p_i$  are exceptional configurations  $\mathcal{E}_i$  and we can

order points  $p_1, \dots, p_{10}$  in such a way that

$$\boxed{\text{exconf}} \quad (5.1) \quad \mathcal{E}_i := \pi^{-1}(x_i) = E_i + R_1^{(i)} + \dots + R_{k_i}^{(i)}, \quad E_i^2 = -1,$$

and  $\mathcal{E}_{i+1} - \mathcal{E}_i > 0$  if and only if  $p_{i+1}$  is infinitely near to  $p_i$ .

Since  $C_0 \in |-2K_S|$ , the image of  $C_0$  in the plane is an irreducible rational curve  $B$  of degree 6, the points  $p_i$  are its ordinary double points (recall that some of them may be infinitely near points), thus  $B$  has simple singularities of types  $A_{k_i}$ , where  $k_i$  is the number of irreducible components in the exceptional configuration  $\mathcal{E}_i$  over  $x_i$  and  $\sum_{i=1}^{10} k_i = 10$ .

Let  $e_0 = c_1(\pi^* \mathcal{O}_{\mathbb{P}^2}(1))$  and  $e_i = [\mathcal{E}_i]$ . The ordered set  $(e_0, \dots, e_{10})$  defines a basis of  $\text{Pic}(S)$  which we call a *geometric basis*. It defines a lattice isomorphism  $\phi : \mathbb{1}^{1,10} \rightarrow H^2(S, \mathbb{Z})$  such that  $\phi(\mathbf{e}_i) = e_i, i = 0, \dots, 10$ , and  $\phi(\mathbf{k}_{10}) = K_S$ . This shows that the restriction of  $\phi$  to  $\mathbf{k}_{10}^\perp$  defines a lattice isomorphism

$$\boxed{\text{markcoble}} \quad (5.2) \quad \phi : \mathbf{E} \rightarrow K_S^\perp.$$

This is the first property shared by Coble and Enriques surfaces.

We extend the notion of a marked Enriques surface to Coble surfaces continuing to assume that  $K_S^2 = -1$ . A *marking* is an isomorphism  $(\text{markcoble } 5.2)$ . Two markings  $\phi, \phi'$  are equivalent if there exist  $g \in \text{Aut}(S)$  and  $w \in W_S^{\text{nod}}$  such that  $\phi' = w \circ g^* \circ \phi$  (note that we require that  $\phi(\mathbf{k}_{10}) = K_S$  and since  $-2K_S$  is effective, we cannot compose a marking with  $-\text{id}_{\text{Pic}(S)}$ ). Here, as in the case of Enriques surfaces,  $W_S^{\text{nod}}$  denotes the subgroup of the orthogonal group of  $\text{Pic}(S)$  generated by the reflections in the classes of  $(-2)$ -curves. By the adjunction formula, the classes of such curves belong to  $K_S^\perp$ .

A marking is called a *geometric marking* if it is defined by a geometric basis of  $\text{Pic}(S)$ .

We say that a Coble surface  $S$  is *unnodal* if it does not contain  $(-2)$ -curves and *nodal* otherwise.

**Proposition 5.1.** *Assume that  $S$  is an unnodal Coble surface. Any marking is isomorphic to a marking given by a geometric basis. Two geometric markings are isomorphic if and only if they differ by a composition with  $g^*$  for some  $g \in \text{Aut}(S)$ .*

*Proof.* (Igor: Changed the proof, it were the gaps in the previous proof. Please check!) By the assumption, any smooth rational curve on  $S$  different from  $C_0$  is a  $(-1)$ -curve. By Riemann-Roch, any  $e_i$  or  $-e_i$  is an effective divisor class. However, since  $e_i \cdot C_0 = -2e_i \cdot K_S = 2$ , we may assume that  $e_i$  is the class of an effective divisor  $E_i$  with  $E_i \cdot K_S = -1$ . Since  $C_0 \cdot C \geq 0$  for any irreducible curve  $C \neq C_0$ , we have  $C \cdot (-K_S) \geq 0$ , hence writing  $E_i$  as the sum of irreducible components, we obtain that  $E_i$  has a unique irreducible component  $R_0^{(i)}$  with  $R_0^{(i)} \cdot K_S = -1$  and all other components  $R_j^{(i)}$  satisfy  $R_j^{(i)} \cdot K_S = 0$ . Since  $S$  has no  $(-2)$ -curves, by adjunction formula,

$R_0^{(i)2} \geq -1, R_j^{(i)2} \geq 0, i \neq 0$ . This gives

$$-1 = E_i^2 = (R_0^{(i)} + \sum_{j \neq 0} R_j^{(i)})^2 \geq -1 + \sum_{j \neq 0} R_0^{(i)} \cdot R_j^{(i)} + \sum_{j \neq 0} R_j^{(i)2} \geq -1.$$

It follows that  $R_0^{(i)}$  is a  $(-1)$ -curve, and  $R_j^{(i)2} = 0 = R_0^{(i)} \cdot R_j^{(i)} = 0$ . By Hodge-Index Theorem, all  $R_j^{(i)}$  for a fixed  $i$  are linearly equivalent, so we can write  $E_i = R_0^{(i)} + m_i R^{(i)}$ , where  $R_0^{(i)}$  is a  $(-1)$ -curve and  $R^{(i)2} = R^{(i)} \cdot R_0^{(i)} = 0$ . After we blow down  $R_0^{(i)}$ , we obtain a rational elliptic surface  $S_i$  with one multiple fiber of multiplicity 2 (see [PZ, Theorem 2.5]). It follows that the image of  $R^{(i)}$  is an irreducible curve of arithmetic genus 1 that either moves in a pencil or taken with multiplicity 2 moves in a pencil. Since  $R^{(i)}$  does not intersect  $R_0^{(i)}$ , we obtain that  $|R^{(i)}|$  or  $|2R^{(i)}|$  is a pencil. One of its fibers is  $2R_0^{(i)} + B_0^{(i)}$ , where  $B_0^{(i)}$  is the proper transform of an irreducible singular fiber of the elliptic fibration on  $S_i$ .

Since the image of  $R_0^{(j)}$  on  $S_i$  is not contained in a fiber of the elliptic fibration on  $S_i$ , we obtain  $R_0^{(j)} \cdot R^{(i)} > 0$  for  $i \neq j$ , and we get  $E_i \cdot E_j > 0$  contradicting the definition of the  $E_i$ 's. Thus all  $m_i = 0$  and we obtain that each  $E_i$  is the class of a  $(-1)$ -curve. This defines a geometric marking of  $S$ . The pre-image of a line in  $\mathbb{P}^2$  is linearly equivalent to  $e_0$ .  $\square$

*Remark 5.2.* The assertion is still true without assumption that  $S$  is unnodal. But the proof is more involved.

It is shown in [PZ] that blowing down any  $(-1)$ -curve on a  $S$ , we obtain a rational elliptic surface with one double fiber (an *Halphen elliptic surface* of index 2). Conversely, any unnodal  $S$  is obtained by blowing up the singular point of a non-multiple irreducible fiber of an Halphen elliptic surface. The surface  $S$  is unnodal if and only if all fibers of the Halphen surface are irreducible. We refer to [Cantat, Theorem 3.2] for 496 explicit conditions for ten nodes of a rational plane sextic that are necessary and sufficient for  $S$  to be unnodal.

*Remark 5.3.* As in the case of Enriques surfaces, the group of automorphisms of an unnodal Coble surface contains a subgroup isomorphic to  $W_{2,3,7}(2)$  and it coincides with it when  $S$  is general enough [Cantat, Theorem 3.5, Remark 3.11]. This shows that the set of isomorphisms of markings of a general Coble surface is a torsor under the group  $O^+(10, \mathbb{F}_2)$ .

Fix a geometric marking  $\phi$  defined by a geometric basis  $(e_0, \dots, e_{10})$  and let  $(f_1, \dots, f_{10}) = (\phi(\mathbf{f}_1), \dots, \phi(\mathbf{f}_{10}))$  be the isotropic 10-sequence defined as in [Z2]. We have

$$f_i = 3e_0 - (e_1 + \dots + e_{10}) + e_i = -K_S + e_i, \quad i = 1, \dots, 10.$$

Suppose  $e_{i_1}, \dots, e_{i_c}$  are represented by some  $(-1)$ -curves  $E_{i_k}$ . Then  $e_{i_k+j}, 0 < j < i_{k+1} - i_k$  are represented by the part  $E_{i_k} + \sum_{j=i_k+1}^{i_k+j} R_{i_k}^j$  of the exceptional configuration  $\mathcal{E}_{i_k}$  from (5.1).

We can also define a *Fano polarization*. It is given by the divisor class  $\Delta$  such that

$$\text{fanopol} \quad (5.3) \quad \Delta = \frac{1}{3}(f_1 + \dots + f_{10}) = 10e_0 - 3e_1 - \dots - 3e_{10},$$

where  $(f_1, \dots, f_{10})$  is a canonical isotropic sequence. Arguing as in the proof of [13, Theorem 4.6.1], one proves that  $\Delta \cdot f \geq 3$ , for any nef isotropic divisor class, and the linear system  $|\Delta|$  defines a birational morphism

$$\Phi_\Delta : S \rightarrow \bar{S} \subset \mathbb{P}^5$$

onto a normal surface of degree 10 in  $\mathbb{P}^5$ . We call this surface a *Fano model of a Coble surface*. The morphism  $\Phi_\Delta$  is an isomorphism outside the proper transform of the sextic if and only the isotropic sequence is nondegenerate, or equivalently,  $S$  is obtained by blowing up ten nodes of a rational sextic none of which is infinitely near. Note that there is also the adjoint Fano polarization  $\Delta + K_S$ . The adjoint linear system

$$|\Delta + K_S| = |7e_0 - 2e_1 - \dots - 2e_{10}|$$

maps  $S$  onto a surface  $\bar{S}$  of degree 9 in  $\mathbb{P}^5$ . It maps the divisor  $C_0 \in |-2K_S|$  to a conic spanning a plane  $\Pi$ . The union  $\bar{S} \cup \Pi$  should be considered as a degeneration of a Fano model of an Enriques surface.

twenty

**5.2. The twenty planes.** Let  $\Delta$  be a Fano polarization defined by a nondegenerate isotropic sequence  $(f_1, \dots, f_{10})$  in  $\text{Pic}(S)$ . As in the case of Enriques surfaces, we see that the image of each curve  $F_i$  representing the divisor class  $f_i$  under the map  $\Phi_\Delta$  is a plane cubic curve in  $\mathbb{P}^5$  that spans a plane  $\Lambda_i$ . Since  $f_i \cdot f_j = 1$ , the planes pairwise intersect.

Since  $\Delta \cdot C_0 = 0$ ,  $\phi_\Delta$  blows down  $C_0$  to a singular point. Since  $C_0^2 = -4$ , it is locally isomorphic to the singular point of the affine cone over a Veronese curve of degree 4. It is also the quotient singularity of type  $\frac{1}{4}(1, 1)$ .

Since  $\Delta \cdot e_i = 3$  and  $C_0 \cdot e_i = 2$ , the image of the  $(-1)$ -curve  $E_i$  representing  $e_i$  is a rational curve of degree 3 with a singular point at  $\Phi_\Delta(C_0)$ . Thus we have another set of ten planes  $\Lambda_{-i}$  intersecting each other at the same point. Since  $F_i \cdot F_j = f_i \cdot e_j = 1$  if  $i \neq j$ , we see that  $\Lambda_i \cap \Lambda_{-j} \neq \emptyset$ .

In the polarization  $\Delta + K_S$ , we have again the ten planes corresponding to the curves  $F_i$ 's and the ten planes spanned by the images of the exceptional curves (they are mapped to conics in  $\mathbb{P}^5$ ). The peculiarity of this set of 20 planes is that the plane spanned by the image of the sextic  $B$  intersects all the planes from the second set of 10 planes.

The proof of the following proposition follows almost word by word the proof of Proposition 2.9 and will be omitted.

twenty1

**Proposition 5.4.** *Let  $S$  be a Fano model of a Coble surface defined by a Fano polarization satisfying  $3\Delta \sim F_1 + \dots + F_{10}$ . Let  $\Lambda_i, i = 1, \dots, 10$ , be the*

planes spanned by the plane cubic curves  $F_i$  and  $\Lambda_{-i}$  be the planes spanned by the curves  $E_i$ .

- (i) For any  $i \neq j$ , the planes  $\Lambda_i$  and  $\Lambda_j$  intersect at one point. All planes  $\Lambda_{-i}$  intersect at a common point, and  $\Lambda_i \cap \Lambda_j \neq \emptyset$  if and only if  $i + j \neq 0$ .

Assume additionally that  $S$  is an unnodal Coble surface. Then

- (ii) The intersection points  $F_j \cap F_i = \Lambda_i \cap \Lambda_j, j \neq i$  are all distinct and form the base locus of an Halphen pencil of index 2.
- (iii) If we replace one of the planes  $\Lambda_j$  with  $\Lambda_{-j}$ , then the 9 intersection points with  $\Lambda_i$  are the base points of a pencil of cubic curves in  $\Lambda_i$ .

## 6. LAGRANGIAN 10-TUPLES OF PLANES IN $\mathbb{P}^5$

**6.1. The Enriques-Fano cubic fourfold.** Let  $S \subset \mathbb{P}^5$  be a Fano model of an Enriques surface. A choice of an ordered set of representatives  $(F_1, \dots, F_{10})$  of a canonical isotropic sequence  $(f_1, \dots, f_{10})$  such that

tenplanes

$$(6.1) \quad \mathcal{O}_S(3) \cong \mathcal{O}_S(F_1 + \dots + F_{10})$$

defines an ordered set  $(\Lambda_1, \dots, \Lambda_{10})$  of ten planes such that  $\Lambda_i \cap S = F_i$ . We call this set a *Fano set of planes*. The curves  $F_{-i} \in |F_i + K_S|$  define another ordered set of planes  $(\Lambda_{-1}, \dots, \Lambda_{-10})$  which we call the *adjoint set*. We know that a choice of the isomorphism (6.1) is equivalent to a choice of a supermarking of  $S$ .

P6.2

**Proposition 6.1.** *Let  $\underline{\Lambda} = (\Lambda_1, \dots, \Lambda_{10})$  be a Fano 10-tuple of planes and  $(\Lambda_{-1}, \dots, \Lambda_{-10})$  its adjoint set of planes. Assume  $S$  is general in moduli sense. Then*

- (i) *There exists a unique cubic hypersurface containing all 10 planes  $\Lambda_i$ .*
- (ii) *The linear system of cubic hypersurfaces passing through the 45 points  $p_{ij}$  is of dimension 10 and the generic hypersurface in it does not contain  $S$ .*
- (iii) *For any  $i$  there exists a pencil of cubic hypersurfaces containing the planes  $\Lambda_j, j \neq i$ . It contains a unique cubic hypersurface  $\mathcal{C}_i$  vanishing on the Enriques surface  $S$ . The hypersurfaces  $\mathcal{C}_i$  span the 9-dimensional linear system of cubic hypersurfaces containing  $S$ .*

*Proof.* (i) The linear system of cubic hypersurfaces in  $\mathbb{P}^5$  is of dimension 55. Thus the dimension of the linear system of cubics containing the 45 points  $p_{ij}$  is of dimension  $\geq 10$ . Using Proposition 2.9 (ii) we see that there is a unique cubic curve through any subset  $\mathcal{P}_i$  of 45 points lying in a plane  $\Lambda_i$ . Taking a general point in each plane we obtain that there exists a cubic hypersurface that intersects each plane along a cubic curve and a point outside it. Obviously, it must contain each plane. We will prove below in Theorem 6.3 that such a cubic hypersurface must be smooth for a general Enriques surface. Since a pencil of hypersurfaces always contains a singular member we find that the cubic is unique.

(ii) This follows from (i). In fact, counting constants we see that the dimension of the linear system is at least 10. Suppose it is greater than 10. Then for a general point  $p$  on one of the planes  $\Lambda_i$  we can find a cubic curve passing through  $\mathcal{P}_i$  and  $p$ . This shows that cubic curves passing through  $\mathcal{P}_i$  form at least a pencil. By Proposition 2.9 (iii) this implies that the cubic contains the plane  $\Lambda_i$ . Thus there is at least a pencil of cubic hypersurfaces containing all 10 planes. This contradicts (i).

(iii) Consider the 10-dimensional linear system of cubics from (ii). Take a general point  $q_i$  in each  $\Lambda_i, i \neq j$ . By Proposition 2.9 (ii), a cubic hypersurface in the linear system which passes through the additional nine points must contain the planes  $\Lambda_i$ . Thus the dimension of the linear system containing the planes  $\Lambda_i, i \neq j$ , is greater or equal than 1. If the dimension is greater than 1, we can find a pencil containing all 10 planes contradicting property (i). Since  $\dim |3\Delta - F_1 - \dots - F_9 - F_{10} + F_j| = \dim |F_j| = 0$ , the restriction of the pencil of hypersurfaces from (ii) to  $\Lambda_j$  contains a unique hypersurface  $\mathcal{C}_i$  vanishing on  $S$ . We know from Corollary 4.14 that the dimension of the linear system of cubics containing  $S$  is equal to 9. Suppose the cubics  $\mathcal{C}_i = V(\Phi_i), i = 1, \dots, 10$ , generate a linear system of smaller dimension. This means that there is a linear dependence  $a_1\Phi_1 + \dots + a_{10}\Phi_{10} = 0$  with some  $a_i \neq 0$ . Taking a general point  $q_i$  in  $\Lambda_i$ , we obtain that  $\Phi_i(q_i) = 0$ , hence  $\mathcal{C}_i$  contains all 10 planes. This means that  $\mathcal{C}_i$  coincides with the unique cubic hypersurface containing  $S$ . In fact 9.1.6 from the Appendix we show an example of a surface  $S$  for which there are no cubics containing  $S$  and the 10 planes. So the same must be true for a general  $S$ .  $\square$

Let  $\mathcal{C}(S, \underline{\Lambda})$  be the cubic hypersurface containing a set  $\underline{\Lambda} = (\Lambda_1, \dots, \Lambda_{10}) \in \mathbb{G}^{10}$  of 10 planes. We call it the *Enriques-Fano cubic*. We have already proved its uniqueness (for a general  $S$ ). We will later see that a general  $S$  is not contained in  $\mathcal{C}(S, \underline{\Lambda})$ .

Below we will prove that  $\mathcal{C}(S, \underline{\Lambda})$  is smooth for general  $S$ . But first we need the following well-known lemma.

**L6.2** **Lemma 6.2.** *Let  $V$  be an irreducible cubic threefold in  $\mathbb{P}^4$  that contains a plane  $\Lambda$ . Then  $V$  is singular at some point of  $\Lambda$ , and the set-theoretical intersection  $\text{Sing}(V) \cap \Lambda$  of the singular locus of  $V$  with  $\Lambda$  is one of the following: either the whole plane, or a conic, or a line, or a line plus a point, or  $k$  points in  $\Lambda$  with  $k \in \{1, 2, 3, 4\}$ .*

*Proof.* Choose coordinates  $x_0, \dots, x_4$  in such a way that the equation of the plane is  $x_0 = x_1 = 0$ . Then we can write the equation of the cubic in the form  $x_0q_0 + x_1q_1 = 0$ , where  $q_0$  and  $q_1$  are quadratic forms. Taking the partial derivatives, we obtain that the singular points in the plane are common solutions of equations  $x_0 = x_1 = q_0 = q_1 = 0$ . This shows that the singular locus in  $\Lambda$  is either the whole plane, or the conic  $V(q_0) \cap \Lambda = V(q_1) \cap \Lambda$ , or else the base locus of the pencil of conics generated by  $V(q_0) \cap \Lambda, V(q_1) \cap \Lambda$ . The latter may be either four points counted with multiplicities, or a common line of the conics  $V(q_i) \cap \Lambda$ , possibly plus one extra point.  $\square$

smoothness

**Theorem 6.3.** *Let  $\mathcal{C}(S, \underline{\Lambda})$  be a cubic hypersurface containing the ten planes defined by a Fano model of a general Enriques surface. Then  $\mathcal{C}(S, \underline{\Lambda})$  is smooth.*

*Proof.* For a general Fano model  $S$  this follows from Fact [9.1.3](#) in the [nonsing-cub](#) appendix. We also give a proof below that does not rely on computer computations.

Step 1:  $\mathcal{C}(S, \underline{\Lambda})$  is irreducible.

Suppose it is not, i.e.  $\mathcal{C}(S, \underline{\Lambda})$  is the union of a hyperplane  $H$  and a quadric  $Q$ , maybe reducible. Since the Fano polarization is not that of Reye type,  $S$  is not contained in a quadric. Thus assuming  $\Lambda_1, \dots, \Lambda_k \subset H$ ,  $\Lambda_{k+1}, \dots, \Lambda_{10} \subset Q$ , we see that there are effective divisors  $D \in |\Delta - \sum_{i=1}^k F_i|$ ,  $D' \in |2\Delta - \sum_{i=k+1}^{10} F_i|$ , where  $F_i = \Lambda_i \cap S$  are as above. Since  $(\Delta - F_1 - \dots - F_m)^2 = 10 + 2m(m-1) - 6k = 2(m^2 - 4m + 5) \neq 0$ , the divisors  $D$  and  $D'$  are not the zero divisors and since  $D + D' \sim 3\Delta - F_1 + \dots + F_{10} \sim 0$ , they cannot be both effective.

