

ON ISOGENOUS PRINCIPALLY POLARIZED ABELIAN SURFACES

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ABSTRACT. We study a relationship between two genus 2 curves whose jacobians are isogenous with kernel equal to a maximal isotropic subspace of p -torsion points with respect to the Weil pairing. For $p = 3$ we find an explicit relationship between the set of Weierstrass points of the two curves extending the classical results of F. Richelot (1837) and G. Humbert (1901) in the case $p = 2$.

1. INTRODUCTION

Let $A = \mathbb{C}^g / \mathbb{Z}^g + \tau \mathbb{Z}^g$ be a complex algebraic torus with the period matrix τ in the Siegel space Z_g . Replacing τ by $p\tau$ for some integer p defines an isogenous torus with the kernel of isogeny equal to $F = (\mathbb{Z}/p\mathbb{Z})^g$. In the case of elliptic curves this construction leads to the theory of modular equations and, for $p = 2$, to the Gauss algebraic-geometrical mean. The construction can be made independent of a choice of τ and, in fact, can be defined for any ordinary abelian principally polarized variety A over an algebraically closed field \mathbb{K} of arbitrary characteristic. We take a maximal isotropic subspace K in the group $A[p]$ of p -torsion points with respect to the Weil pairing defined by the principal polarization. Then the quotient $B = A/F$ is a principally polarized abelian variety. When $\mathbb{K} = \mathbb{C}$, the isomorphism class of B can be defined by the period matrix $p\tau$, where τ is a period matrix of A .

In the case $A = \text{Jac}(C)$ is the jacobian variety of a smooth algebraic curve of genus g over \mathbb{K} one may ask whether $B = \text{Jac}(C')$ for some other curve C' of genus g , and if so, what is the precise relationship between the moduli of C and C' . For $g = 1$ the answer is given by exhibiting an explicit modular equation relating the absolute invariants of the two elliptic curves (see a survey of classical and modern results in [BB]). In the case $p = 2$ this was done by Gauss in his work about the algebraic-geometrical mean (see [Cox]). For $g = 2$ and $p = 2$, the explicit geometric moduli relationship between the two curves of genus 2 was found by Richelot [Ric] and Humbert [Hum] (see a modern account in [BM]). It was extended to the case $g = 3$ by Donagi-Livné [DL] and Lehavi-Ritzenthaler [LR].

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In the present paper we study the case $g = 2$ and $p > 2$ and assume that the ground field \mathbb{K} is an algebraically closed field of characteristic $\neq 2$. Our main result is the following

Theorem 1.1. *Let C be a smooth genus 2 curve. Let p be an integer coprime to the characteristic of \mathbb{K} and F be a maximal isotropic subgroup of $\text{Jac}(C)[p]$ with respect to the Weil pairing defined by the natural principal polarization of the jacobian. For any $e, -e \in F \setminus \{0\}$ let $\{x_e, y_e\}$ be a unique, up to the hyperelliptic involution ι , pair of points on C such that $x_e - y_e = \pm e$. Let $\phi : C \rightarrow R_{2p} \subset \mathbb{P}^{2p}$ be the degree two map onto a rational norm curve given by the linear system $|2pK_C|^t$. Let (c_e, d_e) be the images of the pairs (x_e, y_e) in R_{2p} and $\ell_e = \overline{c_e, d_e}$ be the corresponding secant lines of R_{2p} . There exists a unique hyperplane \mathcal{H} in \mathbb{P}^{2p} containing the images w_1, \dots, w_6 of the six Weierstrass points such that the intersection points of \mathcal{H} with the secants ℓ_e are contained in a subspace L of \mathcal{H} of codimension 3. The images of the points w_i under a projection from L to \mathbb{P}^3 are contained on a conic (maybe reducible), and the double cover of the conic ramified at these points is a stable curve C' of arithmetic genus 2 such that $J(C') \cong \text{Jac}(C)/F$.*

In the case $\mathbb{K} = \mathbb{C}$ and $p = 3$ we give an effective algorithm for determining the curve C' in terms of C . The assertion of the theorem is also valid in the case $p = \text{char } \mathbb{K}$ if we assume that $F = \text{Jac}(C)[p] \cong (\mathbb{Z}/p\mathbb{Z})^2$, i.e. $\text{Jac}(C)$ is an ordinary abelian variety. In this case the algorithm for finding C' can be made even more effective.

Note that G. Humbert also considered the case $g = 2, p = 3$ in [Hum] but his solution is different from ours and cannot be made effective (see Remark 3.2).

2. PRELIMINARIES

2.1. Polarized abelian varieties. Let A be a g -dimensional abelian variety over an algebraically closed field \mathbb{K} . Let \mathcal{L} be an invertible sheaf on A and $\pi : \mathbb{V}(\mathcal{L}) \rightarrow A$ be the corresponding line bundle, the total space of \mathcal{L} . One defines the *theta group scheme* $G(\mathcal{L})$ whose S -points are lifts of translation automorphisms $t_a, a \in A(S)$, of $A_S = A \times_{\mathbb{K}} S$ to automorphisms of $\mathbb{V}(\mathcal{L})_S$. It fits in the canonical central extension of group schemes

$$1 \rightarrow \mathbb{G}_m \rightarrow G(\mathcal{L}) \rightarrow K(\mathcal{L}) \rightarrow 1,$$

where $K(\mathcal{L})(S)$ is the subgroup of $A(S)$ of translations which send \mathcal{L}_S to isomorphic invertible sheaf on A_S . The extension is determined by the *Weil pairing*

$$e^{\mathcal{L}} : K(\mathcal{L}) \times K(\mathcal{L}) \rightarrow \mathbb{G}_m$$

defined by the commutator in $G(\mathcal{L})$. A subgroup K of $K(\mathcal{L})$ is isotropic with respect to the Weil pairing if and only if the extension splits over K .