Step 2:  $\mathcal{C}(S, \underline{\Lambda})$  is a normal hypersurface.

Assume  $\mathcal{C}(S, \underline{\Lambda})$  is not normal. Then its singular locus is a 3-dimensional projective space  $H$ . Since  $\Lambda_i \cap \Lambda_j$  is a single point  $x_{ij}$ , we can find two planes, say  $\Lambda_1$  and  $\Lambda_2$  that are not contained in  $H$  and  $x_{12} \notin H$ . Let  $Q$  be the polar quadric of  $\mathcal{C}(S, \underline{\Lambda})$  with pole at  $x_{12}$ . By the properties of the polar hypersurfaces [\[19, Theorem 1.1.5\]](#),  $Q$  contains  $H$  and contains the planes  $\Lambda_1, \Lambda_2$ . Thus  $Q = H \cup H'$  is reducible and  $\Lambda_1, \Lambda_2$  are contained in  $H'$ , hence intersect along a line. This contradiction proves the assertion.

Step 3:  $\mathcal{C}(S, \underline{\Lambda})$  is not a cubic cone.

Assume that it is a cone, so that the projection of  $\mathcal{C}(S, \underline{\Lambda})$  from its triple singular point  $p$  is a cubic threefold  $V_3$  in  $\mathbb{P}^4$ . Let  $p_{ij} = \Lambda_i \cap \Lambda_j$ ; we will denote by  $\bar{p}_{ij}$ ,  $\bar{\Lambda}_i$  the projections of  $p_{ij}$ , resp.  $\Lambda_i$  in  $V_3$  whenever they are defined.

By Proposition [6.1](#), [P6.2](#), no three planes intersect at one point, hence  $p$  is contained in at most two planes. Assume first that  $p$  lies on at most one of the ten planes  $\Lambda_i$ . Without loss of generality, we may assume that it does not lie on  $\Lambda_1, \dots, \Lambda_9$ . The cubic  $V_3$  is irreducible and contains 9 planes  $\bar{\Lambda}_1, \dots, \bar{\Lambda}_9$ . If two planes  $\bar{\Lambda}_i$  and  $\bar{\Lambda}_j$  ( $i, j \neq 10$ ) intersect at one point, this point is  $\bar{p}_{ij}$ , and by evaluating the dimension of the tangent space of  $V_3$  at  $\bar{p}_{ij}$  we see that  $\bar{p}_{ij}$  is a singular point of  $V_3$ . A non-normal cubic of dimension 4 has a  $\mathbb{P}^3$  as its singular locus in codimension 1 (see [\[42, Lemma 2.4\]](#)). Thus we may assume that at most one of the planes  $\bar{\Lambda}_i$  is contained in the singular locus. So, we may assume that  $\bar{\Lambda}_1$  is not contained in  $\text{Sing}(V_3)$ . Suppose  $\bar{\Lambda}_1$  has 1-point intersections with 6 other planes, say,  $\bar{\Lambda}_1 \cap \bar{\Lambda}_i = \bar{p}_{1i}$  for  $i = 2, \dots, 7$ . Then these 6 points lie in  $\text{Sing}(V_3)$  and applying the previous Lemma, we obtain that the intersection points lie on a conic, or 5 of the intersection points lie on a line. This implies that the points  $p_{1i}$ ,  $i = 1, \dots, 6$ ,



lie on a conic in the plane  $\Lambda_1$ , or 5 points are collinear. By Proposition [2.9](#), <sup>twentyplanes1</sup> this contradicts the assumption that  $S$  is general.

So, we may assume that  $\bar{\Lambda}_1$  intersects three other planes, say  $\bar{\Lambda}_2, \bar{\Lambda}_3, \bar{\Lambda}_4$  along the lines  $\ell_{12}, \ell_{13}, \ell_{14}$ . This means that the 3-dimensional spaces  $\langle p, \Lambda_1 \rangle$  and  $\langle p, \Lambda_i \rangle$  intersect along a plane  $\Pi_{1i}$ . But then the line  $\langle p, p_{23} \rangle$  intersects the plane  $\Pi_{12}$  and hence the intersection point of  $\ell_{12}$  and  $\ell_{13}$  coincides with  $\bar{p}_{23}$ . Similarly,  $\bar{p}_{24} = \ell_{12} \cap \ell_{14}$ , so that the three points  $\bar{p}_{12}, \bar{p}_{23}, \bar{p}_{24}$  in the plane  $\bar{\Lambda}_1$  are collinear,  $\bar{p}_{12} \in \ell_{12} = \langle \bar{p}_{23}, \bar{p}_{24} \rangle$ . But this implies that the points  $p_{12}, p_{23}, p_{24}$  in  $\Lambda_2$  are collinear as well. Indeed, if  $p_{12}, p_{23}, p_{24}$  generate  $\Lambda_2$ , then  $\bar{\Lambda}_2 = \langle \bar{p}_{12}, \bar{p}_{23}, \bar{p}_{24} \rangle = \ell_{12}$  is a line, which means that  $p \in \Lambda_2$ , but we assumed  $p \notin \Lambda_i$  for  $j \neq 10$ . Thus  $p_{12}, p_{23}, p_{24}$  are collinear. But this contradicts Proposition [6.1](#): for a general  $S$  no three points  $p_{ij}$  in one plane are collinear.

It remain to consider the case when the center of the projection  $p$  coincides with one of the points  $p_{ij}$ , say  $p_{9,10}$ . Then if  $\bar{\Lambda}_1$  intersects  $\bar{\Lambda}_2$  along a line, then  $p$  belongs to  $\mathbb{P}^4 = \langle \Lambda_1, \Lambda_2 \rangle$ . This implies that three curves  $F_1, F_2$  and  $F_{12} \in |\Delta - F_1 - F_2|$  intersect at one point  $p$ . The numerical classes of the three curves form an isotropic 3-sequence that can be extended to a nondegenerate canonical isotropic sequence. <sup>6.2</sup> It will define another Fano polarization that does not satisfy Proposition [6.1](#). So, we find a contradiction with the assumption that  $S$  is general enough.

Step 4:  $\mathcal{C}(S, \underline{\Lambda})$  is smooth.

By the previous steps we may assume that  $\mathcal{C}(S, \underline{\Lambda})$  is normal and not a cone, hence it is either smooth or has only double points as singularities. Assume that  $p$  is a double singular point. Without loss of generality, we may assume that  $p = (1, 0, 0, 0, 0)$  and hence the equation of the cubic is

$$\text{eqcone} \quad (6.2) \quad x_0 F_2(x_1, \dots, x_5) + F_3(x_1, \dots, x_5) = 0.$$

The projection of  $\mathcal{C}(S, \underline{\Lambda})$  from  $p$  to the hyperplane  $H = \{x_0 = 0\}$  defines a regular birational map from the blowup of  $\mathcal{C}(S, \underline{\Lambda})$  at  $p$  contracting the exceptional divisor, isomorphic to a  $\mathbb{P}^1$ -bundle over the surface  $X = V_H(F_2) \cap V_H(F_3)$ . In other words,  $\mathcal{C}(S, \underline{\Lambda})$  is the image of  $H$  under the rational map  $\Phi$  given by cubics in  $H$  containing  $X$ . Let  $\Lambda_j$  be one of the ten planes that does not contain the point  $p$  and  $W_j = \langle \Lambda_j, p \rangle$  the span of  $\Lambda_j$  and  $p$ . If  $W_j$  is contained in  $\mathcal{C}(S, \underline{\Lambda})$ , then it is contained in the tangent cone  $V(F_2)$ . Thus we can always choose  $j$  in such a way that  $W_j$  intersects  $\mathcal{C}(S, \underline{\Lambda})$  along a cubic surface containing  $\Lambda_j$  and singular at  $p$ . The cubic surface is the union of  $\Lambda_j$  and a quadric cone  $Q_j$  with its singular point  $p$ . The quadric  $Q_j$  is a cone over a conic  $R_j$  in  $X$  lying in the plane  $\Lambda_j$ . If  $p$  is contained in  $\Lambda_i$ , then the projection of  $\Lambda_i$  is a line in  $X$ .

It follows from the equation that if  $X$  is not normal then  $\mathcal{C}(S, \underline{\Lambda})$  is not normal. By Step 2, we may assume that  $X$  is normal. Since  $p$  is contained in at most two planes  $\Lambda_i$ , the ten curves  $R_1, \dots, R_{10}$  in  $X$  are either 10 conics, or 9 conics and a line, or 8 conics and two lines.



Let us see how the 10 curves  $R_1, \dots, R_{10}$  intersect. Suppose  $p \notin \Lambda_i$ . Without loss of generality we may assume that  $i = 1$ . Suppose  $R_1$  intersects  $R_2$ . Then the intersection point  $p_{12} = \Lambda_1 \cap \Lambda_2$  lies on  $X$  and hence equation (6.2) shows that  $\mathcal{C}(S, \underline{\Lambda})$  contains the line  $\langle p, p_{12} \rangle$ . Let  $p_{ij} = \Lambda_i \cap \Lambda_j$ . Then  $\mathcal{C}(S, \underline{\Lambda})$  contains the lines  $\langle p, p_{12} \rangle$ , and  $\langle p_{12}, p_{13} \rangle \subset \Lambda_1$ . Thus the plane spanned by these lines intersects the cubic along these lines and the third line  $\langle p, p_{13} \rangle$ . Continuing in this way we see that the cubic contains all the lines  $\langle p, p_{1t} \rangle, t = 2, \dots, 10$ . This implies that the residual quadric cone  $Q_1$  of the plane  $\Lambda_1$  intersects  $\Lambda_1$  along a conic containing the nine points. But we know that these points do not lie on a conic (for a general Enriques surface). Since  $R_i$  does not intersect  $R_j$  if  $p_{ij} = p$ , we obtain that all curves are disjoint.

Since  $X$  has no triple points, all its singularities are double points. It follows from analysis of GIT-stability of cubic fourfolds from [Laz04] that the projective equivalence classes of cubic fourfolds whose associated K3 surface of degree 6 has non-rational double point have a smaller dimension than 10. Thus we may assume that our double points are rational double points. Thus a minimal resolution of  $X$  is a K3 surface. The divisor classes of the proper transforms of  $R_i$  are linearly independent. By the usual period mapping argument, we obtain that the projective isomorphism class of  $X$  depends on  $\leq 9$  parameters. So, for a general Enriques surface, this case cannot happen.  $\square$

**6.2. Fano sets of 10-planes.** Let  $\mathbb{G} = \text{Gr}_2(\mathbb{P}^5)$  be the Grassmannian of planes in  $\mathbb{P}^5$  and let  $\text{Hilb}_{\text{Fano}}$  be the Hilbert scheme of Fano models of Enriques surfaces. Let  $\text{Hilb}_{\text{Fano}} \rightarrow \text{Hilb}_{\text{Fano}} // \text{PGL}(6) \cong \mathcal{M}_{\text{Enr}, \text{Fano}}^a$  be the projection map. Let

$$\text{Hilb}_{\text{Fano}}^{\text{sm}} := \text{Hilb}_{\text{Fano}} \times_{\mathcal{M}_{\text{Enr}, \text{Fano}}^a} \mathcal{M}_{\text{Enr}}^{\text{sm}, a} \rightarrow \mathcal{M}_{\text{Enr}}^{\text{sm}, a}$$

be the base change map corresponding to the map (3.11)  $\mathcal{M}_{\text{Enr}}^{\text{sm}, a} \rightarrow \mathcal{M}_{\text{Enr}}^{\text{m}, a}$ .

The projection  $\text{Hilb}_{\text{Fano}}^{\text{sm}} \rightarrow \text{Hilb}_{\text{Fano}}$  is a Galois cover with the Galois group  $(\mathbb{Z}/2\mathbb{Z})^9 \times \mathfrak{S}_{10}$ . Adding to the group the involution  $\text{adj}$  that changes the polarization to the adjoint polarization we obtain the group  $\tilde{G} = (\mathbb{Z}/2\mathbb{Z})^{10} \rtimes \mathfrak{S}_{10}$  that acts on  $\text{Hilb}_{\text{Fano}}^{\text{sm}}$  with quotient  $\mathcal{M}_{\text{Enr}, \text{Fano}}^{\text{sm}}$ .

Any point of  $\text{Hilb}_{\text{Fano}}^{\text{sm}}$  is a pair  $(S, (F_1, \dots, F_{10}))$ , where  $\mathcal{O}_S(3) \cong \mathcal{O}_S(F_1 + \dots + F_{10})$ . Or equivalently, a pair  $(S, \underline{\Lambda})$ , where  $\underline{\Lambda} = (\Lambda_1, \dots, \Lambda_{10})$  is an ordered set of planes  $\lambda_i$  with  $\Lambda_i \cap S = F_i$ .

We denote by  $\mathbb{G}_{\text{Fano}}^{10}$  the image of the map

$$(6.3) \quad \text{Hilb}_{\text{Fano}}^{\text{sm}} \rightarrow \mathbb{G}^{10}, (S, \underline{\Lambda}) \rightarrow \underline{\Lambda}.$$

**Theorem 6.4.**  $\mathbb{G}_{\text{Fano}}^{10}$  is an irreducible variety of dimension 45 birationally isomorphic to  $\text{Hilb}_{\text{Fano}}^{\text{sm}}$ .

*Proof.* By Theorem 3.7,  $\mathcal{M}_{\text{Enr}}^{\text{sm}, a}$  is irreducible, since the base change projection is generically  $\text{PGL}(6)$ -bundle, we obtain that  $\text{Hilb}_{\text{Fano}}^{\text{sm}}$  is irreducible

hilbertmap

T6.5

irreducibility

variety. By Proposition [3.3](#), its dimension is equal to 45. It follows from property (iii) of Proposition [6.1](#) that  $\underline{\Lambda}$  uniquely determines  $S$ . This shows that the map [\(6.3\)](#) is birational onto its image.  $\square$

**VarEnr10**

**6.3. The variety of Fano 10-tuples of planes.** Let  $\mathbb{P}^5 = \mathbb{P}(V)$  for some 6-dimensional linear space  $V = V_6$ . Consider a nondegenerate symplectic form on  $\bigwedge^3 V$  defined by the wedge product

**sympform** (6.4) 
$$\bigwedge^3 V \times \bigwedge^3 V \rightarrow \bigwedge^6 V \cong \mathbb{C}$$

and a choice of a volume form on  $V$ . Each plane in  $\mathbb{P}(V)$  is defined by a line in  $\bigwedge^3 V$ , so we can speak about a subspace of  $\bigwedge^3 V$  spanned by a set of planes. The bilinear form [\(6.4\)](#) is a nondegenerate symplectic form on a linear space of dimension 20. As is well known, its maximal isotropic subspace, which will be called a *Lagrangian subspace*, is of dimension 10.

**newlemma**

**Lemma 6.5.** *Fano 10-tuple of planes  $(\Lambda_1, \dots, \Lambda_{10})$  and its adjoint set of plane  $(\Lambda_{-1}, \dots, \Lambda_{-10})$  each define a basis of a Lagrangian subspace  $U$  (resp.  $U'$ ) of  $\bigwedge^3 V$ . Moreover*

$$\bigwedge^3 V = U \oplus U'.$$

*Proof.* Since  $\Lambda_i \cap \Lambda_j \neq \emptyset$ , the set  $\{\Lambda_1, \dots, \Lambda_{10}\}$  spans an isotropic subspace, so to prove the first assertion it suffices to show that they are linearly independent. Let  $v_{\pm 1}, \dots, v_{\pm 10}$  represent  $\Lambda_{\pm i}$  in  $\bigwedge^3 V$ . We scale them to assume that  $v_i \wedge v_{-i} = 1$ . Suppose there exist scalars  $\lambda_1, \dots, \lambda_{10}$  such that  $v := \lambda_1 v_1 + \dots + \lambda_{10} v_{10} = 0$ . Since  $v \wedge v_{-i} = \lambda_i$ , we see that all  $\lambda_i$  must be equal to zero. This proves the first assertion. The second assertion is obvious since we can identify the span of  $v_{-1}, \dots, v_{-10}$  with the dual space of  $U$  and obtain that the matrix of the symplectic form in the basis  $(v_1, \dots, v_{10}, v_{-1}, \dots, v_{-10})$  is the standard symplectic matrix.  $\square$

**def\_slg**

**Definition 6.6.** An ordered 10-tuple of planes  $(\Lambda_1, \dots, \Lambda_{10})$  is called *semi-Lagrangian* if  $\Lambda_i \cap \Lambda_j \neq \emptyset$  for all  $i, j$ . A semi-Lagrangian set is called *Lagrangian* if there exists a semi-Lagrangian set  $(\Lambda_{-1}, \dots, \Lambda_{-10})$  such that

**lagrangeset**

(6.5) 
$$\Lambda_i \cap \Lambda_j \neq \emptyset \Leftrightarrow i + j \neq 0.$$

holds.

It follows from the proof of Lemma [6.5](#) that its assertion holds for any Lagrangian 10-tuple of planes.

Let  $\mathbb{G} = \text{Gr}(2, \mathbb{P}^5)$  be the Grassmannian of planes in  $\mathbb{P}^5$ . Let

$$\mathbb{G}_{\text{Fano}}^{10} \quad (\text{resp. } \mathbb{G}_{\text{lag}}^{10}, \quad \text{resp. } \mathbb{G}_{\text{slag}}^{10})$$

denote the ordered sets of Fano (resp. Lagrangian, resp. semi-Lagrangian) 10-tuples of planes. We have

$$\mathbb{G}_{\text{Fano}}^{10} \subset \mathbb{G}_{\text{lag}}^{10} \subset \mathbb{G}_{\text{slag}}^{10},$$

where  $\mathbb{G}_{\text{slag}}^{10}$  is a closed subvariety of  $\mathbb{G}^{10}$  and  $\mathbb{G}_{\text{Fano}}^{10}$  and  $\mathbb{G}_{\text{lag}}^{10}$  are its constructible subsets.

Note that a naive count of parameters gives that the expected dimension of  $\mathbb{G}_{\text{slag}}^{10}$  is equal to 45. In fact, all planes in  $\mathbb{P}^5$  intersecting a fixed plane are parameterized by a hyperplane section of  $G_2(\mathbb{P}^5)$ , hence they depend on 8 parameters. So we have 9 parameters for choosing  $\Lambda_1$ , then 8 parameters to choose  $\Lambda_2$ , 7 parameters to choose  $\Lambda_3$ , and so on. This gives  $9 + \dots + 1 = 45$  parameters.

Let us compute the tangent space of  $\mathbb{G}_{\text{slag}}^{10}$  at a given point  $\underline{\Lambda} = (\Lambda_1, \dots, \Lambda_{10})$ . We view  $\mathbb{P}^5$  as the projectivization of a 6-dimensional space  $V = U \oplus U^\vee$  with the standard symplectic form  $\langle (u, \xi), (u', \xi') \rangle = \xi'(u) - \xi(u')$ . Let  $\Lambda_i = \mathbb{P}(V_i)$  for some 3-dimensional subspace  $V_i$  of  $V$ . Without loss of generality we may assume that  $V_i \cap U^\vee = \{0\}$ , hence the projection map to  $p_i : V_i \rightarrow U$  is an isomorphism. Let us fix a basis  $\bar{u} = (u_1, u_2, u_3)$  in  $U$  and the dual basis  $\underline{u}^* = (u_1^*, u_2^*, u_3^*)$  in  $U^\vee$ . Let  $\underline{v} = (v_1^{(i)}, v_2^{(i)}, v_3^{(i)})$  be a basis in  $V_i$  such that  $p_i(v_j^{(i)}) = u_j$ . Then  $V_i$  can be identified with the column space of a  $6 \times 3$ -matrix  $\begin{pmatrix} I_3 \\ A_i \end{pmatrix}$ , where  $A_i$  is the matrix of the projection  $q_i : V_i \rightarrow U^\vee$  with respect to the bases  $\bar{v}$  and  $\bar{u}^*$ . We identify  $V$  with  $V^\vee$  by means of the symplectic form. Let  $V_i^\perp$  be the annihilator subspace of  $V_i$  in  $V^\vee = V$ . We have  $V_i^\perp \cap U^\vee = \{0\}$ , and the projection  $q_i : V_i^\perp \rightarrow U$  is an isomorphism. We choose a basis in  $V_i^\perp$  to identify  $V_i^\perp$  with the column space of the matrix  $\begin{pmatrix} -{}^t A_i \\ I_3 \end{pmatrix}$ .