From now we assume that \mathcal{L} is ample. In this case $K(\mathcal{L})$ is a finite group scheme and the Weil pairing is non-degenerate. Recall that the algebraic equivalence class of an ample invertible sheaf on A is called a *polarization*

on A . An abelian variety equipped with a polarization is called a *polarized abelian variety*.

Let A^\vee be the dual abelian variety representing the connected component of the Picard scheme of A . Any invertible sheaf \mathcal{L} defines a homomorphism of abelian varieties

$$\phi_{\mathcal{L}} : A \rightarrow A^\vee, a \mapsto t_a^*(\mathcal{L}) \otimes \mathcal{L}^{-1}.$$

It depends only on the algebraic equivalence class of \mathcal{L} and its kernel is isomorphic to the group $K(\mathcal{L})$. In particular, λ is an isogeny if and only if \mathcal{L} is ample. We say that \mathcal{L} defines a *principal polarization* if $\phi_{\mathcal{L}}$ is an isomorphism. This is also equivalent to that \mathcal{L} is ample and $h^0(\mathcal{L}) = 1$.

The proof of the following proposition can be found in [Mu2], §23.

Proposition 2.1. *Let $\lambda : A \rightarrow B$ be a separable isogeny of abelian varieties. There is a natural bijective correspondence between the following sets*

- *the set of isomorphism classes of invertible ample sheaves \mathcal{M} such that $\lambda^*\mathcal{M} \cong \mathcal{L}$;*
- *the set of homomorphisms $\ker(\lambda) \rightarrow G(\mathcal{L})$ lifting the inclusion $\ker(\lambda) \hookrightarrow A$.*

Under this correspondence $K(\mathcal{M}) = \ker(\lambda)^\perp / \ker(\lambda)$. In particular, \mathcal{M} defines a principal polarization on B if and only if $\ker(\lambda)$ is a maximal isotropic subgroup.

Assume that \mathcal{L} defines a principal polarization on A . Then $K(\mathcal{L}^n) = A[n] = \ker([n]_A)$, where $[n]_A$ is the multiplication map $x \mapsto nx$ in A . Applying the previous proposition to \mathcal{L}^n , we obtain

Corollary 2.2. *Assume $(n, \text{char } \mathbb{K}) = 1$. Let F be a maximal isotropic subgroup of $A[n]$ and $\lambda : A \rightarrow B = A/F$ be the quotient map. Then B admits a principal polarization \mathcal{M} such that $\lambda^*\mathcal{M} \cong \mathcal{L}^n$.*

2.2. Kummer varieties. Let A be a principally polarized abelian variety. Since $h^0(\mathcal{L}) = 1$, there exists a unique effective divisor Θ such that $\mathcal{L} \cong \mathcal{O}_A(\Theta)$. The divisor Θ is called a *theta divisor* associated to the polarization. It is defined only up to a translation. One can always choose a theta divisor satisfying $[-1]_A^*\Theta = \Theta$, a *symmetric theta divisor*. Two symmetric theta divisors differ by a translation $t_a, a \in A[2]$.

The proof of the following result over $\mathbb{K} = \mathbb{C}$ can be found in [LB], Chapter IV, §8 and in [Du] in the general case.

Proposition 2.3. *Let A be a principally polarized abelian variety and Θ be a symmetric theta divisor. Then the map $\phi_{2\Theta} : A \rightarrow |2\Theta|^*$ factors through the projection $\phi : A \rightarrow A/\langle [-1]_A \rangle$ and a morphism $j : A/\langle [-1]_A \rangle \hookrightarrow |2\Theta| \cong \mathbb{P}^{2g-1}$. If A is not the product of principally polarized varieties of smaller dimension and $\text{char } \mathbb{K} \neq 2$, then j is a closed embedding.*

We assume that $\text{char } \mathbb{K} \neq 2$. The quotient variety $A/\langle [-1]_A \rangle$ is denoted by $\text{Km}(A)$ and is called the *Kummer variety* of A . In the projective embedding

$\text{Km}(A) \hookrightarrow \mathbb{P}^{2g-1}$ its degree is equal to $2^{g-1}g!$. The image of any $e \in A[2]$ in $\text{Km}(A)$ is a singular point P_e , locally (formally) isomorphic to the affine cone over the second Veronese variety of \mathbb{P}^{g-1} . For any $e \in A[2]$, the image of $\Theta_e := t_e^*(\Theta)$ in $\text{Km}(A) \subset \mathbb{P}^{2g-1}$ is a subvariety T_e cut out by a hyperplane $2\Theta_e$ with multiplicity 2. It is called a *trope*. Since each Θ_e is symmetric, the corresponding trope T_e is isomorphic to the quotient $\Theta_e/\langle[-1]_A\rangle$.

The configuration of the singular points P_e and the tropes T_e form an abstract symmetric configuration $(2^{2g}, 2^{g-1}(2^g - 1))$. This means that each trope contains $2^{g-1}(2^g - 1)$ singular points and each singular point is contained in the same number of tropes.

The Kummer variety $\text{Km}(A)$ admits a resolution of singularities

$$\pi : \mathcal{K}(A) \rightarrow \text{Km}(A)$$

with the exceptional locus equal to the union of $E_e = \pi^{-1}(P_e), e \in A[2]$. Each E_e is isomorphic to \mathbb{P}^{g-1} and the self-intersection E_e^g is equal to the degree of the Veronese variety $\nu_2(\mathbb{P}^{g-1})$ taken with the sign $(-1)^{g-1}$, that is, the number $(-2)^{g-1}$.

Let $p > 2$ be a prime number and K be a maximal isotropic subgroup in $A[p]$. If $p \neq \text{char } \mathbb{K}$, then $A[p] \cong \mathbb{F}_p^{2g}$ and the number of such K 's is equal to $\prod_{i=1}^g (p^i + 1)$. If $p = \text{char } \mathbb{K}$, we assume that A is an ordinary abelian variety, i.e. $A[p]_{\text{red}} \cong \mathbb{F}_p^g$. In this case $F = A[p]_{\text{red}}$ is unique.

Proposition 2.4. *Let $\lambda : A \rightarrow B = A/F$ be the quotient isogeny defined by \mathcal{L}^p . There exists a symmetric theta divisor Θ on A and a symmetric theta divisor Θ' on B such that $\lambda^*\Theta' \in |p\Theta|$ and $\lambda(\Theta) \in |p^{g-1}\Theta'|$. Let D be the proper transform of $\lambda(\Theta)$ in $\mathcal{K}(B)$. Let m_e be the multiplicity of Θ at a 2-torsion point e . Then*

$$2D \in \left| p^{g-1}H - \sum_{e \in A[2]} m_e E_e \right|,$$

where H is the divisor class of the pre-image of a hyperplane in \mathbb{P}^{2g-1} under the composition map $\mathcal{K}(B) \xrightarrow{\sigma} \text{Km}(B) \xrightarrow{j} \mathbb{P}^{2g-1}$, where the first map is a resolution of singularities and the second map is induced by the map $\phi_{2\Theta}$.

Proof. As we observed earlier there exists an ample invertible sheaf \mathcal{M} on B defining a principal polarization such that $\lambda^*\mathcal{M} \cong \mathcal{L}^p$. Let Θ' be a theta divisor on B defined by \mathcal{M} . We have $\lambda^*\Theta' \in |p\Theta|$ and

$$\lambda^*(\lambda(\Theta)) = \sum_{e \in K} t_e^*(\Theta) \equiv p^g \Theta.$$

Since the canonical map of the Neron-Severi groups $\lambda^* : \text{NS}(B) \rightarrow \text{NS}(A)$ is injective, we obtain that $p^{g-1}\Theta'$ and $\lambda(\Theta)$ are algebraically equivalent divisors on B . Since they are both symmetric divisors, they differ by a translation with respect to a 2-torsion point e . Replacing Θ' by $t_e^*(\Theta)$ we obtain the linear equivalence of the divisors.

It remains to prove the second assertion. Let $\sigma : B' \rightarrow B$ be the blow-ups of 2-torsion points on B . We have a commutative diagram

$$\begin{array}{ccc} B' & \xrightarrow{\phi'} & \mathcal{K}(B) \\ \sigma \downarrow & & \downarrow \pi \\ B & \xrightarrow{\phi} & \mathrm{Km}(B). \end{array}$$

It is clear that λ defines a bijection $A[2] \rightarrow B[2]$. Since λ is a local isomorphism, for any $e \in A[2]$, the multiplicity m_e of Θ at e is equal to the multiplicity of $\lambda(\Theta)$ at $\bar{e} = \lambda(e)$. Thus $2\lambda(\Theta)$ belongs to the linear system $\left| p^{g-1}(2\Theta') - 2 \sum_{e \in A[2]} m_e \bar{e} \right|$ of divisors in $|2p^{g-1}\Theta'|$ passing through the 2-torsion points \bar{e} with multiplicities $2m_e$. Let D' be the proper transform of $\lambda(\Theta)$ in B' . Then $D' \sim \sigma^*(p^{g-1}\Theta' - \sum m_e \sigma^{-1}(\bar{e}))$. On the other hand, since ϕ' ramifies over each $E_{\bar{e}}$ with multiplicity 2, we have

$$2D' \sim \sigma^*(p^{g-1}2\Theta' - 2 \sum m_e \sigma^{-1}(\bar{e})) \sim \phi'^*(p^{g-1}\pi^*(H) - \sum_e m_e E_{\bar{e}}).$$

This shows that the proper transform of the image of $2\lambda(\Theta)$ in $\mathcal{K}(B)$ is linearly equivalent to $p^{g-1}\pi^*(H) - \sum_e m_e E_{\bar{e}}$. \square

Remark 2.5. It is known that a theta divisor on general principally polarized abelian variety has no singular points at 2-torsion points. Thus $m_e = 1$ for $2^{g-1}(2^g - 1)$ points and $m_e = 0$ at the remaining 2-torsion points. Also note that $\mathrm{Pic}(\mathcal{K}(B))$ has no 2-torsion, so there is only one half of the divisor class $p^{g-1}\pi^*(H) - \sum_e m_e E_{\bar{e}}$.

2.3. Theta level structure. The main reference here is [Mu1] (see also [Bo], [LB] where only the case $\mathbb{K} = \mathbb{C}$ is considered). Let A be an ordinary abelian variety of dimension g and $\mathcal{L} \cong \mathcal{O}_A(\Theta)$ be an ample invertible sheaf defining a symmetric principal polarization. The theta divisor Θ defines a function

$$q_\Theta : A[2] \rightarrow \mu_2, \quad x \mapsto (-1)^{\mathrm{mult}_x(\Theta) + \mathrm{mult}_0(\Theta)}.$$

This function is a quadratic form whose associated bilinear form is the Weil pairing. We call Θ *even* (resp. *odd*) if the quadratic form is even (resp. odd). Recall that the latter means that $\#q^{-1}(0) = 2^{g-1}(2^g + 1)$ (resp. $\#q^{-1}(1) = 2^{g-1}(2^g - 1)$). One can show that Θ is even if and only if $\mathrm{mult}_0(\Theta)$ is even. Also, if we normalize the isomorphism $\mathcal{L} \rightarrow [-1]_A^* \mathcal{L}$ to assume that is equal to the identity on the fibres over the zero point, then Θ is even if and only if $[-1]_A^*$ acts as the identity on $\Gamma(\mathcal{L})$.