We know that  $\Lambda_i \cap \Lambda_j \neq \emptyset$ . In the above matrix interpretation, this means that  $\det(A_i - A_j) = 0$ . Let  $\text{adj}(X)$  denote the adjugate matrix of a matrix  $X$  (the cofactor matrix). We have  $\text{adj}(A_i - A_j) = {}^t \beta^{(ij)} \cdot \alpha^{(ij)}$ , where  $V_i \cap V_j = \mathbb{C}\alpha^{(ij)}$ ,  $V_i^\perp \cap V_j^\perp = \mathbb{C}\beta^{(ij)}$ , unless  $\Lambda_i, \Lambda_j$  intersect along a line in which case the adjugate matrix is zero.

tangent

**Lemma 6.7.** *Denote  $\Lambda = \mathbb{P}(U^\vee)$ , a fixed plane in  $\mathbb{P}^5$ . With the above notation we have the following:*

(1) *Let  $\mathbb{G}_{\text{slag},0}^{10}$  be the open subset of  $\mathbb{G}_{\text{slag}}^{10}$  parameterizing  $\underline{\Lambda} = (\Lambda_1, \dots, \Lambda_{10})$  such that  $\Lambda_i \cap \Lambda = \emptyset$  for all  $i$ . Then*

$$\mathbb{G}_{\text{slag},0}^{10} = \{(A_1, \dots, A_{10}) \in \text{Mat}_{3 \times 3}^{10} : \det(A_i - A_j) = 0, 1 \leq i < j \leq 10\}.$$

(2) *The tangent space of  $\mathbb{G}_{\text{slag}}^{10}$  at a point  $\underline{\Lambda} \in \mathbb{G}_{\text{slag},0}^{10}$  is the linear space*

tanspace

$$(6.6) \quad \{(X_1, \dots, X_{10}) \in \text{Mat}_{3 \times 3}^{10} : \text{Tr}(\text{adj}(A_i - A_j) \cdot (X_i - X_j)) = 0, \\ 1 \leq i < j \leq 10\}$$

*of dimension  $\geq 45$ .*

*Proof.* (1) The first assertion follows from the above discussion where we take  $\Lambda = \mathbb{P}(U)$ .

(2) We choose the coordinates  $x_1, \dots, x_6$  to assume that  $\Lambda$  is given by  $x_1 = x_2 = x_3 = 0$ . Then we choose a basis  $\Lambda_k$  spanned by vectors  $v_i^{(k)}$ ,  $i =$

1, 2, 3, with coordinates

$$(1, 0, 0, a_{11}^{(k)}, a_{12}^{(k)}, a_{13}^{(k)}), (0, 1, 0, a_{21}^{(k)}, a_{22}^{(k)}, a_{23}^{(k)}), (0, 0, 1, a_{31}^{(k)}, a_{32}^{(k)}, a_{33}^{(k)}).$$

This defines 10 matrices  $A_k = (a_{ij}^{(k)})$ . The tangent space of  $\text{Gr}_2(\mathbb{P}^5)^{10}$  at the point  $\underline{\Lambda}$  is isomorphic to the set of 10-tuples  $(A_1 + \varepsilon B_1, \dots, A_{10} + \varepsilon B_{10})$ , where  $\varepsilon^2 = 0$ . We have the additional condition that  $\det(A_i - A_j + \varepsilon(B_i - B_j)) = 0$ ,  $1 \leq i, j \leq 10$  that expresses the condition that the planes pairwise intersect. A well-known formula for computing the determinant of the sum of two matrices proves the assertion. The system of linear equations has 90 variables given by the entries of matrices  $X_i$  and  $\binom{10}{2} = 55$  equations. Thus its null-space is of dimension  $\geq 45$ .  $\square$

two-comps

**Theorem 6.8.** *The closure of  $\mathbb{G}_{\text{Fano}}^{10}$  in  $\mathbb{G}_{\text{lag}}^{10}$  is an irreducible component of  $\mathbb{G}_{\text{lag}}^{10}$  of dimension 45. Any other irreducible component of  $\mathbb{G}_{\text{lag}}^{10}$  has dimension  $\geq 45$ .*

*Proof.* We already know that  $\mathbb{G}_{\text{Enr}}^{10}$  is an irreducible variety of dimension 45. Its closure in  $\mathbb{G}_{\text{lag}}^{10}$  is an irreducible subvariety that contains a point with the tangent space of dimension 45. By Lemma 6.7, the tangent space of  $\mathbb{G}_{\text{lag}}^{10}$  at this point coincides with the tangent space of  $\mathbb{G}_{\text{Fano}}^{10}$  at this point. Its dimension is equal to 45. It follows from Lemma 6.7 that the dimension of the tangent space at a point in  $\mathbb{G}_{\text{Fano}}^{10}$  is of dimension  $\geq 45$ . Thus the closure of  $\mathbb{G}_{\text{Enr}}^{10}$  in  $\mathbb{G}_{\text{lag}}^{10}$  is an irreducible component. It follows from the previous Lemma that any other irreducible component has dimension  $\geq 45$ .  $\square$

Let

$$c : \mathbb{P}(\bigwedge^3 V) \rightarrow \mathbb{P}(\bigwedge^3 V)^\vee = \mathbb{P}(\bigwedge^3 V^\vee)$$

be the correlation isomorphism defined by the symplectic form (6.4). It defines bijections between the sets

$$(6.7) \quad c_{\text{lag}} : \text{Gr}(10, \bigwedge^3 V)_{\text{lag}}^{10} \leftrightarrow \text{Gr}(10, \bigwedge^3 V^\vee)_{\text{lag}}^{10},$$

$$(6.8) \quad c_{\text{slag}} : \text{Gr}(10, \bigwedge^3 V)_{\text{slag}}^{10} \leftrightarrow \text{Gr}(10, \bigwedge^3 V^\vee)_{\text{slag}}^{10}$$

For any  $\underline{\Lambda} \in \text{Gr}(10, \bigwedge^3 V)_{\text{lag}}^{10}$  the image  $c_{\text{lag}}(\underline{\Lambda})$  is linearly isomorphic to  $\underline{\Lambda}$ . Hence it satisfies all the properties from Proposition 2.9, 5.4 and 6.1. This shows that the correlation isomorphism defines a birational isomorphism

$$s_{\text{Fano}} : \text{Gr}(10, \bigwedge^3 V)_{\text{Fano}}^{10} \dashrightarrow \text{Gr}(10, \bigwedge^3 V^\vee)_{\text{Fano}}^{10}.$$

In particular, there exists a unique cubic hypersurface containing  $\underline{\Lambda}^\perp$  and an Enriques (or Coble) surface in the dual space defined by the linear system of cubic hypersurfaces through 45 intersection points of the dual planes.

dualconfig

**Proposition 6.9.** *Let  $\Delta$  be an ample Fano polarization of an Enriques or Coble surface  $S$  and  $\phi_\Delta : S \rightarrow |\Delta^*| \cong \mathbb{P}^5$  be the corresponding embedding. Let  $\underline{\Lambda}$  be the corresponding Fano set of planes in  $|\Delta|^*$  spanned by the images of the curves  $F_1, \dots, F_{10}$  such that  $F_1 + \dots + F_{10} \in |3\Delta|$ . Then the linear subsystem  $F_i + |\Delta - F_i|$  in  $|\Delta|$  is a plane  $\Lambda_i^\perp$  in the dual space  $|\Delta|$ . The intersection  $\Lambda_i^\perp \cap \Lambda_j^\perp, i \neq j$ , consists of one point.*

*Proof.* We leave to the reader to use the definition of the maps given by linear system to check that  $F_i + |\Delta - F_i|$  is indeed the dual plane to  $\Lambda_i$ . The linear system  $|\Delta - F_i|$  consists of hyperplane sections of  $\phi_\Delta(S)$  that contain the image of  $F_i$ , and hence contain the plane  $\Lambda_i$ . The dimension of this linear system is equal to 2.

The intersection  $\Lambda_i^\perp \cap \Lambda_j^\perp$  consists of the linear system  $|\Delta - F_i - F_j|$ . Since  $(\Delta - F_i - F_j)^2 = 0$  and  $(\Delta - F_i - F_j) \cdot F_k = 1, k \neq i, j$  it consists of an isolated curve of arithmetic genus one.  $\square$

**6.4. Morin sets of planes.** It is natural to ask whether a semi-Lagrangian set of 10 planes can be extended to a larger set of  $n$  planes with similar property. It is obvious, that if  $\Lambda_{-n}, \dots, \Lambda_{-1}, \Lambda_1, \dots, \Lambda_n$  is a configuration of planes in  $\mathbb{P}^5 = \mathbb{P}(V)$  satisfying (6.5) then  $n \leq 10$ . Indeed, the matrix of products of the Plücker 3-vectors  $v_{-i}, v_i$  representing  $\Lambda_{-i}, \Lambda_i$  with respect to the alternating bilinear form (6.4) is non-degenerate, hence  $2n \leq \dim \bigwedge^3 V = 20$ . It is harder to answer the question about extensions in the class of  $n$ -tuples of pairwise intersecting planes  $\Lambda_1, \dots, \Lambda_n$ . We will call such a  $n$ -tuple an isotropic  $n$ -tuple of planes. We will call an isotropic  $n$ -tuple of planes *complete* (or a *Morin set*, see [36]), if it does not extend to an isotropic  $(n+1)$ -tuple of planes. Note that a Morin configuration of ten planes is a (unordered) semi-Lagrangian 10-tuple of planes.

Let  $\mathbb{L}\mathbb{G}(\bigwedge^3 V_6)$  be the closed subvariety of  $\text{Gr}(10, \bigwedge^3 V)$  that parameterizes Lagrangian subspaces of  $\bigwedge^3 V$ . It is a smooth homogeneous space  $G/P$  of dimension 55 for the group  $\text{Sp}(20)$  and its maximal parabolic subgroup  $P$ . We view points in  $\text{Gr}(10, \bigwedge^3 V)$  both as 10-dimensional linear spaces or 9-dimensional projective spaces, hoping that no confusion arises. For any  $A \in \mathbb{L}\mathbb{G}(\bigwedge^3 V_6)$  we let

$$(6.9) \quad \Theta_A := \{\Lambda \in \mathbb{G} = \text{Gr}(3, V) : \Lambda \subset A\}$$

We consider  $\Theta_A$  as the intersection scheme of  $\mathbb{P}(A) \cap \mathbb{G}$ .

We will use the following fact [55, Claim 3.2].

ogradyclaim

**Lemma 6.10.** *A set of pairwise intersecting planes is a Morin set if and only if it spans some  $A \in \mathbb{L}\mathbb{G}(\bigwedge^3 V_6)$  and coincides with  $\Theta_A$ .*

(Igor:added) It follows from the Lemma that the number of planes in a Morin set is larger than or equal to 10. It is known that the largest possible number is 20 [55, Proposition 4.8]. Examples with 20 planes are given in [36] and [25].

Moringopen

**Theorem 6.11.** *Let  $\underline{\Lambda} \in \mathbb{G}_{\text{Fano}}^{10}$  that corresponds to a Fano model of a general Enriques surface. Then  $\underline{\Lambda}$  is a Morin set of 10 planes.*

*Proof.* Let  $\underline{\Lambda} \in \mathbb{G}_{\text{Fano}}^{10}$  correspond to a general Enriques surface  $S \subset \mathbb{P}^5$ . It spans  $A \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)$ . Suppose  $\Theta_A$  contains another plane  $\Lambda$ . Counting constants, we find a cubic hypersurface  $C_1$  that contains 10 planes  $\Lambda, \Lambda_2, \dots, \Lambda_{10}$ . By property (vi) of Proposition 6.1,  $C_1$  belongs to the pencil of cubic surfaces containing the planes  $\Lambda_i, i \neq 1$ , that contains a unique cubic hypersurface passing through  $S$ . It follows from Corollary 7.3 that  $C_1 \neq \mathcal{C}(S, \underline{\Lambda})$  (otherwise  $\mathcal{C}(S, \underline{\Lambda})$  contains 11 mutually intersecting planes). By the uniqueness of  $\mathcal{C}(S, \underline{\Lambda})$ , the cubic  $C_1$  does not contain  $\Lambda_1$ . Its restriction to  $\Lambda_1$  contains nine points  $p_{1j} = \Lambda_1 \cap \Lambda_j$  and the intersection  $\Lambda \cap \Lambda_1$ . It follows from Proposition 2.9 that there is only one cubic curve  $F_1$  in  $\Lambda_1$  through the points  $p_{1j}$ . This implies that  $\Lambda \cap \Lambda_1 \subset F_1$ . Since  $F_1$  is irreducible, we obtain that  $\Lambda \cap \Lambda_1$  consists of a point  $p_1$  lying on  $F_1$ . Replacing  $\Lambda_1$  with another  $\Lambda_j$ , we obtain that  $\Lambda$  intersects other  $F_j$  at one point  $p_j$ .

Let us now consider the intersection  $B = S \cap \Lambda$ . Assume that  $B$  is a curve passing through the points  $p_1, \dots, p_{10}$ . Then  $B \cap F_i = \Lambda \cap \Lambda_i$  intersects each  $F_i$  with multiplicity 1. But then  $B \cdot \Delta = \frac{10}{3}$ , a contradiction. Thus we may assume that  $B$  is the set of points  $p_i \in F_i$ , some of which may coincide. Consider the linear system  $|D| \subset |\Delta|$  of hyperplane sections containing the plane  $\Lambda$ . It has the points  $p_1, \dots, p_{10}$  as the base points. We have  $D \cdot F_i = \Delta \cdot F_i - 1 = 2$ , hence  $10 = D^2 = D \cdot \Delta = \frac{20}{3}$ , a contradiction.  $\square$

C6.10

**Corollary 6.12.** *The map*

$$(6.10) \quad \Psi : \mathbb{G}_{\text{Fano}}^{10} \rightarrow \mathbb{L}\mathbb{G}(\wedge^3 V_6)$$

*that assigns to  $(\Lambda_1, \dots, \Lambda_{10}) \in \mathbb{G}_{\text{Fano}}^{10}$  the span of the planes  $\Lambda_i$  in  $\mathbb{P}(\wedge^3 V)$  factors through the map*

$$(6.11) \quad \Psi^s : \mathbb{G}_{\text{Fano}}^{(10)} := \mathbb{G}_{\text{Fano}}^{10} / \mathfrak{S}_{10} \rightarrow \mathbb{L}\mathbb{G}(\wedge^3 V_6),$$

*which is birational onto its image.*

**Corollary 6.13.** *There exists an irreducible component of  $\mathbb{G}_{\text{lag}}^{10}$  of dimension 45 that contains an open subset of Morin configurations of 10 planes.*

morinlist

*Remark 6.14.* If  $S$  is embedded in a smooth quadric  $Q$  by an ample Reye polarization, then any Fano 10-tuple of planes is not complete. One can show that any two 10-tuples of planes from the same family of planes in  $Q$  span the same Lagrangian subspace  $A(Q)$  and  $\Theta_A$  consists of all planes from this family. This is a 3-dimensional complete family from Morin's classification of complete continuous families of planes in  $\mathbb{P}^5$ . Under the map  $\Psi$  the 24-dimensional locus of Lagrangian 10-tuples arising from Reye-Fano models of Enriques surfaces lying on the same quadric  $Q$  (isomorphic to the Hilbert

scheme of Reye congruences) is blown down to the point  $A(Q) \in \mathbb{L}\mathbb{G}(\Lambda^3 V)$ . Thus the codimension one divisor of Reye-Fano models is blown down to a 20-dimensional subvariety of Lagrangian subspaces of the form  $A(Q)$ .

Another Morin's continuous family is the 2-dimensional family of planes tangent to a Veronese surface in  $\mathbb{P}^5$ . For any fixed Veronese surface the tangent planes span the same Lagrangian subspace  $A$ . We will see later in Appendix, that 10-tuples  $\underline{\Lambda}$  from this family arise from a Cayley-Fano model of a Coble surface. The cubic hypersurface  $\mathcal{C}(S, \underline{\Lambda})$  is the discriminant cubic hypersurface in  $\mathbb{P}^5$  parameterizing singular quadrics with a Veronese surface as its singular locus. Both of these cases exhibit divisors in  $\text{Hilb}_{\text{Fano}}^{\text{sm}}$  that are blown down to linear subspaces in  $\mathbb{L}\mathbb{G}(\Lambda^3 V)$ .

Exa5

**Example 6.15.** (Igor:changed the wording) One can find examples of Morin sets of  $k$  planes with any  $11 \leq k \leq 20$  in [36]. Here we provide a different construction of a Morin 13-tuple. Let  $\Pi_1, \Pi_2, \Pi_3$  be 3-dimensional subspaces of  $\mathbb{P}^5$  which contain a plane  $\Lambda$ . Set  $\Upsilon_{ij} = \langle \Pi_i, \Pi_j \rangle$ ,  $1 \leq i, j \leq 3$ ,  $i \neq j$ . Choose three smooth quadric surfaces  $Q_i \subset \Pi_i$ ,  $i = 1, 2, 3$ , meeting  $\Lambda$  along three distinct smooth conics  $C_i$  such that  $C_i \cap C_j$  consists of four distinct points  $p_1^{ij}, p_2^{ij}, p_3^{ij}, p_4^{ij}$  if  $i \neq j$ . We do not require that the twelve points of intersection are distinct; it is even possible that  $p_k^{ij}$  do not depend on  $i, j$ , so that the  $C_i$  belong to one pencil. We define the plane  $\Lambda(p)^{ij}$  to be the tangent plane of the quadric  $Q_i$  at one of the points  $p = p_k^{ij}$ ,  $k = 1, 2, 3, 4$ . This gives twelve planes  $\Lambda_k^{ij}$ . Since  $\Lambda_k^{ij} \cap \Lambda$  contains the tangent line of the conic  $C_i$  at the point  $p_k^{ij}$ , they pairwise intersect at a point in  $\Lambda$ . Together with  $\Lambda$ , they form the desired configuration of 13 planes.

Let us now show that this configuration is maximal, that is: there is no 14th plane  $\Sigma$  meeting each one of the 13 planes. Assume that such  $\Sigma$  exists. Then  $\Sigma$  meets the 4 planes  $\Lambda_k^{ij}$  ( $k = 1, 2, 3, 4$ ) lying in the same 4-space  $\Upsilon_{ij}$ . Choose  $i, j$  such that  $\Sigma$  is not contained in  $\Upsilon_{ij}$ . Then the intersection  $\Sigma \cap \Upsilon_{ij}$  is a line, say,  $\ell_{ij}$ . The three lines  $\ell_{12}, \ell_{23}, \ell_{31}$  form a triangle with vertices  $q_j = \ell_{ij} \cap \ell_{jk} \in \Pi_j$ .

Let us fix  $(ij)$  for a while; write  $p_k$  instead of  $p_k^{ij}$ . As an element of the Grassmannian  $\text{Gr}(2, 5) = \text{Gr}_1(\Upsilon_{ij})$ , the line  $\ell_{ij}$  is in the intersection  $W$  of four Schubert varieties  $\bigcap_{k=1}^4 \Omega(\Upsilon_k^{ij})$  of lines intersecting  $\Upsilon^{ij}$ . They are special linear complexes, i.e. hyperplanes in the Plücker space which are tangent to  $\text{Gr}(2, 5)$ . Denote by  $Z$  the locus of the intersections  $\ell \cap \Pi_i$  as  $\ell$  runs over  $W$ . Then,  $q_i \in Z$ .

Counting constants in our construction (we have  $42 = 9 + 6 + 3 \times 9$  of them), we may assume that the 4 planes  $\Lambda_k^{ij}$ , considered as points in the Plücker space span a general 3-dimensional subspace. Then the variety  $W$  is the intersection of  $\text{Gr}(2, 5)$  with four special linear complexes  $\Omega(\Lambda_k^{ij})$  and is an anti-canonical Del Pezzo surface of degree 5. This is a well-known classical fact [19, Proposition 8.5.3]. On the other hand, it is known that the degree of the 4-dimensional Schubert variety  $\Omega(\text{line})$  of lines intersecting



a line is equal to 3. Since  $W$  is the intersection of  $\text{Gr}(2, 5)$  with a linear subspace of codimension 4, we see that the union of lines parametrized by  $W$  in  $\mathbb{P}^4$  is a cubic hypersurface  $\mathcal{S}_3$  (in fact, a Segre cubic primal with 10 nodes, see [19, Example 10.2.20]). The intersection of  $\mathcal{S}_3$  with  $\Pi_i$  is equal to  $Z$  and it must be a cubic surface. Since  $W$  contains a ruling of lines on  $Q_i$ , we see that  $Z$  contains  $Q_i$ . The residual component of  $Z$  must be the fifth plane  $\Sigma_{ij}$ . It is contained in  $\Pi_i$ , and similarly, in  $\Pi_j$ . Thus  $\Sigma_{ij}$  does not depend on  $(ij)$  and coincides with  $\Lambda = \Pi_i \cap \Pi_j$ . This shows that all the three points  $q_i$  are in  $\Lambda$ , so  $\ell_{ij} \subset \Lambda$ , and hence  $\Sigma = \Lambda$ . Hence the configuration of 13 planes  $\Lambda, \Lambda_k^{ij}$  is maximal.