Let $G(\mathcal{L}^n)$ be the theta group of \mathcal{L}^n . A *level n theta structure* on A is a choice of an isomorphism of group schemes

$$\theta : G(\mathcal{L}^n) \rightarrow \mathcal{H}_g(n),$$

where $\mathcal{H}_d(n)$ is the *Heisenberg group scheme* defined by the exact sequence

$$1 \rightarrow \mathbb{G}_m \rightarrow \mathcal{H}_g(n) \rightarrow (\mathbb{Z}/n\mathbb{Z})^g \oplus \mu_n^g \rightarrow 1.$$

It is required that the restriction of θ to the center is the identity. If $(\lambda, a, b) \in \mathbb{K}^* \times (\mathbb{Z}/n\mathbb{Z})^g \times \mu_n^g$ represents a point of $\mathcal{H}_g(n)$, then the law of composition is

$$(\lambda, a, b) \cdot (\lambda', a', b') = (\lambda\lambda'b'(a), a + a', bb'),$$

where we identify μ_n^g with $\text{Hom}((\mathbb{Z}/n\mathbb{Z})^g, \mathbb{K}^*)$.

A theta level n structure defines an n -level on A , i.e. an isomorphism of symplectic group schemes

$$\bar{\theta} : (A[n], e^{\mathcal{L}^n}) \rightarrow ((\mathbb{Z}/n\mathbb{Z})^g \times \mu_n^g, E),$$

where

$$E : (\mathbb{Z}/n\mathbb{Z})^{2g} \times \mu_n^g \rightarrow \mathbb{G}_m$$

is the standard symplectic form $((a, b), (a', b')) \mapsto b'(a)/a'(b)$. In particular, $\bar{\theta}^{-1}((\mathbb{Z}/n\mathbb{Z})^g)$ is a maximal isotropic subgroup in $A[n]$.

The choice of a theta structure of level n defines a representation of the Heisenberg group $\mathcal{H}_g(n)$ on the linear space $V_n(g) = \Gamma(A, \mathcal{L}^n)$, called the *Schrödinger representation*. In this representation the space $V_n(g)$ admits a basis $\eta_\sigma, \sigma \in (\mathbb{Z}/n\mathbb{Z})^g$, such that $(\lambda, a, b) \in \mathcal{H}_g(n)$ acts by sending η_σ to $\lambda b(\sigma + a)\eta_{\sigma+a}$. We will explain how to build such a basis from theta functions when we discuss the case $\mathbb{K} = \mathbb{C}$.

If $n \geq 3$, the map $\phi : A \rightarrow \mathbb{P}(V_n(g)^*)$ given by the complete linear system $|\mathcal{L}^n|$ is a closed embedding and the Schrödinger representation defines a projective linear representation of the abelian group scheme $(\mathbb{Z}/n\mathbb{Z})^g \oplus \mu_n^g$ in $\mathbb{P}(V_n(g)^*) \cong \mathbb{P}^{g^n-1}$ such that the image of A is invariant, and the action on the image is the translation by n -torsion points.

Let \mathcal{L} be a symmetric principal polarization. The automorphism $[-1]_A$ of $A[n]$ can be extended in a canonical way to an automorphism δ_{-1} of $G(\mathcal{L}^n)$. A theta structure is called *symmetric* if, under the isomorphism $G(\mathcal{L}^n) \rightarrow \mathcal{H}_g(n)$, the automorphism δ_{-1} corresponds to the automorphism D_{-1} of $\mathcal{H}_g(n)$ defined by $(t, a, b) \mapsto (t, -a, b^{-1})$. This defines an action of D_{-1} in $V_n(g)$.

From now on we assume that $n > 1$ is odd.

Since D_{-1} is of order 2, the vector space $V_n(g)$ decomposes into the direct sum of two eigensubspaces $V_n(g)^+$ and $V_n(g)^-$ with eigenvalue 1 and -1 , respectively. If \mathcal{L} is defined by an even theta divisor Θ , then $D_{-1}(\eta_\sigma) = \eta_{-\sigma}$ and we can choose a basis $y_\sigma = \eta_\sigma + \eta_{-\sigma}, \sigma \in (\mathbb{Z}/n\mathbb{Z})^g$ in V_n^+ and the basis $z_\sigma = \eta_\sigma - \eta_{-\sigma}, \sigma \in (\mathbb{Z}/n\mathbb{Z})^g$ in V_n^- . In particular,

$$\dim V_n(g)^\pm = (n^g \pm 1)/2.$$

If \mathcal{L} is defined by an odd theta divisor Θ , then $D_{-1}(\eta_\sigma) = -\eta_{-\sigma}$ and we have

$$\dim V_n(g)^\pm = (n^g \mp 1)/2.$$

The two projectivized subspaces form the fixed loci of the projective involution D_{-1} . We will call the subspace of dimension $(n^g - 1)/2$ the *Burkhardt*

space and denote it by \mathbb{P}_{Bu} . The other subspace of dimension $(n^g - 3)/2$ we call the *Maschke subspace* and denote it by \mathbb{P}_{Ma} .