## 7. CUBIC FOURFOLDS

sectcub

**7.1. The Fano variety of lines.** We refer for most of the facts found in this section to [5]. Let  $X$  be a smooth cubic fourfold in  $\mathbb{P}^5$  and let  $F = F(X)$  be the variety of lines in  $X$ . Its dimension is equal to 4 and it admits a natural structure of a holomorphic symplectic manifold. The holomorphic 2-form is defined as follows. One considers the universal family of lines

$$U = \{(x, \ell) \in X \times F : x \in \ell\}$$

which is equipped with two projections  $p : U \rightarrow X$ ,  $q : U \rightarrow F$ . Then one considers the Abel-Jacobi map (also called the cylinder map)

abeljacobi

$$(7.1) \quad \Phi = q_* p^* : H^4(X, \mathbb{Z}) \rightarrow H^2(F, \mathbb{Z})$$

which is compatible with the Hodge structures. Let  $h \in H^2(X, \mathbb{Z})$  be the class of a hyperplane section of  $X$  and  $\sigma = \sigma_1|_F$  the class of a hyperplane section of  $F$  in its Plücker embedding. We denote by  $H^4(X, \mathbb{Z})_0$  the orthogonal complement of  $h^2$  and by  $H^2(F, \mathbb{Z})_0$  the orthogonal complement of  $\sigma^3$  in  $H^2(F, \mathbb{Z})$ . We have

polarization

$$(7.2) \quad \Phi(h^2) = \sigma, \quad \Phi : H^4(X, \mathbb{Z})_0 \xrightarrow{\cong} H^2(F, \mathbb{Z})_0.$$

We have  $(\sigma^4)_F = 108$ , the degree of  $F$  in its Plücker embedding. The *Beauville-Bogomolov form*, or the *BB-form* for short is a symmetric bilinear form on  $H^2(F(X), \mathbb{Z})$  satisfying the following properties:

- $(\Phi(a), \Phi(b))_{\text{BB}} = -a \cdot b$ , for any  $a, b \in H^4(X, \mathbb{Z})_0$ ;
- $6(u, v)_{\text{BB}} = \sigma^2 \cdot u \cdot v$ , for any  $u, v \in H^2(F(X), \mathbb{Z})_0$ ;
- $(\sigma, \sigma)_{\text{BB}} = 6$ .

It follows from loc. cit. that  $F$  is an irreducible holomorphic symplectic manifold (IHS manifold for short) deformation equivalent to the Hilbert square of a K3-surface  $K3^{[2]}$ , so the *Beauville-Fujuki identity* relating the cup-product on  $H^2(F(X), \mathbb{Z})$  and the BB-form [3, Sect. 9, Lemma 1] reads as follows:

$$(a^4)_F = 3(a, a)_{\text{BB}}^2.$$

By polarization of the fourth power, one obtains:

BeauFu

$$(7.3) \quad (a \cdot b \cdot c \cdot d)_F = (a, b)_{\text{BB}}(c, d)_{\text{BB}} + (a, c)_{\text{BB}}(b, d)_{\text{BB}} + (a, d)_{\text{BB}}(b, c)_{\text{BB}}.$$



The BB-form of an IHS is invariant under deformations of complex structure, so to compute  $(H^2(F, \mathbb{Z}), (\cdot, \cdot)_{\text{BB}})$ , we can replace  $F$  by the Hilbert square  $Y^{[2]}$  of a K3 surface  $Y$ . Thus the BB lattice of  $F$  is isomorphic to the even lattice of signature  $(3, 20)$

**k3squarelattice**

$$(7.4) \quad \mathbf{L}_{K3^{[2]}} := \mathbf{E} \oplus \mathbf{E} \oplus \mathbf{U} \oplus \langle -2 \rangle$$

The cup-product quadratic form on  $H^4(X, \mathbb{Z})$  is isomorphic to a unimodular odd lattice  $\mathbf{l}^{21,2}$  of rank 23 and signature  $(21, 2)$ . Under this isomorphism, the image of  $h^2$  is a *characteristic vector*  $\mathbf{h}$  of  $\mathbf{l}^{21,2}$  of square-norm 3 characterized among all vectors of square-norm 3 by the property that its orthogonal complement  $\mathbf{h}^\perp$  is an even lattice of signature  $(20, 2)$ . It is isomorphic to the lattice

**lattice**

$$(7.5) \quad \mathbb{L} := \mathbf{E}(-1) \oplus \mathbf{E}(-1) \oplus \mathbf{A}_2(-1),$$

the unique even lattice of signature  $(20, 2)$  and discriminant  $-3$ .

We have  $h^{3,1}(X) = 1$ ,  $h^{2,2}(X) = 21$  and  $\Phi(H^{3,1}(X)) = H^{2,0}(F) = \mathbb{C}\alpha$ , where  $\alpha$  is a holomorphic symplectic 2-form. More explicitly,  $H^{3,1}(X)$  is generated by  $\alpha = \text{Res}_X \frac{\Omega}{P^2}$ , where  $\Omega$  is a section of  $\omega_{\mathbb{P}^5}(6)$  and  $X = V(P)$ . Integrating  $\alpha$  along 4-cycles on  $X$ , we get the period map

**periodmap**

$$(7.6) \quad \wp : \mathcal{M}_{\text{cub}}^{\text{ns}} \rightarrow \Gamma \backslash \mathcal{D}_{\mathbb{L}},$$

where  $\mathcal{M}_{\text{cub}}^{\text{ns}}$  is the moduli space of non-singular cubic hypersurfaces,  $\mathcal{D}_{\mathbb{L}}$  is the disjoint union of two Hermitian symmetric spaces of orthogonal type corresponding to the quadratic space  $\mathbb{L}_{\mathbb{R}}$  and  $\Gamma = \text{O}(\mathbb{L})^* = \text{Ker}(\text{O}(\mathbb{L}) \rightarrow \text{O}(\mathbb{L}^\vee/\mathbb{L}))$  is the subgroup of index 3 of the orthogonal group of  $\mathbb{L}$  (which coincides with the stabilizer subgroup  $\text{O}(\mathbf{l}^{21,2})_{\mathbf{h}}$  Hassett [30, 2.2]). According to Voisin [60], the period map is an isomorphism on its image equal to  $\Gamma \backslash \mathcal{D}_{\mathbb{L}}^0$ , where  $\mathcal{D}_{\mathbb{L}}^0$  is an open subset of  $\mathcal{D}_{\mathbb{L}}$ .

**7.2. Cubic fourfolds with a Lagrangian set of planes.** Recall that a general point  $\underline{\Lambda}$  of  $\mathbb{G}_{\text{Fano}}^{10}$  defines a smooth cubic fourfold  $\mathcal{C}(S, \underline{\Lambda})$  containing  $\underline{\Lambda}$ . We called it an *Enriques-Fano cubic fourfold*. Here we extend this definition and reserve this name for any smooth cubic fourfold containing a Lagrangian set of 10 planes. We will prove later that 10 is the maximal number of pairwise intersecting planes a smooth cubic fourfold may contain. An Enriques-Fano cubic fourfold with an ordered Lagrangian set of planes is called a *marked Enriques-Fano cubic fourfold*.

Let  $X_{\underline{\Lambda}}$  denote an Enriques-Fano cubic fourfold containing a Lagrangian set of planes  $\underline{\Lambda}$ . Let  $P_i = [\Lambda_i]$  be the class of  $\Lambda_i \in \underline{\Lambda}$  in  $H^4(X, \mathbb{Z})$ . Intersecting  $X$  with a 3-dimensional subspace containing  $\Lambda_i$  we find that

**latticeM**

$$(7.7) \quad h^2 \cdot P_i = 1, \quad P_i \cdot P_j = 1, \quad P_i^2 = (h^2)^2 = h^4 = 3.$$

Let  $M$  be the primitive sublattice of  $H^4(X, \mathbb{Z})$  spanned by  $h^2, P_1, \dots, P_{10}$ . The symmetric bilinear form of  $M$  in the basis  $(h^2, P_1, \dots, P_{10})$  is given by the matrix  $2I_{11} + \mathbb{1}_{11}$ , where  $I_{11}$  is the identity matrix and  $\mathbb{1}_{11}$  a square matrix of size 11 with all entries equal to 1.

One immediately checks that this matrix is positive definite and its discriminant is equal to  $2^{10} \cdot 13$ . Its orthogonal complement  $T = M^\perp$  in  $H^4(X, \mathbb{Z})$  is an even lattice of signature  $(10, 2)$  and discriminant  $-2^{10} \cdot 13$ .

Choose an isomorphism  $\phi : H^4(X, \mathbb{Z}) \rightarrow \mathbb{I}^{21,2}$ . We know that the image of  $h^2$  is a characteristic vector of  $\mathbb{I}^{21,2}$ . It is known that vectors of norm 3 in  $\mathbb{I}^{21,2}$  form two orbits with respect to  $O(\mathbb{I}^{21,2})$  [Wal1, Theorem 6], one of them consists of characteristic vectors. We fix such a vector  $\mathbf{h}$  with orthogonal complement  $\mathbb{L}$  given in (7.5). So, changing an isomorphism  $\phi$ , we will assume that  $\phi(h^2) = \mathbf{h}$ , and hence  $\phi(M)$  contains  $\mathbf{h}$ .

**L7.1** **Lemma 7.1.** *Let  $T$  be the orthogonal complement of  $M$  in  $H^4(X, \mathbb{Z})$ . There is an isomorphism of lattices*

$$T \cong E(-2) \oplus \begin{pmatrix} 2 & 3 \\ 3 & -2 \end{pmatrix}.$$

*Proof.* We choose a basis in  $\mathbb{I}^{21,2}$  by considering  $\mathbb{I}^{21,2}$  as the orthogonal sum  $\mathbb{I}^{1,10}(-1) \oplus \mathbb{I}^{1,10}(-1) \oplus \langle 1 \rangle$ . Here

$$\langle m \rangle$$

denotes a lattice of rank 1 generated by a vector of square-norm equal to  $m$ . In each direct summand  $\mathbb{I}^{1,10}(-1)$ , we choose an orthogonal basis  $(\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_{10})$  and  $(\mathbf{e}'_0, \mathbf{e}'_1, \dots, \mathbf{e}'_{10})$  as in (2.1). We check that the vector

$$\mathbf{h} = (\mathbf{k}_{10}, \mathbf{k}'_{10}, \mathbf{e})$$

is of square-norm 3 and its orthogonal complement is even. Thus we may construct one such embedding  $\iota : M \hookrightarrow \mathbb{I}^{21,2}$  by

$$\iota(h^2) = \mathbf{h}, \quad \iota(P_i) = (\mathbf{e}_i, \mathbf{e}'_i, -\mathbf{e}).$$

The orthogonal complement of this lattice contains a sublattice  $N'$  of rank 12 generated by vectors

$$(\mathbf{e}_i, -\mathbf{e}'_i, 0), \quad i = 0, \dots, 10, \quad (\Delta, \mathbf{e}'_0, -3\mathbf{e}).$$

The first 11 vectors generate a sublattice isomorphic to  $\mathbb{I}^{1,10}(-2)$ . It is easy to see that

**orthogcomplement**

$$(7.8) \quad N' = E(-2) \oplus \mathbb{Z}(\mathbf{k}_{10}, -\mathbf{k}'_{10}, 0) + \mathbb{Z}(\Delta, \mathbf{e}'_0, -3\mathbf{e}),$$

where  $E(-2) \subset \mathbb{I}^{1,10}(-2)$  is equal to  $(\mathbf{k}_{10}, -\mathbf{k}'_{10}, 0)^\perp$ . Computing the Gram matrix of the vectors  $(\mathbf{k}_{10}, -\mathbf{k}'_{10}, 0)$  and  $(\Delta, \mathbf{e}'_0, -3\mathbf{e})$  we find that it is equal to the matrix from the assertion of the lemma. The discriminant of  $N'$  is equal to  $2^{10} \cdot 13$ . Since  $\iota(M)$  has the discriminant equal to  $-2^{10} \cdot 13$ , and  $\iota(M) \oplus N'$  is a sublattice of a unimodular lattice, we obtain that  $N' = \iota(M)^\perp \simeq T$ . The discriminant quadratic form on  $A_M$  is the negative of the discriminant quadratic form on  $A_T$ . We easily find from (7.8) that there is an isomorphism of discriminant quadratic forms

$$A_T \cong A_{E(2)} \oplus A_{\langle 13 \rangle}.$$

Applying [Nikulin 49, Theorem 1.14.12] (we leave to the reader to check that its assumptions are satisfied in our case), we obtain that such a lattice is determined uniquely up to isomorphism by its signature and its discriminant quadratic form. The lattice from the assertion certainly has the same signature and the same discriminant quadratic form as  $T$ , hence it is isomorphic to  $M$ .  $\square$

unique\_emb

**Lemma 7.2.** *Any two primitive embeddings  $\phi : M \hookrightarrow \mathbb{I}^{21,2}$  and  $\psi : M \hookrightarrow \mathbb{I}^{21,2}$  such that  $\phi(h) = \psi(h) = \mathbf{h}$  are conjugate by an isometry of  $\mathbb{I}^{21,2}$  fixing  $\mathbf{h}$ .*

*Proof.* It is clear that a primitive embedding of  $M$  in  $\mathbb{I}^{21,2}$  such that the image contains  $\mathbf{h}$  is equivalent to a primitive embedding of its orthogonal complement  $T$  into the even lattice  $\mathbb{L}$  from [Nikulin 7.5]. Its orthogonal complement in  $\mathbb{L}$  will be isomorphic to the orthogonal complement  $M'$  of  $h^2$  in  $M$ . Its discriminant quadratic form is isomorphic to  $A_M \oplus A_{(3)}$ .

Next we apply Nikulin's result [Nikulin 49, Proposition 15.1] about primitive embeddings of an even lattice  $S$  into an even non-unimodular lattice  $L$ . We take  $S = T$  and  $L = \mathbb{L}$ . According to loc. cit., a primitive embedding of  $M'$  into  $\mathbb{L}$  is defined by the data  $(H_{M'}, H_{\mathbb{L}}, \gamma, K, \gamma_K)$ , where  $H_{M'}$  is a subgroup of  $A_{M'}$ ,  $H_q$  is a subgroup of  $A_{\mathbb{L}}$ ,  $\gamma$  is an isomorphism  $H_{M'} \rightarrow H_q$  preserving the restrictions of the discriminant quadratic forms to them,  $K$  is a lattice of signature  $(1, 10)$  with discriminant form  $\delta \cong \left( (A_{M'}, q_{M'}) \oplus (A_{\mathbb{L}}, -q_{\mathbb{L}}) \Big|_{\Gamma_{\gamma}^{\perp}} \right) / \Gamma_{\gamma}$ , where  $\Gamma_{\gamma}$  is the graph of  $\gamma$  in  $A_{M'} \oplus A_{\mathbb{L}}$ , and  $\gamma_K$  is an isomorphism  $(A_K, q_K) \rightarrow (\Gamma_{\gamma}^{\perp} / \Gamma_{\gamma}, -\delta)$ .

Two such data  $(H_{M'}, H_{\mathbb{L}}, \gamma, K, \gamma_K), (H'_{M'}, H'_{\mathbb{L}}, \gamma', K', \gamma'_K)$  define isomorphic primitive embeddings if and only if  $H_{M'} = H'_{M'}$  and there exist an element  $\xi \in O(M')$  and an isomorphism of lattices  $\psi : K \rightarrow K'$  for which  $\gamma' = \xi \circ \gamma$  and  $\bar{\xi} \circ \gamma_K = \gamma'_{K'} \circ \bar{\psi}$ , where  $\bar{\xi}, \bar{\psi}$  are the restrictions to the discriminant forms.

To apply this proposition we take  $H_{M'}$  to be the 3-torsion subgroup of  $A_{M'}$  isomorphic to  $\mathbb{Z}/3\mathbb{Z}$  and take  $\gamma$  to be an isomorphism  $H_{M'} \rightarrow A_{\mathbb{L}}$ . We see that  $\delta \cong (A_M, -q_{A_M})$  and we take  $\gamma_K : (A_M, -q_{A_M}) \rightarrow (T, q_T)$ , so that we may take  $K = T$ . To prove the assertion, it remains to use that by the above,  $T$  is defined uniquely up to an isomorphism, and apply [Nikulin 49, Theorem 1.14.2] according to which the homomorphism  $r_T : O(T) \rightarrow O(A_T)$  is surjective (since  $H_{M'} \cong \mathbb{Z}/3\mathbb{Z}$ , we choose the above  $\xi$  to be the identity).  $\square$

C7.3

**Corollary 7.3.** *Let  $X$  be a smooth cubic fourfold. Then  $X$  has at most 10 planes with one point pairwise intersections.*

*Proof.* Suppose we have 11 planes  $P_i$  as in the assertion. As in the proof of Lemma [Nikulin 7.2], one shows that the discriminant of the lattice of rank 12 spanned by the cohomology classes  $[P_i], [h^2]$  is equal to  $2^{11} \cdot 14$ . In fact, one computes the discriminant group which turns out to be isomorphic to  $(\mathbb{Z}/2\mathbb{Z})^{10} \oplus \mathbb{Z}/28\mathbb{Z}$ . In particular, the minimal number of its generators is

equal to 11. The discriminant group of its orthogonal complement  $T'$  in  $l^{21,2}$  (Igor:misprint?) is of rank 11 and the same discriminant group. However, it embeds into the lattice  $\mathbb{L}$  of rank 22 and discriminant 3 with the orthogonal complement  $N$  of rank 11. Since  $A_N \cong A_{T'} \oplus \mathbb{Z}/3\mathbb{Z}$  has 12 generators, we get a contradiction.  $\square$

*Remark 7.4.* In [Laza1] Radu Laza introduced the *algebraic index* of a smooth cubic fourfold  $X$  as the rational number  $\kappa(X) = \frac{2^{\rho(X)}}{d_X}$ , where  $\rho_X, d_X$  are the rank and the discriminant of the lattice  $H^4(X, \mathbb{Z})_{\text{alg}} = H^4(X, \mathbb{Z}) \cap H^{2,2}(X)$ . He calls  $X$  *potentially irrational* if the orthogonal complement  $T_X$  of  $H^4(X, \mathbb{Z})_{\text{alg}}$  in  $H^4(X, \mathbb{Z})$  is not embeddable into the K3-lattice  $\mathbf{L}_{K3}$  and proves that the algebraic index for such  $X$  is less than or equal to 1. In our case  $T = M^\perp$  is of rank 12 and discriminant group with minimal number of generators equal to 11. If it admits a primitive embedding into  $\mathbf{L}_{K3}$ , the orthogonal complement would be of rank 10 and the discriminant group with minimal number of generators equal to 11. This contradiction shows that  $T$  is not primitively embedded in  $\mathbf{L}_{K3}$  and hence a general Enriques-Fano  $X$  is potentially irrational. We also find  $\kappa(X) = \frac{2^{11}}{2^{10} \cdot 13} = \frac{2}{13} < 1$ . It follows from Theorem 1.8 of loc.cit. that a general Enriques-Fano cubic fourfold has no Eckardt point (i.e. points with a cubic tangent cone). Of course, we have no idea how to prove that  $X$  is irrational.

**7.3. Lattice polarization on  $F(X_\Lambda)$ .** For any general  $\Lambda \in \mathbb{G}_{\text{Enr}}^{10}$  we denote by  $X_\Lambda$  the unique smooth cubic hypersurface containing the planes from  $\Lambda$  and let  $F(X_\Lambda)$  be the Fano scheme of lines in  $X_\Lambda$ . We use the notation from Section 7. Besides the hyperplane class  $\sigma_1$ ,  $F(X_\Lambda)$  has ten divisor classes  $D_j = \Phi(P_j) = \Phi([\Lambda_j])$ . Each of them is represented by the 3-dimensional family of lines in  $F(X_\Lambda)$  meeting  $\Lambda_j$ .

**Lemma 7.5.** *Let  $M$  be the sublattice spanned by  $h^2, P_1, \dots, P_{10}$ . Then the sublattice  $M_0$  of  $M$  orthogonal to  $h^2$  is spanned by the classes*

$$h^2 - P_1 - P_2 - P_3, P_1 - P_2, \dots, P_9 - P_{10}.$$

*Its discriminant is equal to  $2^{10} \cdot 3 \cdot 13$ .*

*Proof.* Obviously, all the listed classes are orthogonal to  $h^2$ . We compute the intersection matrix and find that its determinant is equal to  $2^{10} \cdot 3 \cdot 13$ . Since  $(h^2)^2 = 3$  and the discriminant of  $M$  is  $2^{10} \cdot 13$ , the assertion follows.  $\square$

We see that the sublattice

$$N_0 := \langle h^2 \rangle \oplus M_0 = \langle h^2, h^2 - P_1 - P_2 - P_3, P_1 - P_2, \dots, P_9 - P_{10} \rangle$$

is of index 3 in  $M$ .

fanocub

**Theorem 7.6.** *In the above notation,  $F(X_\Lambda)$  is an irreducible symplectic fourfold, deformation equivalent to the Hilbert square of a K3 surface, and*

the BB form on the image  $N = \Phi(M)$  of the sublattice  $M$  of  $H^4(X_\Delta, \mathbb{Z})$  in the basis  $\Phi(h^2), \Phi(P_1), \dots, \Phi(P_{10})$  is given by the matrix

$$A = \begin{pmatrix} 6 & 2 & 2 & \dots & 2 \\ 2 & -2 & 0 & \dots & 0 \\ & \vdots & & \ddots & \vdots \\ 2 & 0 & 0 & \dots & -2 \end{pmatrix},$$

The lattice  $N$  is of signature  $(1, 10)$  and its discriminant group is isomorphic to  $A_{\mathbb{E}(2)} \oplus A_{(26)}$ .