Two different theta structures of level n differ by an automorphism of $\mathcal{H}_g(n)$ which is the identity on \mathbb{K}^* . Let $A(\mathcal{H}_g(n))$ be the group of such automorphisms. Let $(\mathbb{Z}/n\mathbb{Z})^{2g} \rtimes \mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$ be the semi-direct product defined by the natural action of $\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$ on $(\mathbb{Z}/n\mathbb{Z})^{2g}$. There is a natural isomorphism

$$(\mathbb{Z}/n\mathbb{Z})^{2g} \rtimes \mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z}) \rightarrow A(\mathcal{H}_g(n))$$

defined by sending (e, σ) to $(t, u) \mapsto (t[e, u], \sigma(u))$. The group $A(\mathcal{H}_g(n))$ acts simply transitively on the set of theta structures of level n with fixed even symmetric theta divisor. However, if n is odd, the subgroup of $A(\mathcal{H}_g(n))$ preserving the set of symmetric structures consists of elements $(0, \sigma)$, hence isomorphic to $\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$. This shows that a symmetric theta structure (with fixed Θ) is determined uniquely by the level n structure.

Since the Schrödinger representation of $\mathcal{H}_g(n)$ is known to be irreducible, by Schur's Lemma, the group $A(\mathcal{H}_g(n))$ has a projective representation in $V_n(g)$. Under this representation, the normal subgroup $(\mathbb{Z}/n\mathbb{Z})^{2g} \cong \mathcal{H}_g(n)/\mathbb{K}^*$ acts via the projectivized Schrödinger representation. We will identify $\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$ with the subgroup of $A(\mathcal{H}_g(n))$ equal to the centralizer of D_{-1} . The Burkhardt and the Maschke subspaces are invariant with respect to the action of the group $\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$ in $\mathbb{P}(V_n(g))$ with the kernel equal to $\langle D_{-1} \rangle$ and hence define two projective representations of $\mathrm{PSp}(2g, \mathbb{Z}/n\mathbb{Z})$ of dimensions $(n^g - 1)/2$ and $(n^g - 3)/2$.

2.4. The theta map. We assume that $\mathrm{char} \mathbb{K} \neq 2$ and n is invertible in \mathbb{K} . Let $\mathcal{A}_g(n)^\pm$ be the moduli space of principally polarized abelian varieties of dimension g with a symmetric even (resp. odd) theta structure of odd level $n \geq 2$. It is known that it exists and is a fine moduli space. We denote by $\mathcal{X}_g(n)^\pm$ the corresponding universal family. There is a canonical forgetful morphism

$$f_\pm : \mathcal{A}_g(n)^\pm \rightarrow \mathcal{A}_g(n)$$

to the moduli space of principally polarized abelian varieties of dimension g with level n structure. The fibres are bijective to the set of even (resp. odd) theta divisors, hence the degree of the forgetful map is equal to $2^{g-1}(2^g \pm 1)$.

A theta structure defines a basis in $V_n(g) = \Gamma(A, \mathcal{L}^n)$ which is independent of A . This defines a $(\mathbb{Z}/n\mathbb{Z} \times \mu_n)^g \rtimes \mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$ -equivariant morphisms

$$(1) \quad \widetilde{\mathrm{Th}}^\pm : \mathcal{X}_g(n)^\pm \rightarrow \mathbb{P}^{n^g-1} = \mathbb{P}(V_n(g)^*),$$

where the group $(\mathbb{Z}/n\mathbb{Z} \times \mu_n)^g$ acts by translations on the image of each A . By composing (1) with the zero section $\mathcal{A}_g(n)^\pm \rightarrow \mathcal{X}_g(n)^\pm$ we get a $\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$ -equivariant morphism, the *even theta map* and the *odd theta map*

$$(2) \quad \mathrm{Th}^+ : \mathcal{A}_g(n)^+ \rightarrow \mathbb{P}_{Bu}, \quad \mathrm{Th}^- : \mathcal{A}_g(n)^- \rightarrow \mathbb{P}_{Ma}.$$

Here we use that the value at the origin of any section from the subspace $V_n(g)^-$ is equal to zero.

Recall that over \mathbb{C} the coarse moduli space \mathcal{A}_g of principally polarized abelian varieties is isomorphic to the orbit space \mathcal{Z}_g/Γ_g , where \mathcal{Z}_g is the Siegel moduli space of complex symmetric $g \times g$ -matrices $\tau = X + iY$ such that $Y > 0$, and $\Gamma_g = \mathrm{Sp}(2g, \mathbb{Z})$ acting on \mathcal{Z}_g in a well-known manner. The moduli space $\mathcal{A}_g(n)$ is isomorphic to $\mathcal{Z}_g/\Gamma_g(n)$, where $\Gamma_g(n) = \{M \in \Gamma_g : M - I_{2g} \equiv 0 \pmod{n}\}$. It is known that the index of $\Gamma_g(n)$ in Γ_g is equal to $n^{g^2} \prod_{i=1}^g (n^{2i} - 1)$, the order of the finite symplectic group $\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$. We have a canonical exact sequence

$$1 \rightarrow \Gamma_g(2n) \rightarrow \Gamma_g(n) \rightarrow \mathrm{Sp}(2g, \mathbb{F}_2) \rightarrow 1$$

defined by the natural inclusion of the groups $\Gamma_g(2n) \subset \Gamma_g(n)$. Comparing the indices with the order of $\mathrm{Sp}(2g, \mathbb{F}_2)$, we see that the last map is surjective. It is well-known that the group $\mathrm{Sp}(2g, \mathbb{F}_2)$ contains the even and the odd orthogonal subgroups $\mathrm{O}(2g, \mathbb{F}_2)^\pm$ of indices $2^{g-1}(2^g \pm 1)$. Let $\Gamma_g(n)^\pm$ be the pre-image in $\Gamma_g(n)$ of the subgroup $\mathrm{O}(2g, \mathbb{F}_2)^\pm$. Then