*Proof.* Using the properties of the BB-form, we find that the image of  $N_0 = \langle h^2 \rangle \oplus M_0$  in  $H^2(F(X_\Delta), \mathbb{Z})$  is the lattice

$$\sigma, \sigma - \Phi(P_1 + P_2 + P_3), \Phi(P_i - P_{i+1}), i = 1, \dots, 9.$$

Its discriminant is equal to  $2^{10} \cdot 6 \cdot 13$ . It is equal to the sublattice of  $\Phi(M)$  of index 3. This shows that  $\Phi(M)$  is of discriminant  $2^{11} \cdot 13$ .

For the brevity of notation we set  $X = X_\Delta$ . Let  $D_i = \Phi(P_i)$ . It is the cohomology class of the 3-dimensional subvariety of  $F(X)$  formed by lines in  $X$  intersecting the plane  $\Lambda_i$ . By [60, Lemma 3], we have, for any  $x \in H^4(X, \mathbb{Z})_0$ ,

voisinlemma

$$(7.9) \quad 2(x^2)_X = -\sigma \cdot D_j \cdot \Phi(x) \cdot \Phi(x).$$

Applying this to  $x = P_i - P_{i+1}$ , and using the Fujiki identity (7.3), we find

$$8 = -(\sigma, D_k)_{\text{BB}}(D_i - D_{i+1}, D_i - D_{i+1})_{\text{BB}} = 4(\sigma, D_k)_{\text{BB}}.$$

This gives  $(\sigma, D_j)_{\text{BB}} = 2$ . Now,  $h^2 - 3P_k \in H^4(X, \mathbb{Z})_0$ , hence  $24 = (h^2 - 3P_k)^2 = -(\sigma - 3D_k, \sigma - 3D_k)_{\text{BB}} = 6 - 12 + 9(D_k, D_k)_{\text{BB}}$  gives  $(D_k, D_k)_{\text{BB}} = -2$ . Since  $(D_i - D_j, D_i - D_j)_{\text{BB}} = -4$ , we obtain  $(D_i, D_j)_{\text{BB}} = 0$ . Thus, we find that the intersection matrix of the cohomology classes  $\sigma, [D_1], \dots, [D_{10}]$  with respect to the BB-form is equal to the matrix from the assertion theorem.

Computing the determinant of  $M$ , we find that it is equal to  $2^{11} \cdot 3 \cdot 13$ . Since it coincides with the discriminant of  $M$ , we obtain the first assertion.

Since under the isomorphism  $\Phi$  the image of  $T = M^\perp$  is the lattice  $N$  defined in (7.8), we obtain that the orthogonal complement of  $\Phi(M)$  in  $H^2(F(X), \mathbb{Z})$  with respect to the BB-form is a lattice of signature  $(2, 10)$  isomorphic to the lattice  $T(-1)$ , see Lemma 7.1. It follows from (7.4) that the lattice  $H^2(F(X), \mathbb{Z})_{\text{BB}}$  has discriminant group isomorphic to  $A_{(-2)} \cong \mathbb{Z}/2\mathbb{Z}$ . Since  $A_T \cong A_{\mathbb{E}(2)} \oplus A_{(13)}$ , the assertion follows from the comparison the discriminant groups of a primitive sublattice and its orthogonal complement.  $\square$

Let us now invoke the theory of the moduli space of lattice polarized IHS manifolds that is a generalization of theory of lattice polarized K3 surfaces (see [6]). Let  $\mathbf{N}$  be an abstract lattice with a fixed basis  $e_0, e_1, \dots, e_{10}$  and the quadratic form in this basis given by the matrix  $A$  from Theorem 7.6.

We fix a primitive embedding  $\iota : \mathbf{N} \hookrightarrow \mathbf{L}_{K3[2]}$  with orthogonal complement  $\mathbf{N}^\perp = \mathbf{E}(2)^\perp$ . Let  $F$  be an IHS manifold isomorphic to  $F(X)$  for some smooth cubic fourfold  $X$ . A choice of an order on the set of planes  $\Lambda_i \subset X_\Delta$  defines a primitive lattice embedding

$$j : \mathbf{N} \hookrightarrow \text{Pic}(F) \subset H^2(F, \mathbb{Z})_{\text{BB}}$$

such that  $j(e_0) = \sigma, j(e_i) = \Phi([\Lambda_i])$ . A choice of a marking  $\phi : \mathbf{L}_{K3[2]} \rightarrow H^2(F, \mathbb{Z})_{\text{BB}}$  such that  $j = \phi \circ \iota$  defines the period point  $p_{F,j,\phi} \in \mathcal{D}_{\mathbf{E}(2)^\perp}$ . We fix a connected component  $K$  of  $\mathbf{N}_{\mathbb{R}} \setminus \{x \in \mathbf{N}_{\mathbb{R}} : x \cdot \delta = 0, \delta \in \mathbf{N}_{-2}\}$  that contains  $e_0$ . By definition from loc. cit. this defines an *ample lattice- $\mathbf{N}$  polarization* on  $F$ .

**Theorem 7.7.** *There is an isomorphism of the moduli space of ample  $\mathbf{N}$ -polarized varieties  $F$  and the moduli space  $\mathcal{M}_{\text{Enr}} = \Gamma_{\mathbf{E}(2)^\perp} \backslash \mathcal{D}_{\mathbf{E}(2)^\perp}^o$ .*

**7.4. Moduli space of Enriques-Fano cubic fourfolds.** Let us consider the restriction of the period map (7.6) <sup>periodmap</sup> to the locus of smooth Enriques-Fano cubics  $\mathcal{M}_{\text{cub,EF}}^{\text{ns}}$ . Since  $H^4(X, \mathbb{Z})_{\text{alg}}$  contains the sublattice  $M$  spanned by  $h^2$  and the classes of planes  $\Lambda_i$ , consider the abstract lattice  $\mathbf{M}$  isomorphic to  $M$  with a fixed basis  $e_0, e_1, \dots, e_{10}$  which is mapped to a basis  $(h^2, P_1, \dots, P_{10})$  under an isomorphism  $j : \mathbf{M} \rightarrow M$ . We fix also the primitive embedding  $\iota : \mathbf{M} \hookrightarrow \mathbb{I}^{20,3}$  defined in the proof of Lemma 7.1 <sup>7.1</sup>.

Let  $\phi : H^4(X, \mathbb{Z}) \rightarrow \mathbb{I}^{20,3}$  be a marking of the cohomology of  $X$  such that  $\phi \circ j = \iota$ . Then the period point  $p_{X,\phi}$  is contained in  $\mathbb{P}(\mathbf{T})$ , where  $\mathbf{T}$  is the orthogonal complement of  $\mathbf{M}$  in  $\mathbb{I}^{20,3}$ . By Lemma 7.2 <sup>unique\_emb</sup>, there exists a  $g \in \Gamma$  that sends any marking  $\phi$  of  $X$  to a marking with this property. Let

$$\Gamma_{\mathbf{T}} = \{g \in \Gamma : g|_{\mathbf{M}} = \text{id}_{\mathbf{M}}\}.$$

Then the period map defines an isomorphism

$$\wp : \mathcal{M}_{\text{cub,EF}}^{\text{ns,m}} \cong \Gamma_{\mathbf{T}} \backslash \mathcal{D}_{\mathbf{T}}^o,$$

where  $\mathcal{M}_{\text{cub,EF}}^{\text{ns,m}}$  is the moduli space of smooth Enriques-Fano cubic fourfolds with an ordering of the ten planes on it and  $\mathcal{D}_{\mathbf{T}}^o$  is an open subset of  $\mathcal{D}_{\mathbf{T}}$ .

Recall that  $\Gamma = \{g \in \text{O}(\mathbb{I}^{20,3})' : g(\mathbf{h}) = \mathbf{h}\}$ . Since any  $g \in \Gamma_{\mathbf{T}}$  acts identically on the orthogonal complement  $\mathbf{M}$  of  $\mathbf{T}$ , we see that group  $\Gamma_{\mathbf{T}}$  coincides with the kernel of  $\rho : \Gamma \rightarrow \text{O}(A_{\mathbf{T}})$ . We have  $A_{\mathbf{T}} \cong A_{\mathbf{E}(2)} \oplus \{\pm 1\}$  and the image of  $\rho$  coincides with the subgroup of  $A_{\mathbf{E}(2)} \cong \text{O}^+(10, \mathbb{F}_2)$  isomorphic to  $\mathfrak{S}_{10} \cong \text{O}(\mathbf{M})_{\mathbf{h}}$ .

**Theorem 7.8.** *Let  $\mathcal{M}_{\text{cub,EF}}^{\text{ns}}$  be the moduli space of smooth Enriques-Fano cubic fourfolds. Then*

$$\mathcal{M}_{\text{cub,EF}}^{\text{ns}} \cong \tilde{\Gamma}_{\mathbf{T}} \backslash \mathcal{D}_{\mathbf{T}}^o,$$

where  $\mathcal{D}_{\mathbf{T}}^o = \mathcal{D}_{\mathbb{L}}^o \cap \mathcal{D}_{\mathbf{T}}$  and

$$\tilde{\Gamma}_{\mathbf{T}} = \{g \in \Gamma : g(M) = M\}.$$

*Proof.* The only unproved result is the surjectivity of the period map. By Global Torelli-Voisin theorem a point in the right-hand-side is the period point of a smooth cubic fourfold  $X$  such that  $\mathbf{M}$  primitively embeds in  $H^4(X, \mathbb{Z})_{\text{alg}}$ . Let  $(h^2, P_1, \dots, P_{10})$  be the image of the standard basis of  $\mathbf{M}$ . The sublattice of rank 2 generated by  $P_i, h^2 - P_i$  has intersection matrix  $\begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}$ . It follows from [60, §3] that  $P_i$  is the cohomology class of a plane. Since  $P_i \cdot P_j = 1$ , the ten planes pairwise incident, hence  $X'$  is a smooth Enriques-Fano fourfold.  $\square$

**Corollary 7.9.** *There are birational isomorphisms*

$$\mathcal{M}_{\text{cub,EF}}^{\text{ns,m}} \stackrel{\text{bir}}{\cong} \mathcal{M}_{\text{Enr}}^{\text{sm,a}} \stackrel{\text{bir}}{\cong} \mathbb{G}_{\text{Enr}}^{10} // \text{PGL}(6),$$

$$\mathcal{M}_{\text{cub,EF}}^{\text{ns}} \stackrel{\text{bir}}{\cong} \mathfrak{S}_{10} \backslash \mathcal{M}_{\text{Enr}}^{\text{sm,a}} \stackrel{\text{bir}}{\cong} \mathbb{G}_{\text{Enr}}^{(10)} // \text{PGL}(6). \text{ (Igor : misprint?)}$$

This gives another proof of the irreducibility of  $\mathcal{M}_{\text{Enr}}^{\text{sm,a}}$ .

sect\_EPW

## 8. DOUBLE EPW SEXTICS

**8.1. EPW sextic hypersurfaces.** We have associated an irreducible holomorphic symplectic (IHS) manifold of dimension 4 to a generic Fano 10-tuple  $\underline{\Lambda}$ : this is the Fano variety  $F(X_{\underline{\Lambda}})$  of lines on the unique Enriques-Fano cubic fourfold  $X_{\underline{\Lambda}}$  that contains  $\underline{\Lambda}$ . In this section we will associate another 4-dimensional IHS manifold to  $\underline{\Lambda}$ , denoted  $\tilde{X}_{\underline{\Lambda}}$ , and will show that these two families of IHS manifolds are distinct, or more exactly,  $F(X_{\underline{\Lambda}}), \tilde{X}_{\underline{\Lambda}}$  are not even birationally equivalent.

Recall the construction of an EPW sextic surface in  $\mathbb{P}^5$  from [50]. Let  $V_6 = \mathbb{C}^6$  and  $\text{LG}(\wedge^3 V_6)$  be the variety of Lagrangian subspaces in  $\wedge^3 V_6$ . For any  $A \in \text{LG}(\wedge^3 V_6)$  one defines the degeneracy loci:

$$(8.1) \quad Y_A[k] := \left\{ [v] \in \mathbb{P}(V_6) : \dim \left( v \wedge \wedge^2 V_6 \cap A \right) \geq k > 0 \right\},$$

If  $Y_A[1] \neq \mathbb{P}(V_6)$ , then  $Y_A := Y_A[1]$  is a hypersurface of degree 6, first introduced in [24, Example 9.3]. Also,  $Y_A[k+1]$  is contained in the singular locus  $\text{Sing}(Y_A[k])$ .

Let

$$\text{LG}(\wedge^3 V_6)^0 = \{A : \Theta_A = \emptyset, Y_A[3] = \emptyset\}.$$

By [54, Proposition 3.3],

$$Y_A \neq \mathbb{P}(V_6) \text{ if } \dim \Theta_A \leq 2.$$

**Proposition 8.1.** [50, Theorem 1.1] *Assume  $Y_A \neq \mathbb{P}(V_6)$ . For any  $A \in \text{LG}(\wedge^3 V_6)$  there exists a double cover  $\hat{Y}_A \rightarrow Y_A$  which is étale outside the surface  $Y_A[2]$ . If  $A \in \text{LG}(\wedge^3 V_6)^0$ , then  $\text{Sing}(Y_A) = Y_A[2]$  and  $\hat{Y}_A$  is smooth*



and admits a unique structure of an IHS manifold deformation equivalent to the Hilbert square of a K3 surface.

It is also known that, for  $A \in \mathbb{L}\mathbb{G} \left( \Lambda^3 V_6 \right)^0$ , the surface  $Y_A[2]$  is a smooth surface of general type with the Hilbert polynomial equal to  $126P_0 - 120P_1 + 40P_2$ , where  $P_n$  stands for the Hilbert polynomial of  $\mathbb{P}^n$  [51, Theorem 1.1].

For any two  $A, A' \in \mathbb{L}\mathbb{G} \left( \Lambda^3 V_6 \right)^0$ , we have  $Y_A \cong Y_{A'}$  if and only if  $A, A'$  are in the same  $\text{PGL}(V_6)$ -orbit [54, Proposition 1.2.1]. Together with the known fact that  $\mathbb{L}\mathbb{G} \left( \Lambda^3 V_6 \right)$  is an irreducible variety of dimension 55, this proves that the moduli space of irreducible holomorphic symplectic varieties  $Y_A$  is 20-dimensional.

sexitics

**Proposition 8.2.** *Suppose  $Y_A[3] = \emptyset$  and  $\Theta_A$  is a finite set.*

- (i) *The singular locus  $\text{Sing}(Y_A)$  consists of the union of the surface  $X[2]$  and the planes  $\Lambda$  from  $\Theta_A$  [52, Corollary 2.5].*
- (ii) *The intersection  $C_{\Lambda, A} = \Lambda \cap Y_A[2]$  is a curve of degree 6 [56, Subsection 3.2 and Corollary 3.3.7].*
- (iii) *The singular points of  $C_{\Lambda, A}$  are the intersection points with  $C_{\Lambda', A}$  for some other  $\Lambda' \in \Theta_A$  [56, Subsection 3.3].*
- (iv)  *$C_{\Lambda, A}$  has only ordinary double points if and only if no other  $\Lambda', \Lambda'' \in \Theta_A$  intersect at one point in  $\Lambda$  [52, Proposition 4.6].*

Let  $\mathbb{L}\mathbb{G} \left( \Lambda^3 V_6 \right)^{ADE}$  be the locus of  $A \in \mathbb{L}\mathbb{G} \left( \Lambda^3 V_6 \right)$  such that  $Y_A[3] = \emptyset$  and, for each plane  $\Lambda \in \Theta_A$ , the plane sextic curve  $C_{\Lambda, A}$  has only simple singularities. For each  $A \in \mathbb{L}\mathbb{G} \left( \Lambda^3 V_6 \right)^{ADE}$ , the pre-image  $K_\Lambda$  of  $\Lambda$  in  $\hat{Y}_A$  is the double cover of  $\Lambda$  branched along  $C_{\Lambda, A}$ . It is birationally isomorphic to a K3 surface admitting a degree 2 polarization.

Let  $\Sigma_k$  be the Zariski closure of the locus of  $A \in \mathbb{L}\mathbb{G} \left( \Lambda^3 V_6 \right)$  such that  $\#\Theta_A = k$ . By [26, Proposition 1.5],  $\Sigma_k$  is of codimension  $k$  (when non-empty) and smooth outside  $\Sigma_{k+1}$ . It follows from Corollary 6.12 that the map  $\Psi^s : \mathbb{G}_{\text{Fano}}^{(10)} \rightarrow \mathbb{L}\mathbb{G} \left( \Lambda^3 V_6 \right)$  from (6.11) is a birational map onto its image in  $\Sigma_{10}$ .

It follows from [53, Proposition 2.2] that the subset

$$\mathbb{L}\mathbb{G} \left( \Lambda^3 V_6 \right)_0 = \left\{ A \in \mathbb{L}\mathbb{G} \left( \Lambda^3 V_6 \right) : Y_A[3] \neq \emptyset \right\}$$

is an irreducible divisor in  $\mathbb{L}\mathbb{G} \left( \Lambda^3 V_6 \right)$ . The next proposition implies that the image of  $\Psi$  is not contained in  $\mathbb{L}\mathbb{G} \left( \Lambda^3 V_6 \right)_0$ .

P8.3

**Proposition 8.3.** *Let  $\underline{\Lambda} \in \mathbb{G}_{\text{Fano}}^{10}$  be generic and  $A_{\underline{\Lambda}} = \Psi(\underline{\Lambda}) \in \mathbb{L}\mathbb{G} \left( \Lambda^3 V_6 \right)$ . Then  $Y_{\underline{\Lambda}} := Y_{A(\underline{\Lambda})} \neq \mathbb{P}^5$  is a sextic hypersurface,  $Y_{A(\underline{\Lambda})}[3] = \emptyset$  and the singular locus of  $Y_{A(\underline{\Lambda})}$  is the union of the ten planes  $\Lambda_i$  from  $\underline{\Lambda}$  and the irreducible degree 40 surface  $Y_{\underline{\Lambda}}[2]$ .*



*Proof.* The fact that  $Y_{\underline{\Lambda}}[2]$  and all the  $\Lambda_j$  are in  $\text{Sing } Y_{\underline{\Lambda}}$  follows from Proposition 8.2. In the Appendix (Fact 9.1.7, see also Proposition 8.4 below) we present a special Enriques surface for which we check that there is nothing else in  $\text{Sing } Y_{\underline{\Lambda}}$  and that  $Y_{A(\underline{\Lambda})}[3] = \emptyset$ . Hence the same properties hold for an open subset of the 10-dimensional irreducible variety  $\mathbb{G}_{\text{Fano}}^{10}$ .  $\square$

**8.2. Description of  $Y_{\underline{\Lambda}}$ .** Using a Macaulay2 computation, we can construct a  $\underline{\Lambda} \in \mathbb{G}_{\text{Fano}}^{10}$  that defines an EPW-sextic satisfying the listed below properties. Each of these properties is open in the variety  $\mathbb{G}_{\text{Fano}}^{10}$ , hence a general EPW-sextic obtained in this way satisfies all these properties.

**Proposition 8.4.** *Let  $\underline{\Lambda}$  be a generic Fano 10-tuple of planes in  $\mathbb{P}^5$ , and let  $A(\underline{\Lambda})$ ,  $Y_{\underline{\Lambda}}$  be defined as above. Then the following properties hold:*

(i) *Sing  $Y_{\underline{\Lambda}}$  decomposes into 11 irreducible components, of which 10 are the planes  $\Lambda_1, \dots, \Lambda_{10}$ , and the eleventh one is an irreducible surface  $Y_{\underline{\Lambda}}[2]$  of degree 40 with Hilbert polynomial  $126P_0 - 120P_1 + 40P_2$ . The locus  $Y_{\underline{\Lambda}}[3]$  is empty. The singularity of  $Y_{\underline{\Lambda}}$  along  $\Lambda_j \setminus Y_{\underline{\Lambda}}[2]$  ( $j = 1, \dots, 10$ ) and along  $Y_{\underline{\Lambda}}[2] \setminus \left(\bigcup_{i=1}^{10} \Lambda_i\right)$  is an ordinary double point in the transversal slices.*

(ii) *The surface  $Y_{\underline{\Lambda}}[2]$  is non-normal. Its singular locus  $\text{Sing } Y_{\underline{\Lambda}}[2]$  is the union of 10 plane sextic curves  $B_i = Y_{\underline{\Lambda}}[2] \cap \Lambda_i$  ( $i = 1, \dots, 10$ ) of genus 1 with 9 nodes at points  $p_{ij} = \Lambda_i \cap \Lambda_j$  ( $j = 1, \dots, 10, j \neq i$ ). One can obtain  $B_i$  as the intersection of  $\Lambda_i$  with the EPW sextic  $Y_{\underline{\Lambda}_i}$ , where  $\underline{\Lambda}_i$  is obtained from  $\underline{\Lambda}$  by replacing  $\Lambda_i$  with  $\Lambda_{-i}$  (we use the notation from Definition 6.6).*

(iii) *Off of the 45 points  $p_{ij}$ , the singularities of  $Y_{\underline{\Lambda}}[2]$  are those of generically transversal intersection of two smooth branches, and the tangent cone of  $Y_{\underline{\Lambda}}[2]$  at any one of the points  $p_{ij}$  is linearly equivalent to the cone in  $\mathbb{C}^4$  with vertex 0, formed by the four coordinate planes  $\langle e_1, e_2 \rangle$ ,  $\langle e_2, e_3 \rangle$ ,  $\langle e_3, e_4 \rangle$ ,  $\langle e_4, e_1 \rangle$ . The two planes  $\langle e_1, e_3 \rangle$ ,  $\langle e_2, e_4 \rangle$  are the images of  $\Lambda_i$ ,  $\Lambda_j$  respectively, so that the lines  $\langle e_1 \rangle$ ,  $\langle e_3 \rangle$  correspond to the tangents of two branches of  $B_i$  at  $p_{ij}$ , and the two lines  $\langle e_2 \rangle$ ,  $\langle e_4 \rangle$  correspond to the tangents of the two branches of  $B_j$  at  $p_{ij}$ .*

(iv) *The canonical class of  $Y_{\underline{\Lambda}}[2]$  is numerically  $\mathcal{O}_{Y_{\underline{\Lambda}}}(3)$ , so that  $\mathcal{O}_{Y_{\underline{\Lambda}}[2]}(2K_{Y_{\underline{\Lambda}}[2]}) \simeq \mathcal{O}_{Y_{\underline{\Lambda}}[2]}(6)$ . The linear system  $|\mathcal{O}_{Y_{\underline{\Lambda}}[2]}(3)|$  contains the divisor  $X_{\underline{\Lambda}} \cap Y_{\underline{\Lambda}}[2]$ , a double structure on the degree 60 curve  $\bigcup_{i=1}^{10} B_i$ , where  $X_{\underline{\Lambda}}$  denotes, as above, the Fano–Enriques cubic associated to  $\underline{\Lambda}$ . The normalization  $\tilde{Y}_{\underline{\Lambda}}[2]$  has numerically trivial canonical class.*

(v) *O’Grady’s double cover  $\pi : \hat{Y}_{\underline{\Lambda}} \rightarrow Y_{\underline{\Lambda}}$  provides a singular symplectic 4-dimensional variety  $\hat{Y}_{\underline{\Lambda}}$ , whose singular locus is the union of 10 K3 surfaces  $K_j = \pi^{-1}(\Lambda_j)$  ( $j = 1, \dots, 10$ ) with an ordinary double point at the intersections  $\hat{p}_{ij} = \bar{B}_i \cap \bar{B}_j$  (thus each  $K_j$  has nine ordinary double points). The map  $\sigma : \tilde{Y}_{\underline{\Lambda}} \rightarrow \hat{Y}_{\underline{\Lambda}}$  obtained by blowing up  $\text{Sing } \hat{Y}_{\underline{\Lambda}}$  is a symplectic resolution of singularities, and  $\tilde{Y}_{\underline{\Lambda}}$  is a symplectic fourfold with a generically 2-to-1 morphism  $\tilde{\pi} = \sigma \circ \pi : \tilde{Y}_{\underline{\Lambda}} \rightarrow Y_{\underline{\Lambda}}$  onto the EPW sextic. The ten exceptional*

divisors  $D_j$  are  $\mathbb{P}^1$ -bundles over the K3 surfaces  $\tilde{K}_j$  obtained by resolving the 9 singular points of  $K_j$ , and  $D_i$  meets  $D_j$  transversely along the quadric  $\sigma^{-1}(\hat{p}_{ij}) \simeq \mathbb{P}^1 \times \mathbb{P}^1$ .

*Proof.* The evidence for all the assertions is the Macaulay2 computation showing that the wanted properties are verified for a special (nodal) Enriques surface  $X$  defined over a finite field, see Appendix (Fact [9.1.7](#)). The linearized picture of the double cover  $\pi$  in (v) in the neighborhood of  $p_{ij}$  is that of the double cover  $\mathbb{C}^4/H \rightarrow \mathbb{C}^4/G$ , where  $G$  is the order-8 group of sign changes of even number of coordinates in  $\mathbb{C}^4$ , and  $H \subset G$  is the index-2 subgroup generated by the two involutions  $\text{diag}(-1, 1, -1, 1)$ ,  $\text{diag}(1, -1, 1, -1)$ . It is easy to check (via the toric geometry, for example) that the singularities of  $\mathbb{C}^4/H$  are resolved by a single blowup of the singular locus with its reduced structure, which is the linearized version (near  $p_{ij}$ ) of the blowup of the 10 singular K3 surfaces in  $\hat{Y}_\Lambda$ . Alternatively one can give the local analytic model of  $Y_\Lambda$  at  $p_{ij}$  by one equation  $u_5^2 = u_1 u_2 u_3 u_4$  in  $\mathbb{C}^5$ , and then the double cover  $\hat{Y}_\Lambda$  is obtained by adding two more local parameters  $v, w$  and the additional equations  $v^2 = u_2 u_4$ ,  $w^2 = u_1 u_3$ ,  $u_5 = vw$ .  $\square$

*Conjecture 8.5.* The assertions of Proposition [8.4](#) hold for any Fano 10-tuple of planes in  $\mathbb{P}^5$  and not only for a generic one.

conj

*Remark 8.6.* Presumably,  $\tilde{Y}_\Lambda[2]$  is birationally isomorphic to an Enriques surface, and  $Y_\Lambda[2]$  is the projection of  $\tilde{Y}_\Lambda[2]$  defined by a linear subsystem of quadrics on its Fano model. An evidence for these two assertions is given by Theorem [4.9](#), where we introduced the bitangent surface  $\text{Bit}(D_W)$  of the discriminant surface of a general web of quadrics whose normalization is isomorphic to a general nodal Enriques surface realized as the Reye congruence of  $W$ . The difference is that the sextic curves  $B_i$  forming the singular locus of  $\text{Bit}(D_W)$  are reducible: they are unions of two plane cubics meeting at nine points  $p_{ij}$  ( $j = 1, \dots, 10$ ,  $j \neq i$ ).

R8.6

Even a stronger evidence is provided by the work of Ferretti [\[26\]](#). Starting from any irreducible quartic surface  $Y_0$  with  $k$  ordinary nodes, he shows that the Hilbert square  $Y_0^{[2]}$  admits a deformation  $\pi : \mathcal{X} \rightarrow U$  over a smooth variety of dimension  $20 - k$  such that  $\mathcal{X}_0$  is birationally isomorphic to  $Y_0^{[2]}$ . For generic  $t \in U$ , the fibre  $\mathcal{X}_t$  is a singular double EPW-sextic  $X_{A(t)}$  which contains  $k$  planes in its singular locus and  $X_{A(t)}[2]$  is birationally isomorphic to the bitangent surface  $\text{Bit}(Y_0)$ . By taking  $Y_0$  to be a quartic symmetroid associated to an excellent web  $W$  of quadrics in  $\mathbb{P}^3$ , he shows that the Hilbert square of its minimal resolution admits a degree 6 map to the quadric  $Q$  in  $\mathbb{P}^5$  that contains the Reye congruence  $\text{Rey}(W)$  defined by  $W$ . The involution of  $Y_0^{[2]}$  defined by passing lines through a pair of points on  $Y_0$  has the fixed locus isomorphic to the bitangent surface  $\text{Bit}(S)$  and the fixed locus of its lift to  $Y_0^{[2]}$  is isomorphic to  $S$ . This defines the normalization map  $\nu : \text{Rey}(W) \rightarrow$

$\text{Bit}(D_W)$  which we discussed in Subsection 4.3. He finds an irreducible 10-dimensional variety  $U'$  which parametrizes EPW sextics  $Y_A$  with the singular locus equal to the union of 10 mutually intersecting planes  $\Lambda_i \in \underline{\Lambda}$  generating a Lagrangian subspace  $A$  with  $Y_A[2]$  birationally isomorphic to an Enriques surface. There exists a symplectic resolution  $\tilde{Y}_A$  of the double cover of  $Y_A$  which provides a holomorphic symplectic manifold which contains 10 divisors  $D_i$ , each is isomorphic to a  $\mathbb{P}^1$ -bundle over a K3 surfaces  $K_i$ . Each  $K_i$  is birationally isomorphic to the double cover of  $\Lambda_i$  along the sextic curve  $\Lambda_i \cap Y_{\underline{\Lambda}}$ . The fixed locus of the deck involution of the double cover is birationally isomorphic to an Enriques surface. All of this suggests that the 10-dimensional irreducible family of double EPW sextics constructed by Ferretti coincides with our 10-dimensional family coming from  $\mathbb{G}_{\text{Enr}}^{10}$ .

However, it is unlikely that the surface  $Y_{\underline{\Lambda}}[2]$  is birationally isomorphic to the original Enriques surface  $S$ , as follows from the computation in Fact 9.3. A degenerate case is described there, in which  $S$  is a union of three quadrics and 4 planes, but  $Y_{\underline{\Lambda}}[2]$  has a component of degree 16.

**8.3. The lattice polarization on  $\tilde{Y}_{\underline{\Lambda}}$ .** In this subsection we work over the open part of the 10-dimensional family of EPW sextics for which the description given in Proposition 8.4 holds. If we assume Conjecture 8.5, this open part is the whole family of  $Y_{\underline{\Lambda}}$  parametrized by the 10-tuples  $\underline{\Lambda} \in \mathbb{G}_{\text{Fano}}^{10}$ . The resolutions  $\tilde{Y}_{\underline{\Lambda}}$  of the double EPW sextics  $\hat{Y}_{\underline{\Lambda}}$  form a 10-dimensional family of irreducible holomorphic symplectic fourfolds over  $\mathbb{G}_{\text{Fano}}^{10}$ . They possess the following properties: the underlying EPW sextic  $Y_{\underline{\Lambda}}$  contains exactly 10 planes (Proposition 8.3), and the pre-images of the ten planes in  $\tilde{Y}_{\underline{\Lambda}}$  are ten exceptional divisors  $D_j$  which are  $\mathbb{P}^1$ -bundles over some K3 surfaces  $K_j$  with ODP  $\hat{p}_{ij} = K_i \cap K_j$ .

As we mentioned in Remark 8.6, Ferretti constructed another 10-dimensional family of irreducible symplectic varieties IHS manifolds  $\tilde{Y}_{\underline{\Lambda}}$ , where  $\underline{\Lambda}$  are parameterized by some variety  $\Sigma_{10}$ . They possess exactly the same properties. By Theorem 6.4  $\mathbb{G}_{\text{Fano}}^{10}$  is an irreducible 45-dimensional variety and the same is true for  $\Sigma_{10}$ , thus giving ten moduli parameters modulo  $\text{PGL}(6)$ . We do not know whether  $\mathbb{G}_{\text{Fano}}^{10}$  is equal to  $\Sigma_{10}$ , though both families are polarized by the same diagonal lattice  $l^{1,10}(2)$ , in which the first basis vector of the lattice, the one of square 2, is the pullback  $h_{\text{EPW}}$  of the hyperplane section of  $Y_{\underline{\Lambda}}$ , and the remaining ten are the classes of the  $D_j$ 's.

latticeEPW

**Lemma 8.7.** *A variety  $\tilde{Y}_{\underline{\Lambda}}$  constructed as in part (v) of Proposition 8.4 is a deformation of the Hilbert square of a K3 surface, and the BB form on the elements  $h_{\text{EPW}}, D_1, \dots, D_{10}$  coincides with  $l^{1,10}(2)$ .*

*Proof.* The first assertion follows from [47], Theorem (2.2). It implies that the BB form of  $\tilde{Y}_{\underline{\Lambda}}$  is even and satisfies the Beauville–Fujiki identity (7.3) relating the BB form to the quadruple intersection products.

The linear system of  $h_{\text{EPW}}$  sends  $\tilde{Y}_\Lambda$  onto a sextic in  $\mathbb{P}^5$  via a degree two map. Each surface  $D_i$  is a double cover of the plane  $\mathbb{P}^2$ , hence  $h_{\text{EPW}}$  cuts out on  $D_i$  the class of the pre-image of a line. This gives  $(h_{\text{EPW}} \cdot D_i)^2 = -4$ . The quadruple self-intersection is  $(h_{\text{EPW}}^4) = 12$ , hence  $(h_{\text{EPW}}, h_{\text{EPW}})^2 = 4$  and  $(h_{\text{EPW}}, h_{\text{EPW}}) = 2$ . Further, as  $\tilde{\pi}(D_i)$  is not a divisor,  $(h_{\text{EPW}}^3 \cdot D_j) = 0$ , hence  $(h_{\text{EPW}}, D_j) = 0$ . It follows from (7.3) that  $(D_i, D_i) = -2$ .  $\square$

**Proposition 8.8.** *For generic  $\underline{\Lambda} \in \mathbb{G}_{\text{Fano}}^{10}$ ,  $F(X_\Lambda)$  is not birationally isomorphic to  $\tilde{Y}_\Lambda$ .*

*Proof.* Indeed, their BB forms on  $H^2(F(X_\Lambda, \mathbb{Z})_{\text{alg}})$  and  $H^2(\tilde{Y}_\Lambda, \mathbb{Z})_{\text{alg}}$  are non-equivalent as they have different discriminants:  $2^{11}$  for  $l^{1,10}(2)$  and  $13 \cdot 2^{11}$  for the matrix  $A$  from Theorem 7.6. However, two birationally isomorphic IHS manifolds have isomorphic second cohomology lattices equipped with the BB-form and the Hodge structures [31, Lemma 2.6]. This implies that their sublattices of algebraic cycles are isomorphic, but it is not true in our case.  $\square$

*Remark 8.9.* As follows from Appendix 9.2, we can degenerate an Enriques surface  $S$  to a Coble surface in such a way that  $F(\mathcal{C}_\Lambda)$ ,  $X_\Lambda$  behave quite differently. When  $S$  is the icosahedral Coble surface embedded by plane decimics with 10 triple points,  $\mathcal{C}_\Lambda$  is the determinantal cubic  $\Delta$ , that is the union of secant lines of the Veronese surface  $V_4$  in  $\mathbb{P}^5$ , and  $X_\Lambda$  is this cubic doubled.

The Fano scheme of lines of the determinantal cubic  $\Delta$  is the union of two components. The first one is isomorphic to the Hilbert square  $V_4^{[2]} \simeq (\mathbb{P}^2)^{[2]}$ , parametrizing the secant lines of  $V_4$ . It is isomorphic to the blow-up of the secant variety of  $V_4$  along  $V_4$ . The second one is the family of lines contained in the tangent planes to  $V_4$ . It is isomorphic to the product  $\mathbb{P}^2 \times \mathbb{P}^{2*}$ . Indeed, if we interpret the projective space  $\mathbb{P}^5$  as the linear system of conics in  $\mathbb{P}^2$ , so that  $V_4$  parametrizes the double lines, then  $\Delta$  is the locus of reducible conics. The lines of  $\mathbb{P}^5$  contained in the tangent planes to  $V_4$  are the pencils of reducible conics of the form  $\ell + \sigma(p)$ , where  $\ell$  is a fixed line in  $\mathbb{P}^2$  and  $\sigma(p)$  is the pencil of lines in  $\mathbb{P}^2$  passing through a fixed point  $p$ . The pair  $(p, \ell)$  that determines such a pencil runs over  $\mathbb{P}^2 \times \mathbb{P}^{2*}$ . The two components intersect along a 3-dimensional variety which can be identified with the exceptional divisor of the Chow map  $(\mathbb{P}^2)^{[2]} \rightarrow (\mathbb{P}^2)^{(2)}$  in the first component and with the projectivization of the tangent bundle of  $\mathbb{P}^2$  in the second component.

The reducible Fano variety is an example of a “constructible” holomorphically symplectic variety; it is interesting to obtain more such examples in the spirit of Kulikov-Morrison-Pinkham semistable degenerations of K3 surfaces.

*Remark 8.10.* Let  $A^\perp \in \mathbb{LGr}(\wedge^3 V_6^\vee)$  be the dual Lagrangian subspace in  $\wedge^3 V_6^\vee$ . If  $\Theta_A = \emptyset$ , O’Grady shows that  $Y_{A^\perp}$  is the dual hypersurface  $Y_A^*$  of  $Y_A$  [51, Proposition 3.1]. He also remarks that for a general  $A$ , the EPW-sextic  $Y_A$  is not self-dual. So this gives a non-trivial birational involution

on the projective equivalence classes of EPW-sextics. We know that  $\underline{\Lambda} \in \text{Gr}(3, V_6)_{\text{lag}}^{10}$  defines the dual configuration  $\underline{\Lambda}^\perp \in \text{Gr}(\wedge^3 V_6^\vee)_{\text{lag}}^{10}$  and the dual of a Fano-Enriques configuration is a Fano-Enriques configuration. What one can say about  $Y_{A(\underline{\Lambda})}$  and  $Y_{A(\underline{\Lambda}^\perp)} = Y_{A(\underline{\Lambda}^\perp)}$  in this case?

## 9. APPENDIX: NUMERICAL EXPERIMENTS WITH TENS OF PLANES

All the computations described in this Appendix were realized in the computer algebra system Macaulay2 [28] over one of the base fields  $\mathbb{F}_p$ ,  $\mathbb{Q}$  or their finite extensions. None of the computations was approximate, only exact arithmetic operations were used, so the properties verified by these computations may serve as rigorous proofs of those properties for special input data. When a property that we verify for special data is an open one, we can conclude that it holds for generic data of the same type.

**9.1. Cayley-Fano model of an Enriques surface.** Here we show by using computer computation the following.

**Proposition 9.1.** *Let  $\text{Hilb}_{\text{Cay}}^{\text{sm}}$  be the subvariety of  $\text{Hilb}_{\text{Fano}}^{\text{sm}}$  of Cayley polarized Enriques surface with a choice of a supermarking. Then  $\text{Hilb}_{\text{Cay}}^{\text{sm}}$  contains an open non-empty subset of pairs  $(S, \underline{\Lambda})$  such that the assertions of Propositions 2.9, 6.16.3 are still true.*

To show this it suffices to give one example of a Reye embedded Enriques surface  $S$  and a set of planes  $\underline{\Lambda}$  cutting out the curves  $F_i$  on it such that all the assertions from the Propositions and the Theorem are valid.

We consider the Cayley model of an Enriques surface  $S$  given by a symmetric  $4 \times 4$ -matrix  $\mathcal{A}$  with linear forms in 6 variables  $x_0, \dots, x_5$  as its entries. We refer to section 4.4 for the description of the 20 planes  $\Lambda_k^c$  ( $k \in \{\pm 1, \dots, \pm 10\}$ ) in  $\mathbb{P}^5$  spanned by the twenty cubic curves on  $S$ . We will omit the superscript  $c$  referring to the Cayley model in the rest of this section. We computed several examples, in which  $\mathcal{A}$  is defined over a prime field  $\mathbb{F}_p$ . Usually the twenty planes  $\Lambda_k$  are not defined over the same field. We found most convenient the symmetric case when they split in two Galois orbits of length 10 over  $\mathbb{F}_p$ , that is when the  $\Lambda_k$  are defined over  $\mathbb{F}_{p^{10}}$ . Here is one such example, for which we realized the computations described in

the sequel:  $p = 17$  and  $\mathcal{A} = (a_{ij})$ , where

$$\begin{aligned}
a_{11} &= -6x_0 + 4x_1 + 5x_2 + 8x_3 + 4x_4 \\
a_{12} &= 2x_0 + x_1 + x_2 + 8x_3 - 6x_4 - 5x_5 \\
a_{13} &= 8x_0 - 8x_1 - x_2 - 5x_3 - 2x_4 \\
a_{14} &= -5x_0 - 2x_1 - 7x_2 - 3x_3 - 8x_4 \\
a_{22} &= x_0 + 7x_1 - 2x_2 - 6x_3 - x_4 - 5x_5 \\
a_{23} &= -6x_0 + 3x_1 - x_2 + x_3 - 2x_4 - 6x_5 \\
a_{24} &= 6x_0 + 4x_1 + 4x_2 + 2x_3 - x_4 + 2x_5 \\
a_{33} &= x_0 + 2x_1 - 7x_2 - 2x_3 - 4x_4 - 7x_5 \\
a_{34} &= -x_0 + 3x_1 - 6x_2 + 8x_3 + 6x_5 \\
a_{44} &= 6x_0 + 2x_1 - 8x_3 + 2x_4 + x_5
\end{aligned}$$

To determine the twenty planes  $\Lambda_k$  corresponding to these data, we use the interpretation of the Fano model  $S$  as the locus in  $\mathbb{P}(L^\perp)$  of reducible quadrics in  $\mathbb{P}^3 = \mathbb{P}(V_4)$ , so that the union of trisecants to  $S$  is the quartic hypersurface  $D_4$  in  $\mathbb{P}(L^\perp)$  parametrizing singular quadrics (see Theorem 14.11 and 4.12). The variables  $x_i$  above are the coordinates on  $\mathbb{P}(L^\perp)$ . The planes  $\Lambda_k$  are the spans of 20 cubic curves contained in  $S$ , hence they are swept by the trisecants and are contained in  $D_4$ . Passing to the incidence correspondence

$$\tilde{D}(L^\perp) = \{(x, Q) \in \mathbb{P}(V_4^\vee) \times \mathbb{P}(L^\perp) : x \in \text{Sing}(Q)\}$$

we see that the image  $P$  of the planes contained in  $D(L^\perp)$  under the first projection  $\text{pr}_1$  to  $\mathbb{P}(V_4^\vee)$  is the locus of 20 points that are the images of the planes  $\Lambda_\pm$  under the duality correspondence. It can be determined by means of linear algebra as follows:  $\mathcal{A}$  defines a map  $L^\perp \rightarrow S^2((V_4^\vee)^\vee) = S^2(V_4)$ . After polarization, we get a map  $V_4^\vee \rightarrow \text{Hom}(L^\perp, V_4)$ , so we can consider  $\mathcal{A}$  as a web of linear maps  $L^\perp \rightarrow V_4$  parametrized by  $\mathbb{P}(V_4^\vee)$ , and  $P$  is the rank  $\leq 3$  degeneracy locus of this web. This is the departure point of the computation. The ideal  $I_P$  of  $P \subset \mathbb{P}(L^\perp)$  is computed as that of order 4 minors of a  $6 \times 4$  matrix of linear forms on  $\mathbb{P}(V_4^\vee)$  constructed from  $\mathcal{A}$ . It turns out that  $I_P$  is of colength 20, that it splits into two primes  $P^+$ ,  $P^-$  of degree 10 each over  $\mathbb{F}_p$ , and that over  $\mathbb{F}_{p^{10}}$  it defines 20 distinct points:  $P^+ = \{q_1, \dots, q_{10}\}$ ,  $P^- = \{q_{-1}, \dots, q_{-10}\}$ . A verification of the incidence relations between the planes  $\Lambda_k$  corresponding to  $q_k$  under the first projection  $D(L^\perp) \rightarrow \mathbb{P}(V_4^\vee)$  shows that  $\Lambda_1, \dots, \Lambda_{10}$  is a Lagrangian ten (and hence any ten obtained by switching the signs of a number of subscripts  $\Lambda_i \rightarrow \Lambda_{-i}$  is also Lagrangian).