$$\mathcal{Z}_g/\Gamma_g(n)^\pm \cong \mathcal{A}_g(n)^\pm.$$

A choice of a symmetric theta structure is defined by a line bundle \mathcal{L} whose space of sections is generated by a Riemann theta function $\vartheta \left[\begin{smallmatrix} m \\ m' \end{smallmatrix} \right] (z; \tau)$ with theta characteristic $(m, m') \in (\mathbb{Z}/n\mathbb{Z})^g \times (\mathbb{Z}/n\mathbb{Z})^g$. The even (resp. odd) structure corresponds to the case when $m \cdot m' \equiv 0 \pmod{2}$ (resp. $\equiv 1 \pmod{2}$). A basis of the space $\Gamma(\mathcal{L}^n)$ is given by the functions $\vartheta \left[\begin{smallmatrix} m+\sigma \\ m' \end{smallmatrix} \right] (nz, n\tau)$, where $\sigma \in (\mathbb{Z}/n\mathbb{Z})^g$. It follows from the standard properties of the Riemann theta function that

$$(3) \quad \vartheta \left[\begin{smallmatrix} m+\sigma \\ m' \end{smallmatrix} \right] (-z; \tau) = (-1)^{m \cdot m'} \vartheta \left[\begin{smallmatrix} m-\sigma \\ m' \end{smallmatrix} \right] (z; \tau).$$

The theta map (2) is defined by the theta constants $x_\sigma = \vartheta \left[\begin{smallmatrix} m+\sigma \\ m' \end{smallmatrix} \right] (0, n\tau)$. They span the space of modular forms of weight $1/2$ with respect to the group $\Gamma(n)^\pm$ and some character $\chi : \Gamma(n)^\pm \rightarrow \mathbb{C}^*$. It follows from (3) that the functions $y_\sigma = x_\sigma + x_{-\sigma}$ (resp. $z_\sigma = x_\sigma - x_{-\sigma}$) are identical zero if (m, m') is odd (resp. even). This shows that the theta maps have the same target spaces as in (2).

Proposition 2.6. *Assume $\mathbb{K} = \mathbb{C}$. The even theta map*

$$\mathrm{Th}^+ : \mathcal{A}_g(n)^+ \rightarrow \mathbb{P}_{Bu}$$

is an embedding for $n = 3$.

The proof can be found in [SM], p. 235.

3. ABELIAN SURFACES

3.1. Kummer surfaces. Now we specialize to the case when A is a principally polarized abelian surface. It is known that A is not the product of two elliptic curves if and only if Θ is an irreducible divisor. In this case Θ

is a smooth curve of genus 2 and A is isomorphic to its Jacobian variety $\text{Jac}(\Theta)$. By adjunction formula $K_\Theta \cong \mathcal{O}_\Theta(\Theta)$ and the map $\phi_{2\Theta}$ restricts to the bicanonical map of Θ onto the corresponding trope of $\text{Km}(A)$. Let C be a genus 2 curve and $\text{Jac}^1(C)$ be its Picard scheme of degree 1. Fix a Weierstrass point w_0 to identify $\text{Jac}^1(C)$ with $\text{Jac}(C)$. Then one can take for Θ the translate of the divisor W of effective divisors of degree 1, naturally identified with C . Under this identification Θ contains the six 2-torsion points $w_i - w_0$, where $w_0 = w_1, w_2, \dots, w_6$ are the six Weierstrass points on C . None of them is a singular point of Θ .

Assume $A = \text{Jac}(C)$. In this case $\text{Km}(A)$ is isomorphic to a quartic surface in \mathbb{P}^3 . It has 16 nodes as singularities and its tropes are conics passing through 6 nodes. The surface $\mathcal{K}(A)$ is a K3 surface with 16 disjoint smooth rational curves $E_e, e \in A[2]$. The proper transform of a trope T is a smooth rational curve \bar{T} in the divisor class $\frac{1}{2}(H - \sum_{e \in T} E_e)$.

Assume A is the product of two elliptic curves $F \times F'$. In this case $\text{Km}(A)$ is the double cover of a nonsingular quadric $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ branched over the union B of eight lines, four in each family. Their preimages on $\text{Km}(A)$ is the union of two sets of four disjoint smooth rational curves. The tropes T on $\text{Km}(A)$ are the unions of a curve T_1 from one set and a curve T_2 from another set. Each component of a trope has four 2-torsion points, one point is common to both components. The proper transform of a trope on $\mathcal{K}(A)$ is the disjoint union of two smooth rational curves from the divisor class $\frac{1}{2}(H - \sum_{e \in T_1 + T_2} E_e - 2E_{T_1 \cap T_2})$.

3.2. Main result I. We employ the notations of Proposition 2.4.

Proposition 3.1. *Assume $A = \text{Jac}(C)$. Then $\lambda(\Theta)$ is an irreducible curve of arithmetic genus $p^2 + 1$ with $p^2 - 1$ ordinary double points. Its image D in $\text{Km}(B)$ is a rational curve of arithmetic genus $\frac{1}{2}(p^2 - 1)$ with $\frac{1}{2}(p^2 - 1)$ ordinary double points.*

Proof. We know that $\lambda(\Theta) \in |p\Theta'|$. Thus $\lambda(\Theta)^2 = 2p^2$ and the first assertion follows from the adjunction formula. Since the isogeny $A \rightarrow B$ is a local isomorphism in étale topology, the curve $\lambda(\Theta)$ has only ordinary multiple points corresponding to the intersection of an orbit of K with Θ . Let $\Gamma_a \subset A \times A$ be the graph of the translation map $t_a, a \in K$. It is algebraically equivalent to the diagonal Δ_A of $A \times A$. Let $C \times C \subset A \times A$ embedded via the Cartesian product of the Abel-Jacobi map. A point in the intersection $(C \times C) \cap \Gamma_a$ is a pair of points (x, y) on C such that $[x - y] = a$. By the intersection theory,

$$(C \times C) \cdot \Gamma_a = (C \times C) \cdot \Delta_A = \Delta_C^2 = 2.$$

Thus, for any nonzero $a \in K$, there exists two ordered pairs of points on C such that the difference is linearly equivalent to a . Since $x - y \sim x' - y'$ implies that $x + y' \sim x' + y \sim K_C$, the two pairs differ by the hyperelliptic involution. If we forget about the order we get p^2 unordered pairs of points in a coset of K . This shows that $\lambda(\Theta)$ has $p^2 - 1$ ordinary double points.