We verified the following properties:

betti\_numbers

**Fact 9.1.1.** *Let  $D$  be the quartic polynomial in  $x_i$  defining  $D(L^\perp)$ . Then  $\text{Cay}(S)$  is defined, scheme theoretically, by the 6 cubic polynomials  $\frac{\partial D}{\partial x_0}, \dots, \frac{\partial D}{\partial x_5}$ . The ideal generated by the 6 polar cubics is not saturated,*



and its saturation  $I_S$  is generated by 10 cubics. The ideal  $I_S$  has a minimal resolution <sup>resolution</sup> (4.9) of the same type as the minimal resolution of the ideal of the singular locus of the universal quartic symmetroid in  $\mathbb{P}^9$ .

**Lambda**

**Fact 9.1.2.** Let  $\mathbb{k} = \mathbb{F}_p$ ,  $K = \mathbb{F}_{p^{10}}$ ,  $V = \mathbb{k}^6$ ,  $V_K = K^6$ ,  $\mathbb{P}^5 = \mathbb{P}(V)$ . Let  $v_k$  denote the 3-vectors in  $\bigwedge^3 V_L$  representing the Plücker images of  $\Lambda_k$  in  $\text{Gr}(3, 6) = \text{Gr}(3, V_K) \subset \mathbb{P}(\bigwedge^3 V_K)$ . Then the 3-vectors  $v_1, \dots, v_{10}$ , defined over  $K$ , are linearly independent and generate a Lagrangian 10-dimensional subspace  $A$  of  $\bigwedge^3 V$ , defined over  $\mathbb{k}$ .

**nonsingimub**

**Fact 9.1.3.** There is a unique, up to proportionality, cubic  $\mathcal{C}_0 \in \mathbb{P}(S^3 V^\vee)$  vanishing on  $\Lambda_1, \dots, \Lambda_{10}$ . This cubic is nonsingular and the scheme-theoretic intersection of  $\mathcal{C}_0 \cap S$  is the union of ten plane cubic curves  $C_i = \Lambda_i \cap S$ ,  $i = 1, \dots, 10$ .

**dim10**

**Fact 9.1.4.** There are 45 distinct intersection points  $p_{ij} = \Lambda_i \cap \Lambda_j$ , and there are 11 linearly independent cubics  $F_0, \dots, F_{10}$  vanishing on the 45 points. We can choose them in such a way that  $F_0$  is the cubic from the previous fact. The common zero locus of the 11 cubics is the union of 10 plane cubic curves  $C_i$  ( $i = 1, \dots, 10$ ).

**choiceF\_i**

**Fact 9.1.5.** The scheme-theoretic base locus of a generic linear subsystem  $\mathbb{P}^9$  of cubics in the linear system  $\mathbb{P}^{10} = \langle F_0, \dots, F_{10} \rangle$  is  $\bigcup_{i=1}^{10} C_i$ . There are exactly 11 subsystems  $\mathbb{P}_i^9$  ( $i = 0, \dots, 10$ ) of cubics with base locus of dimension 2. For ten of them, say,  $\mathbb{P}_i^9$  ( $i = 1, \dots, 10$ ), the base locus is  $\Lambda_i \cup \bigcup_{j \neq i} C_j$ . The eleventh,  $\mathbb{P}_0^9$ , is the degree 3 part  $I_{S,3}$  of  $I_S$ , and its base locus is  $S$ .

Next we picked up a random point  $p_i \in \Lambda_i \setminus C_i$  for each  $i = 1, \dots, 10$ . Since every cubic from  $I_{S,3}$  vanishes on the ten cubic curves  $C_i \subset \Lambda_i$ , a cubic  $F \in I_{S,3}$  vanishes on  $\Lambda_i$  if and only if it vanishes at  $p_i$ . We have additionally verified the following.

**fact**

**Fact 9.1.6.** The ten conditions  $F(p_j) = 0$  ( $j = 1, \dots, 10$ ) for  $F \in I_{S,3}$  are linearly independent, which is equivalent to the non-existence of a cubic vanishing on  $S \cup \bigcup_{i=1}^{10} \Lambda_i$ .

We are using this observation to specify a choice of a particular basis of  $\langle F_0, \dots, F_{10} \rangle$  which will be used in the next facts: leave  $F_0$  unchanged, and choose for  $F_i$  a cubic vanishing identically on  $S$  and on 9 planes  $\Lambda_j$  ( $j = 1, \dots, 10, j \neq i$ ), but not vanishing on  $\Lambda_i$ . With this choice, we have:

**verify\_conj**

**Fact 9.1.7.** The Eisenbud–Popescu–Walter sextic  $Y_A$ , where  $A \subset \bigwedge^3 V$  <sup>Lambda</sup> EPW-ten-planes-general introduced in Fact 9.1.2, satisfies all the properties listed in Proposition 8.4 <sup>VarEnr10</sup>.

Next we will verify that the variety of tens of Lagrangian planes is smooth and is of expected dimension 45 near the ten  $\Lambda_1, \dots, \Lambda_{10}$  considered in this section. We use the notation from Section 6.3, where we represented each  $\Lambda_i$  by a  $3 \times 3$  matrix  $A_i$  of coordinates in the standard open cell of the

Grassmannian  $\text{Gr}(3, 6)$ , so that  $\Lambda_i$  is spanned by the columns of the matrix  $\begin{pmatrix} I_3 \\ A_i \end{pmatrix}$ .

dim-45

**Fact 9.1.8.** *The elements of the matrices  $A_i$  representing the ten planes  $\Lambda_i$  ( $i = 1, \dots, 10$ ), associated to  $\mathcal{A}$  define an open subset isomorphic to an affine space of dimension 90 of the space of tens of planes. The subvariety of tens of planes in this open set with nonempty pairwise intersections is given by 45 cubic equations  $\det(A_i - A_j) = 0$  ( $1 \leq i < j \leq 10$ ). Let  $(X_i)_{i=1, \dots, 10}$  be a ten of  $3 \times 3$  matrices considered as a tangent vector to the 90-dimensional space of tens of planes at the point  $A = (A_i)_{i=1, \dots, 10}$ . Then the linear system of 45 equations in  $\overset{\text{tanspace}}{(6.6)}$  defining the tangent space  $T_A \Sigma$ ,*

$$\text{Tr}(\text{adj}(A_i - A_j)(X_i - X_j)) = 0, \quad 1 \leq i < j \leq 10,$$

is of rank 45. Thus  $A$  is a smooth point of  $\Sigma$  and the local dimension of  $\Sigma$  at  $A$  is  $90 - 45 = 45$ .

The next interesting facts discovered from the computation are purely experimental, we do not know their theoretical proofs.

**Fact 9.1.9.** *There are  $210 = \binom{10}{3}$  other Enriques surfaces constructed from the cubics  $F_i$  as follows. For each triple  $i, j, k$  such that  $1 \leq i < j < k \leq 10$ , the intersection  $X_{ijk} = \{F_i = F_j = F_k = 0\}$  is a surface of degree 27 which decomposes as  $S \cup S_{ijk} \cup \bigcup_{m \notin \{i, j, k\}} \Lambda_m$ . The component  $S_{ijk}$  is an Enriques surface, different from  $S$ . The incidences of components of  $X_{ijk}$  can be described as follows. Set  $(ijk) = (123)$ ,  $X = X_{ijk}$ ,  $S' = S_{ijk}$  to simplify the notation.*

(a) *Double curves:*

- (1)  $B$ , a genus 4 curve of degree 9, the dimension 1 component of the intersection  $S \cap S'$ . The latter intersection is not pure and, besides  $B$ , contains 21 isolated points  $p_{ij}$  ( $4 \leq i < j \leq 10$ ).
- (2)  $C_i = S \cap \Lambda_i$ , 7 plane cubics ( $i = 4, \dots, 10$ ).
- (3)  $C'_i = S' \cap \Lambda_i$ , 7 plane cubics ( $i = 4, \dots, 10$ ).

(b) *The intersection points of double curves:*

- (1) 21 triple points  $Q_{i1}, Q_{i2}, Q_{i3} \in C_i \cap C'_i \cap B$  ( $i = 4, \dots, 10$ );
- (2) 21 quadruple points  $p_{ij} = C_i \cap C'_i \cap C_j \cap C'_j$  ( $4 \leq i < j \leq 10$ ).

For each  $i$ ,  $C_i$  and  $C'_i$  are two plane cubics in one plane  $\Lambda_i$ , and their nine intersection points are  $Q_{i1}, Q_{i2}, Q_{i3}$  and  $\{p_{ij}\}_{j \in \{4, \dots, 10\} \setminus \{i\}}$ .

“Triple” and “quadruple” in (b) also refers to the number of surface components meeting at these points: the three components meeting at  $Q_{ik}$  are  $S, S', \Lambda_i$ , and the four components meeting at  $p_{ij}$  are  $S, S', \Lambda_i, \Lambda_j$ . The local structure of  $X$  at a  $n$ -uple point ( $n = 3, 4$ ) is the same as that of a  $n$ -gonal pyramid spanning  $L^n$  near its vertex. The intersections are quasi-transversal



in the following sense: the sequence of sheaves

$$\begin{aligned}
0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_S \oplus \mathcal{O}_{S'} \oplus \bigoplus_{i=4}^{10} \mathcal{O}_{\Lambda_i} \xrightarrow{\alpha} \mathcal{O}_B \oplus \bigoplus_{i=4}^{10} (\mathcal{O}_{C_i} \oplus \mathcal{O}_{C'_i}) \xrightarrow{\beta} \\
\longrightarrow \bigoplus_{i=4}^{10} \mathcal{O}_{Q_{ik}} \oplus \bigoplus_{4 \leq i < j \leq 10} \mathcal{O}_{p_{ij}} \longrightarrow 0.
\end{aligned}$$

is exact, where, on each 1-dimensional stratum,  $\alpha$  is the *difference* of restrictions from two two-dimensional components meeting along this stratum, and on each  $n$ -uple point ( $n = 3, 4$ ),  $\beta$  is the *sum* of restrictions from  $n$  curves meeting at this point.

For the sake of completeness, we will also mention the intersections of  $S'$  with  $\Lambda_1, \Lambda_2, \Lambda_3$ :  $S'$  meets  $\Lambda_i$  ( $i = 1, 2, 3$ ) in two points of  $C_i$ , and the triple intersections  $S' \cap \Lambda_i \cap \Lambda_j$  ( $1 \leq i \leq 3, i < j \leq 10$ ) are empty.

coble-comp

## 9.2. Tens of planes related to Fano models of a nodal Coble surface.

Let  $C$  be a plane rational sextic with ten nodes  $\Sigma = \{p_i\}_{i=1, \dots, 10}$ . The Coble surface  $S$  is the blow-up  $\tilde{\mathbb{P}}_\Sigma^2$  of  $\Sigma$  in  $\mathbb{P}^2$ . It has two Fano models: the Reye model is the image of the embedding by the 5-dimensional linear system of septics with ten nodes on  $\Sigma$ , and the Cayley model is the image of the embedding by the linear system of decimics with ten triple points on  $\Sigma$  (see section 5.1). The septics singular at all the points of  $\Sigma$  form a 6-dimensional linear system and embed the blowup  $\tilde{\mathbb{P}}^2$  of  $\mathbb{P}^2$  at  $\Sigma$  into  $\mathbb{P}^5$ . The image is a surface of degree 9. In order to keep the idea that the Reye model should have degree 10, we may say that the Reye model is reducible and contains, besides the above surface  $S_9$  of degree 9, the plane  $\mathbb{P}^2$  defined as the span of the image of  $C$ , which is a conic in  $S_9$ . The Cayley model of  $\tilde{\mathbb{P}}^2$  is singular, as the map by decimics with triple points on  $\Sigma$  blows down  $C$  to a singular point.

First we have to produce a rational sextic with ten nodes. We find one in the pencil of sextic invariants of the icosahedral group  $\mathfrak{A}_5$  in its irreducible representation in  $\mathrm{SL}(3, \mathbb{C})$  (see [DolgachevQuartics](#) [23, Section 7]). It is generated by an invariant triple conic  $V(h_2)$  and the orbit  $V(h_6)$  of 6 lines and can be given by the equation

$$\mu(xy + z^2)^3 + \lambda z(x^5 + y^5 + z^5 + 5x^2y^2z - 5xyz^3 + z^5) = 0.$$

It contains exactly four singular members:

- $\lambda = 0$ :  $C_0$  has 6 nodes and is of genus 4.
- $\lambda = -1$ :  $C_{-1} = V(h_6)$  is the union of 6 lines, polar to the 6 nodes of  $C_0$  w. r. t.  $h_2$ .
- $\lambda = -\frac{32}{27}$ :  $C = C_\lambda$  is Winger's rational sextic with ten nodes [Win](#) [63].
- $\lambda = \infty$ :  $C_\infty$  is a triple conic.

We will realize constructions of tens of planes related to Fano models of a Coble surface starting from Winger's sextic  $C = \{f = 0\}$ , where

$$f = 32h_2^3 - 27h_6 = 32x^6 + 27xy^5 - 120x^4yz + 150x^2y^2z^2 + 5y^3z^3 + 27xz^5.$$

We denote by  $(e_0, e_1, \dots, e_{10})$  the geometric basis of the Picard group of the Coble surface  $S$  corresponding to the set  $\Sigma$  of the nodes of the Winger sextic. For brevity of notation we let  $E = E_1 + \dots + E_{10}$ .

The following facts were verified through symbolic computations with Macaulay2.

**Fact 9.2.1.** *The vector space  $V$  of ternary septic forms  $f_7(x, y, z)$  with  $\text{mult}_{p_i} f_7 \geq 2$  for all  $i = 1, \dots, 10$  is 6-dimensional.*

p2s\_septics

**Fact 9.2.2.** *For any  $i = 1, \dots, 10$ , we have  $h^0(\mathbb{P}^2, \mathcal{I}_{\Sigma-p_i}(3)) = 1$  and  $h^0(\mathbb{P}^2, \mathcal{I}_{\Sigma+p_i}(4)) = 3$ . Let  $\Lambda_i \subset \mathbb{P}(V) = \mathbb{P}^5$  be the projectivized image of  $H^0(\mathbb{P}^2, \mathcal{I}_{\Sigma-p_i}(3)) \otimes H^0(\mathbb{P}^2, \mathcal{I}_{\Sigma+p_i}(4))$  under the multiplication map. Then the ten planes  $\Lambda_i \subset \mathbb{P}^5$  satisfy the following conditions:*

(a)  $\Lambda_i$  meets  $\Lambda_j$  at one point whenever  $i \neq j$ .

(b) The 3-vectors  $v_i \in \wedge^3 V$  associated to the planes  $\Lambda_i$  are linearly independent and span a Lagrangian subspace  $A$  of  $\wedge^3 V$ .

fixed\_quadric

**Fact 9.2.3.** *The ten planes  $\Lambda_i$  all belong to one ruling of planes on a non-singular quadric in  $\mathbb{P}^5$ . The EPW sextic  $Y_A$  is nothing else but this quadric with multiplicity three.*

The configuration of planes  $(\Lambda_1, \dots, \Lambda_{10})$  constructed in [9.2.2](#) lives in  $|7e_0 - 2\Sigma|$ , whilst the Reye model  $S_9$  is in  $|7e_0 - 2E|^*$ . Thus we need the dual configuration of planes, which one obtains by applying annihilators:  $\Lambda_i^\perp \subset |7e_0 - 2E|^*$ .

Geometrically, the planes  $\Lambda_i^\perp$  can be constructed as follows. Let  $\phi: \tilde{\mathbb{P}}^2 \rightarrow \mathbb{P}^5$  be the embedding of the Coble surface by the linear system  $|7e_0 - 2E|$ , so that  $\mathbb{P}^5$  is naturally identified with  $|7e_0 - 2E|^*$ . As  $(7e_0 - 2\sum_j e_j) \cdot (3e_0 - \sum_{j \neq i} e_i) = 3$ , the images of the ten elliptic cubics  $q_i \in |3e_0 - \sum_{j \neq i} e_i|$  under  $\phi$  are of degree 3. Hence they are plane curves, and  $\Lambda_i^\perp = \langle \phi(q_i) \rangle$ .

The following two facts follow from the construction and do not need a computer verification.

**Fact 9.2.4.** *The planes  $\Lambda_i^\perp \subset |7e_0 - 2E|^*$  form a configuration of the same type as the original planes  $\Lambda_i \subset |7e_0 - 2E|$ . In particular:*

(a)  $\Lambda_i^\perp$  meets  $\Lambda_j^\perp$  at one point whenever  $i \neq j$ .

(b) The 3-vectors  $\omega_i^* \in \wedge^3 V^\vee$  associated to the planes  $\Lambda_i^\perp$  are linearly independent and span a Lagrangian subspace  $A^\perp$  of  $\wedge^3 V^\vee$ .

fixed\_quadric2

**Fact 9.2.5.** *The ten planes  $\Lambda_i^\perp$  all belong to one ruling of planes on a non-singular quadric. The EPW sextic  $X_{A^\perp}$  is nothing else but this quadric with multiplicity three.*

**Fact 9.2.6.** *The vector space of cubics containing the 10 planes  $\Lambda_i^\perp$  is 6-dimensional and consists of linear forms times the fixed quadric defined in Fact [9.2.5](#).*

Now we turn to the Cayley model  $S_{10}$  of  $\tilde{\mathbb{P}}^2$ . We start with a configuration of planes  $\Lambda'_i \subset |10e_0 - 3E|$ ,  $i = 1, \dots, 10$ , and the planes discussed in Section [5.2](#) are defined as their annihilators:  $\Lambda'_i{}^\perp \subset |10e_0 - 3E|^*$ .

basis\_decimics

**Fact 9.2.7.** *The vector space  $V'$  of ternary decimic forms  $f_{10}(x, y, z)$  with  $\text{mult}_{p_i} f_{10} \geq 3$  for all  $i = 1, \dots, 10$  is 6-dimensional.*

**Fact 9.2.8.** *The statement of Fact [9.2.2](#) holds upon replacement of  $\mathcal{I}_{\Sigma+p_i}(4)$  by  $\mathcal{I}_{2\Sigma+p_i}(7)$ . In the sequel, we will denote by  $\Lambda'_i$  the projectivized image of  $H^0(\mathbb{P}^2, \mathcal{I}_{\Sigma-p_i}(3)) \otimes H^0(\mathbb{P}^2, \mathcal{I}_{2\Sigma+p_i}(7))$  in  $\mathbb{P}(V')$  and by  $A'$  the associated 10-dimensional Lagrangian subspace in  $\wedge^3 V'$ .*

cubic\_10

**Fact 9.2.9.** *The ten planes  $\Lambda'_i$  lie on a unique cubic hypersurface in  $\mathbb{P}^5$ . The EPW sextic  $X_{A'}$  is nothing else but this cubic with multiplicity two.*

**Fact 9.2.10.** *The cubic in Fact [9.2.9](#) can be identified, via a linear change of variables, with the symmetric determinantal cubic  $\mathcal{C}$ , which is the secant variety of the Veronese surface  $V_4 \subset \mathbb{P}^5$ . Under this identification, the planes  $\Lambda'_i$  meet  $V_4$  along conics that are images of lines in  $\mathbb{P}^2$  via the Veronese map  $\mathbb{P}^2 \rightarrow V_4$ . Thus the 9 points of intersection of  $\Lambda'_i$  with the remaining planes  $\Lambda'_k$  lie on the conic  $\Lambda'_i \cap V_4$ , and the number of linearly independent cubics in  $\Lambda'_i$  passing through these 9 points is equal to 3. This implies, in particular, that the 45 intersection points of the ten planes  $\mathbb{P}_i^{2'}$  fail to impose independent conditions on the cubics in  $\mathbb{P}^5$  passing through them.*

The next two properties are stated for the dual configuration  $(\Lambda_1, \dots, \Lambda_{10})$  in  $|10e_0 - 3E|^*$ .

**Fact 9.2.11.** *The planes  $\Lambda'_i$  are contained in the determinantal cubic  $\mathcal{C}$  of the Veronese surface  $V_4$  and their annihilators  $\Lambda_i = \Lambda'_i{}^\perp$  are contained in the dual cubic  $\mathcal{C}^*$ , which is the chordal variety of the dual Veronese surface  $V_4^*$ . Moreover, as  $\Lambda'_i$  meet  $V_4$  in conics that are images of lines  $\ell_i$  in  $\mathbb{P}^2$ , the  $\Lambda_i$  are tangent to  $V_4^*$  at the points  $\ell_i^\perp \in \mathbb{P}^{2*}$ . The 3-vectors  $v_i^* \in \wedge^3 V'^*$  associated to the planes  $\Lambda_i$  are linearly independent and span the Lagrangian subspace  $A'^\perp$  of  $\wedge^3 V'^*$ , and the EPW sextic  $X_{A'^\perp}$  is the cubic  $\mathcal{C}^*$  squared.*

11-independent

**Fact 9.2.12.** *The 45 intersection points  $\Lambda_i \cap \Lambda_j$  ( $i \neq j$ ) impose independent linear conditions on cubics, so that there is an 11-dimensional linear system of cubics in  $\mathbb{P}^5$  through these points. The analogues of Facts [9.1.3](#) and [9.1.4](#) hold with the Enriques surface  $S$  replaced by the Cayley model  $S_{10}$  of the Coble surface considered in this section.*

deg53

**9.3. Ten planes in three hyperplanes.** Our objective in this section is to construct some configuration of 10 planes in  $\mathbb{P}^5$ , such that any two planes intersect and the associated 3-vectors are linearly independent, out of some

set of data, easier than a Fano model of an Enriques surface. We construct an explicit configuration of ten planes “of type 3-3-3-1”; as follows from [9.3.5](#), this can be seen as the case of a degenerate Enriques surface which decomposes into three quadrics and 4 planes.

We use the construction and the notation from Example [6.15](#). We will fix a basis  $(e_l)$  in the vector space  $V = k^6$  defining  $\mathbb{P}^5 = \mathbb{P}(V)$  so that  $\Lambda = \langle e_0, e_1, e_2 \rangle$  and  $\Upsilon_{ij} = \langle e_0, e_1, e_2, e_{3+i}, e_{3+j} \rangle$ . We have  $\Pi_i = \langle e_0, e_1, e_2, e_{3+i} \rangle$ .

The construction goes as follows.

(a) Choose in each 4-point intersection  $C_i \cap C_j$  a subset  $E_m^{ij}$ ,  $m = 1, 2, 3$  of three of the intersection points. This gives us the set of nine points in  $\Lambda$  which will turn out to be the points of intersection of  $\Lambda_{10}$  with the other nine planes  $\Lambda_i$  of our configuration.

(b) Choose three lines  $\ell_m^{ij}$  from one ruling of the quadric  $Q_i$  passing through the three points from  $E_m^{ij}$ . This is done in such a way that the lines  $\ell_m^{ij}$  and  $\ell_m^{ik}$  belong to different rulings, so that they meet at a point  $p_m^i \in Q_i$ .

(c) Get nine planes  $\Lambda_{km} = \langle p_m^i, p_m^j, q_m^i \rangle$ .

The configuration  $\underline{\Lambda}$  that consists of nine planes  $\Lambda_{km}$ ,  $k, m = 1, 2, 3$  and the plane  $\Lambda$  of ten planes possesses the desired properties. Indeed, any  $\Lambda_{km}$  meets  $\Lambda_{jm}$  at  $p_m^i$  and meets  $\Lambda$  at  $q_m^k$ . Two planes  $\Lambda_m^k, \Lambda_{m'}^k$  meet because they are in the same  $\mathbb{P}^4 = \Upsilon_{ij}$ , where  $\{i, j, k\} = \{1, 2, 3\}$ .

To construct such a configuration, one has to solve some algebraic equations. So, if the conics  $C_i$  are defined over  $\mathbb{k}$ , then the points  $q_m^k$  may be defined over an extension of degree 2, 3 or 6 of  $\mathbb{k}$ , and the search of the lines on the quadrics in (b) may again force us to introduce a quadratic extension of the field. To reduce the volume of computation, we searched for the initial data with the conics  $C_i$  and the points  $q_m^k$  both defined over a smallest possible field  $\mathbb{F}_p$ . The smallest value of  $p$  for which we detected such solutions is  $p = 29$ . Here is one of them:

$$\begin{aligned} C_0 &= x_0^2 - 7x_0x_2 - 12x_1x_2 \\ C_1 &= -4x_0x_1 + 9x_1^2 - 5x_0x_2 - 10x_1x_2 \\ C_2 &= 6x_0x_1 - 14x_0x_2 + 10x_1x_2 + x_2^2 \\ C_1 \cap C_2 &= \{(1, 0, 0), (1, 10, -7), (1, 11, -1), (1, 5, 9)\} \\ C_2 \cap C_0 &= \{(0, 1, 0), (1, 5, 13), (1, 10, 8), (1, -1, -6)\} \\ C_0 \cap C_1 &= \{(0, 0, 1), (1, 6, -11), (1, -11, -13), (1, -4, 12)\} \end{aligned}$$

The computations reported below have been done for these three conics and  $E_k^{ij} = C_i \cap C_j \setminus [e_k]$ . The resulting planes  $\Lambda_{km}$  are defined over the quadratic extension  $\mathbb{F}_{p^2}$  of  $\mathbb{F}_p$ .

**Fact 9.3.1.** *The 3-vectors  $v_1, \dots, v_{10} \in \wedge^3 V$  representing the 10 planes from  $\underline{\Lambda}$  are linearly independent. Hence they generate a Lagrangian 10-dimensional subspace  $A$  of  $\wedge^3 V$ .*

**Fact 9.3.2.** *There is a unique, up to proportionality, cubic  $\mathcal{C}(\underline{\Lambda})$  vanishing on the ten planes, and this cubic is the product of linear forms  $x_3x_4x_5$ .*

**Fact 9.3.3.** *The ideal of  $S^\bullet V^\vee$  defining the 45 intersection points of the 10 planes  $\underline{\Lambda}$  is generated by forms of degrees  $\geq 3$  and contains 11 linearly independent forms of degree 3. These forms, denoted in the sequel by  $F_0, \dots, F_{10}$ , can be chosen in such a way that  $F_0 = x_3 x_4 x_5$ .*

**Fact 9.3.4.** *The scheme-theoretic zero locus of the 11 cubic forms  $F_0, \dots, F_{10}$  is the union of 10 cubic curves  $B_j = B_{km} \subset \Lambda_{km}$ .*

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**Fact 9.3.5.** *The scheme-theoretic base locus of a generic linear subsystem  $\mathbb{P}^9$  of cubics in the linear system  $\mathbb{P}^{10} = \langle F_0, \dots, F_{10} \rangle$  is  $\bigcup_{i=1}^{10} B_i$ . There are subsystems  $\mathbb{P}^9$  of cubics with 2-dimensional base locus. There is exactly one such  $\mathbb{P}^9$  not containing the cubic  $V(F_0)$ . The base locus of this  $\mathbb{P}^9$  is a “degenerate” Enriques surface  $S$ . It is reducible but its Hilbert polynomial is that of the Fano model of an Enriques surface. The irreducible components of  $S$  are four planes, one of them is  $\Lambda$ , and the three nonsingular quadric surfaces  $Q_1, Q_2, Q_3$ . The remaining three components of  $S$  are planes  $H_1, H_2, H_3$  which are not among the ten planes from  $\underline{\Lambda}$ . The plane  $H_i$  meets  $\Lambda$  at one point  $[e_i] \in C_j \cap C_k$ , meets each of the quadrics  $Q_j, Q_k$  along a line passing through  $[e_i]$ , and does not meet  $Q_i$ . The intersections of the  $H_i$  between them are of length one.*

**Fact 9.3.6.** *Let  $A_{\underline{\Lambda}}$  be the Lagrangian subspace spanned by the 3-vectors  $v_1, \dots, v_{10} \in \wedge^3 V$  representing the 10 planes from  $\underline{\Lambda}$ . Then the EPW sextic  $X_{A_{\underline{\Lambda}}}$  is irreducible and is not a linear combination of the products  $F_a F_b$  ( $0 \leq a \leq b \leq 10$ ).*

**Fact 9.3.7.** *Let  $R$  be the singular locus of  $X_{A_{\underline{\Lambda}}}$ . Then the Hilbert polynomial of  $R$  is  $53P_2 - 198P_1 + 322P_0$ , in particular, it is a surface of degree 53 (possibly with other lower-dimensional components and/or embedded points). The following are some of the irreducible components of  $R$ :*

- (1) *The 10 planes  $\underline{\Lambda}$  with multiplicity 1.*
- (2) *The three quadric surfaces  $Q_i$  with multiplicity 1.*
- (3) *An irreducible surface  $S$  of degree 16 not contained in any of coordinate hyperplanes with Hilbert polynomial  $16P_2 - 24P_1 - 9P_0$ . The ideal of  $S$  is generated by 15 quartics.*
- (4) *Each hyperplane  $x_i = 0$  ( $i = 3, 4, 5$ ) contains two-dimensional irreducible components of  $R$  whose degrees sum up to 15. The components enumerated above provide the degree only up to 8 (two quadrics and 4 planes in each hyperplane  $x_i = 0$ ), so there are other components of  $R$  in  $x_i = 0$  with the sum of degrees equal to 7.*

## REFERENCES

ArtinMumford

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1. M. Artin and D. Mumford, *Some elementary examples of unirational varieties which are not rational*. Proc. London Math. Soc. (3) **25** (1972), 75–95.
2. W. Barth and C. Peters, *Automorphisms of Enriques surfaces*. Invent. Math. **73** (1983), 383–411.
3. A. Beauville, *Variétés Kähleriennes dont la première classe de Chern est nulle*, J. Differential Geom. **18** (1983), 755–782.

- Beauville 4. A. Beauville, *Fano threefolds and K3 surfaces*. The Fano Conference, 175–184, Univ. di Torino, Torino, 2004.
- BD 5. A. Beauville and R. Donagi, *La variété des droites d'une hypersurface cubique de dimension 4*, C. R. Acad. Sci. Paris Ser. I Math. **301** (1985), 703–706.
- Camere 6. C. Camere, *Lattice polarized irreducible holomorphic symplectic manifolds*, Ann. Inst. Fourier (Grenoble) **66** (2016), 687–709.
- Cantat 7. S. Cantat, I. Dolgachev, *Rational surfaces with a large group of automorphisms*, J. Amer. Math. Soc. **25** (2012), 863–905.
- Ciliberto 8. C. Ciliberto, C. Galati, A. Knutsen, *Irreducible unirational and uniruled components of moduli spaces of polarized Enriques surface*, arXiv:1809.10569. math.AG.
- CobleBook 9. A. Coble, *Algebraic geometry and theta functions*. Reprint of the 1929 edition. American Mathematical Society Colloquium Publications, 10. American Mathematical Society, Providence, R.I., 1982.
- ConteVerra 10. A. Conte, A. Verra, *Reye constructions for nodal Enriques surfaces*. Trans. Amer. Math. Soc. **336** (1993), 79–100.
- CR 11. F. Cossec, *Reye congruences*. Trans. Amer. Math. Soc. **280** (1983), 737–75.
- CossecDolgachev 12. F. Cossec, I. Dolgachev, *Rational curves on Enriques surfaces*, Math. Ann. **272** (1985), 369–384.
- CD 13. F. Cossec and I. Dolgachev, *Enriques surfaces. I*. Progress in Mathematics, 76. Birkhäuser Boston, Inc., Boston, MA, 1989.
- DR 14. I. Dolgachev and I. Reider, *On rank 2 vector bundles with  $c_1^2 = 10$  and  $c_2 = 3$  on Enriques surfaces*. Algebraic geometry (Chicago, IL, 1989), 39–49, Lecture Notes in Math., 1479, Springer, Berlin, 1991.
- DolgachevMirrors 15. I. Dolgachev, *Mirror symmetry for lattice polarized K3 surfaces*. Algebraic geometry, J. Math. Sci. **81** (1996), 2599–2630.
- DZ 16. I. Dolgachev and De-Qi Zhang, *Coble rational surfaces*. Amer. J. Math. **123** (2001), 79–114.
- DolgachevBAMS 17. I. Dolgachev, *Reflection groups in algebraic geometry*. Bull. Amer. Math. Soc. (N.S.) **45** (2008), 1–60.
- DM-OW 18. I. Dolgachev, D. Markushevich, *Mutually intersecting planes in  $\mathbb{P}^5$ , Enriques surfaces, cubic fourfolds, and EPW-sextics*, Classical algebraic geometry (June 20–26, 2010), Oberwolfach Rep. **7** (2010), 1603–1604.
- CAG 19. I. Dolgachev, *Classical algebraic geometry. A modern view*. Cambridge University Press, Cambridge, 2012.
- DolgachevKondo2 20. I. Dolgachev, S. Kondo, *Rationality of moduli spaces of Coble surfaces and general nodal Enriques surfaces*, Izv. Russ. Akad. Nauk, Ser. Mat. **77** (2013), 77–92.
- DolgachevBrief 21. I. Dolgachev, *A brief introduction to Enriques surfaces*, Development in Moduli theory, Kyoto-2013, Advanced Studies in Pure Math., vol. 69, 2016.
- DolgachevSegre 22. I. Dolgachev, *Corrado Segre and nodal cubic threefolds*. From classical to modern algebraic geometry, 429–450, Trends Hist. Sci., Birkhäuser/Springer, Cham, 2016.
- DolgachevQuartics 23. I. Dolgachev, *Quartic surfaces with icosahedral symmetry*. Adv. Geom. **18** (2018), no. 1, 119–132.
- EPW 24. D. Eisenbud, S. Popescu, and C. Walter, *Lagrangian subbundles and codimension 3 subcanonical subschemes*. Duke Math. J. **107** (2001), 427–467.
- vanGeemen 25. M. Donten-Bury, B. van Geemen, G. Kapustka, M. Kapustka, J. Wi?niewski, *A very special EPW sextic and two IHS fourfolds*. Geom. Topol. **21** (2017), no. 2, 1179–1230.
- F3 26. A. Ferretti, *Special subvarieties of EPW sextics*, Math. Z. **272** (2012), 1137–1164.
- GLM 27. L. Giraldo, A. Lopez, and R. Muñoz, *On the projective normality of Enriques surfaces. With an appendix by Lopez and Alessandro Verra*, Math. Ann. **324** (2002), 135–158.
- M2 28. D. Grayson, and M. Stillman, *Macaulay2, a software system for research in algebraic geometry*, Available at <http://www.math.uiuc.edu/Macaulay2/>.
- GritsenkoHulek 29. V. Gritsenko, K. Hulek, *Moduli of polarized Enriques surfaces. K3 surfaces and their moduli*, 55–72, Progr. Math., 315, Birkhäuser/Springer, [Cham], 2016.

- Hassett** 30. B. Hassett, *Special cubic fourfolds*. *Compositio Math.* **120** (2000), no. 1, 1–23.
- HuybrechtsInv** 31. D. Huybrechts, *Compact hyper-Kähler manifolds: basic results*. *Invent. Math.* **135** (1999), no. 1, 63–113.
- Huybrechts** 32. D. Huybrechts, *Lectures on K3 surfaces*. Cambridge Studies in Advanced Mathematics, 158. Cambridge University Press, Cambridge, 2016.
- HodgePedoe** 33. W. Hodge, D. Pedoe, *Methods of algebraic geometry*. Vol. II. Reprint of the 1952 original. Cambridge University Press, Cambridge, 1994.
- IngallsKuznetsov** 34. C. Ingalls, A. Kuznetsov, *On nodal Enriques surfaces and quartic double solids*. *Math. Ann.* **361** (2015), 107–133.
- Jensen** 35. S. T. Jensen, *Picard schemes of quotients by finite commutative group schemes*, *Math. Scand.* **42** (1978), 197–210.
- KapustkaVerra** 36. G. Kapustka and A. Verra, *On Morin configurations of higher length*, arXiv:1903.07480 [math.AG].
- KeelMori** 37. S. Keel, S. Mori, *Quotients by groupoids*. *Ann. of Math. (2)* **145** (1997), no. 1, 193–213.
- KL** 38. F. Klein, *Vorlesungen Über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade*, Leipzig, Teubner, 1884.
- KondoRationality** 39. S. Kondō, *The rationality of the moduli space of Enriques surfaces*. *Compositio Math.* **91** (1994), no. 2, 159–173.
- Laza** 40. R. Laza, *The moduli space of cubic fourfolds via the period map*. *Ann. of Math. (2)* **172** (2010), no. 1, 673–711.
- Laza1** 41. R. Laza, *Maximally Algebraic potentially irrational Cubic Fourfolds*, math.AG. arXiv:1805.04063.
- Lee** 42. W. Lee, E. Park, P. Schenzel, *On the classification of non-normal cubic hypersurfaces*. *J. Pure Appl. Algebra* **215** (2011), no. 8, 2034–2042.
- MatsusakaMumford** 43. T. Matsusaka, D. Mumford, *Two fundamental theorems on deformations of polarized varieties*. *Amer. J. Math.* **86** (1964), 668–684.
- Morin1** 44. U. Morin, *Sui sistemi di piani a due a due incidenti*, *Ist. Veneto Sci Lett. Arti, Atti II* **89** (1930), 907–926.
- Morin2** 45. U. Morin, *Sur sistemi di  $S_k$  a due a due incidenti e sulla generalizzazione proiettiva di alcune varietà algebriche*, *Ist. Veneto Sci Lett. Arti, Atti II* **101** (1942), 183–196.
- Namikawa** 46. Y. Namikawa, *Periods of Enriques surfaces*. *Math. Ann.* **270** (1985), 201–222.
- Nam** 47. Y. Namikawa, *Deformation theory of singular symplectic  $n$ -folds*, *Math. Ann.* **319** (2001), 597–623.
- Noohi** 48. B. Noohi, *Fundamental groups of algebraic stacks*. *J. Inst. Math. Jussieu* **3** (2004), no. 1, 69–103.
- Nikulin** 49. V. Nikulin, *Integer symmetric bilinear forms and some of their geometric applications*. *Izv. Akad. Nauk SSSR Ser. Mat.* **43** (1979), 111–177.
- OG1** 50. K. O’Grady, *Irreducible symplectic fourfolds and Eisenbud-Popescu-Walter sextics*. *Duke Math. J.* **134** (2006), 99–137.
- OG2** 51. K. O’Grady, *Irreducible symplectic fourfolds numerically equivalent to  $(K3)^{[2]}$* . *Commun. Contemp. Math.* **10** (2008), no. 4, 553–608.
- OG4** 52. K. O’Grady, *EPW-sextics: taxonomy*, *Manuscripta Math.* **138** (2012), 221–272.
- OG6** 53. K. O’Grady, *Double covers of EPW-sextic*, *Michigan Math. J.* **62** (2013), no. 1, 143–184.
- OGPeriods** 54. K. O’Grady, *Periods of double EPW-sextics*. *Math. Z.* **280** (2015), no. 1–2, 485–524.
- OG5** 55. K. O’Grady, *Pairwise incident planes and hyperkähler four-folds*. A celebration of algebraic geometry, 553–566, *Clay Math. Proc.*, 18, Amer. Math. Soc., Providence, RI, 2013.
- OGModuli** 56. K. O’Grady, *Moduli of double EPW-sextics*. *Mem. Amer. Math. Soc.* **240** (2016), no. 1136,
- Salmon** 57. G. Salmon, *A treatise on the analytic geometry of three dimensions*. Revised by R. A. P. Rogers. 7th ed. Vol. 1 Edited by C. H. Rowe. Chelsea Publ. Company, New York 1958.



- Verra** 58. A. Verra, *A short proof of the unirationality of  $\mathcal{A}_5$* . Nederl. Akad. Wetensch. Indag. Math. **46** (1984), no. 3, 339–355.
- Viehweg** 59. E. Viehweg, *Quasi-projective moduli for polarized manifolds*. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 30. Springer-Verlag, Berlin, 1995.
- Voisin** 60. C. Voisin, *Théorème de Torelli pour les cubiques de  $P^5$* . Invent. Math. **86** (1986), 577–601.
- Wall** 61. C. T. C. Wall, *On the orthogonal groups of unimodular quadratic forms*. Math. Ann. **147** (1962), 328–338.
- Weyman** 62. J. Weyman, *Cohomology of vector bundles and syzygies*. Cambridge Tracts in Mathematics, 149. Cambridge University Press, Cambridge, 2003.
- Win** 63. R. Winger, *Self-projective rational sextics*, Amer. J. Math. **38** (1916), 45–56.
- YY** 64. Stephen. T. Yau, Y. Yu, *Gorenstein quotient singularity in dimension three*, Mem. Amer. Math. Soc. 105 (1993), no. 505.
- Zucker** 65. S. Zucker, *The Hodge conjecture for cubic fourfolds*. Compositio Math. **34** (1977), no. 2, 199–209.

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