The last assertion follows from Proposition 2.4. □

Remark 3.2. Note that one can rewrite the formula from Proposition 2.4 in the form

$$(4) \quad \frac{1}{2}(pH - E_1 - \dots - E_6) = \frac{p+1}{2}H - \frac{1}{2}(H - E_1 - \dots - E_6) - (E_1 + \dots + E_6).$$

Assume that Θ' is irreducible, i.e. $B = \text{Jac}(C')$ for some curve C' . In this case we use $|H|$ to realize $\text{Km}(B)$ as a quartic surface in \mathbb{P}^3 with 16 nodes. Formula (4) shows that the image D of $\lambda(\Theta)$ on $\text{Km}(B)$ is cut out by a surface S of degree $(p+1)/2$ containing the trope T . Since D is a rational curve of arithmetic genus $\frac{1}{2}(p^2-1)$ with $\frac{1}{2}(p^2-1)$ ordinary double points, the surface S is tangent to the Kummer surface at $\frac{1}{2}(p^2-1)$ points. Humbert proves this fact in [Hum]. In the case $p=3$ he considers the projection of D to the projective plane from a node of $\text{Km}(B)$ not lying on the trope T . The image of the projection is a rational curve Q of degree 6 that passes through the six nodes of the branch curve, the projections of the double points on the trope T . The curve Q is also tangent to the branch curve B at any other intersection point. The six nodes of the branch curve correspond to 6 parameters of the rational parametrization $\mathbb{P}^1 \rightarrow Q$. They are projectively equivalent to the six Weierstrass points defining the original curve C . Since the Weierstrass points of the curve C' with $\text{Jac}(C') = B$ are defined by the intersection of B with the osculating conic, this gives a “solution” of our problem for $p=3$. We put it in the quotation marks since it seems impossible to find Q explicitly (we tried!). Note that Q contains 10 double points, four of them are the projections of the double points of D , the remaining 6 points are resolved under the double cover. The pre-image of Q under the double cover is equal to the union $D + D'$, where

$$D' \in \left| \frac{1}{2}(9H - 12E - E_1 - \dots - E_6) \right|,$$

and E is the class of the exceptional curve corresponding to the node from which we project the Kummer surface.

The curves D, D' intersect at 24 points, 12 points correspond to the intersection of Q with B outside its singular points, and 12 other points correspond to six double points of Q that are resolved under the double cover. These peculiar properties of the rational curve Q are impossible to fulfill even with a computer help. Note that given such a curve Q its pre-image on the Kummer surface defines a curve D linearly equivalent to the divisor from (4). Its pre-image on the abelian surface B defines a genus 2 curve whose Jacobian is isomorphic to A . Thus there are 40 curves Q with the above properties, each defines an isogeny $A \rightarrow B$. The problem is that we cannot construct any of them.

3.3. The reducible case. Note that the principally polarized abelian surface $B = A/F$ could be reducible, i.e. isomorphic to the product $C_1 \times C_2$ of two elliptic curves. In this case B is isomorphic to $\text{Pic}^0(C')$, where C'

is a stable curve of genus 2 isomorphic to the union of two elliptic curves C_1, C_2 intersecting transversally at one point. The theta divisor Θ' on B is equal to $(\{0\} \times C_2) \cup (C_1 \times \{0\})$, or is its translate by a 2-torsion point. As we mentioned before, the Kummer surface $\text{Km}(B)$ is a double cover of a nonsingular quadric Q . The covering involution σ leaves the divisors H, E_i invariant. However σ does not act identically on the linear system $|\frac{1}{2}(pH - E_1 - \dots - E_6)|$. Using the decomposition (4), we see that the pre-image of the residual part of a divisor from $|\mathcal{O}_Q(\frac{p+1}{2})|$ containing the image of the trope T on Q belongs to the linear system $|D|$. The dimension of this linear system is equal to

$$\dim |\mathcal{O}_Q(\frac{p-1}{2})| = \frac{1}{4}(p^2 + 2p - 3) < \dim |D| = \frac{1}{2}(p^2 - 1).$$

The image D of $\lambda(\Theta)$ in $\text{Km}(B)$ is an irreducible member of the linear system $|\frac{1}{2}(pH - E_1 - \dots - E_6)|$ with $\frac{1}{2}(p^2 - 1)$ nodes. It is easy to see that it cannot come from $|\mathcal{O}_Q(\frac{p-1}{2})|$. Thus D corresponds to a divisor from $|\mathcal{O}_Q(p)|$ that passes through the images of the nodes contained in D and splits under the cover.

Consider the case $p = 3$. Let L_1, \dots, L_4 and L'_1, \dots, L'_4 be the components of the branch divisor of $\text{Km}(B) \rightarrow Q$, the first four belong to the same ruling. Let $p_{ij} = L_i \cap L'_j$. Their pre-images in $\text{Km}(B)$ are the 16 nodes. Without loss of generality we may assume that the curves E_1, \dots, E_6 correspond to the points $p_{12}, p_{13}, p_{14}, p_{21}, p_{31}, p_{41}$. A curve of bi-degree $(3, 3)$ passing through these points splits if and only if it is tangent to each of the lines $L_2, L_3, L_4, L'_2, L'_3, L'_4$ at one point and the tangency points are coplanar. Take any rational 4-nodal curve R of bi-degree $(3, 3)$. These curves depend on $11 = 15 - 4$ parameters. Let $\pi_i : Q \rightarrow \mathbb{P}^1$ be the two rulings on the quadric. The projection $\pi_i : R \rightarrow \mathbb{P}^1$ ramifies over four points. Let f_1, \dots, f_4 be the corresponding fibres of π_1 and f'_1, \dots, f'_4 be the corresponding fibres of π_2 . Let $f_i \cap C = 2x_i + y_i, f'_i = 2x'_i + y'_i$. Suppose y_1, y_2, y_3 lie on a fibre f'_0 of π_2 and y'_1, y'_2, y'_3 lie on a fibre f_0 of π_1 . This imposes 4 conditions. Take $L_i = f_i, L'_i = f'_i, i = 1, 2, 3$, and $L_4 = f_0, L'_4 = f'_0$. Then the double cover of Q branched over the curves $L_i, L'_i, i = 1, \dots, 4$ defines a Kummer surface of a reducible principally polarized abelian variety B . The pre-image of the curve R on $\text{Km}(B)$ splits if the 6 points $L_i \cap L'_j, i, j = 1, 2, 3$, are coplanar. This imposes 3 conditions. Counting parameters we see that a curve R always exists. It defines a 4-nodal divisor D on $\mathcal{K}(B)$ from the linear system $|\frac{1}{2}(3H - E_1 - \dots - E_6)|$, where E_1, \dots, E_6 corresponds to the pre-images of the six points $p_{12}, p_{13}, p_{14}, p_{21}, p_{31}, p_{41}$ in the notation from above. The pre-image of D on B' is a genus 2 curve, the Jacobian of its normalization C is an abelian surface A isogenous to B . The six Weierstrass points of C are projectively equivalent to the six coplanar points p_{ij} on the rational curve R . This shows that the reducible case realizes, however we do not know how to construct the curve R effectively.

3.4. Main result II. Let us return to the general case $p > 2$. Let $A = \text{Jac}(C)$ and F be a maximal isotropic subspace in $A[p]$. Consider the restriction of the isogeny $\lambda : A \rightarrow B = A/F$ to Θ and compose it with the map $\phi_{2\Theta'} : B \rightarrow \text{Km}(B) \subset \mathbb{P}^3$ to obtain a map $f : \Theta \rightarrow \mathbb{P}^3$. Since $\lambda^*(\Theta') \in |p\Theta|$, the map f is given by a linear system contained in $|2p\Theta|$ restricted to Θ . This is the linear system $|2pK_\Theta|$. Since Θ is invariant with respect to the involution $[-1]_A$, the image of f is equal to the projection of a rational norm curve R_{2p} of degree $2p$ in $\mathbb{P}^{2p} = \mathbb{P}(H^0(\Theta, 2pK_\Theta)^{\iota_A})$ from a subspace L of dimension $2p-4$. Let v_1, \dots, v_6 be the images of the six Weierstrass point of Θ in R_{2p} . The divisor $2\lambda^*(\Theta')$ belongs to $|2p\Theta|$ and defines a hyperplane \mathcal{H} in \mathbb{P}^{2p} which cuts out R_{2p} at $2p$ points containing the points v_1, \dots, v_6 . This is because $\lambda^*(\Theta')$ contains $\Theta \cap A[2]$ which we identified with the Weierstrass points. Our main observation is the following.

Theorem 3.3. *Let $(z_i, z'_i), i = 1, \dots, \frac{1}{2}(p^2 - 1)$, be the images on R_{2p} of the pairs of points on Θ belonging to the same coset of K and $\ell_i = \overline{z_i, z'_i}$ be corresponding secant lines of R_{2p} . Then the hyperplane \mathcal{H} intersects the secants at $(p^2 - 1)/2$ points which span a linear subspace contained in $L \cong \mathbb{P}^{2p-4}$. The projection of R_{2p} from L maps the points v_1, \dots, v_6 to a conic Q in \mathbb{P}^3 . If Q is irreducible, the double cover of Q branched along the points v_1, \dots, v_6 is a nonsingular curve C' of genus 2 such that $\text{Jac}(C') \cong B$. If Q is the union of lines then each component has three of the points v_i 's and the double covers of each line branched along the three points and the intersection point of the line components define two elliptic curves E and E' such that $B = E \times E'$.*

Proof. Assume first that $B \cong \text{Jac}(C')$ for some nonsingular curve C' . By Proposition 3.1, the image of the Veronese curve R_{2p} in \mathbb{P}^3 is a rational curve with $(p^2 - 1)/2$ ordinary nodes, the images of the points z_i, z'_i . This means that each secant ℓ_i intersects the center of the projection $L \cong \mathbb{P}^{2p-4}$. Since the divisor $\lambda^*(\Theta')$ is the pre-image of a trope in \mathbb{P}^3 , the hyperplane \mathcal{H} must contain the center of the projection L . This implies that L intersects the secants ℓ_i 's at the points, where \mathcal{H} intersects them. The image of R_{2p} in \mathbb{P}^3 lies on the Kummer quartic surface $\text{Km}(B)$ and intersects the trope $T = \Theta'/\langle [-1]_B \rangle$ at six nodes. The nodes are the images of the Weierstrass points w_1, \dots, w_6 . The conic T and the six nodes determine the isomorphism class of the curve C' such that $\text{Jac}(C') \cong B$.

Next assume that B is the product of elliptic curves $F \times F'$. The argument is the same, only this time the image of R_{2p} lies on the quadric Q , the image of $\text{Km}(B)$ in \mathbb{P}^3 . The trope $T = \Theta'/\langle [-1]_B \rangle$ is mapped to the union of two lines $l_1 \cup l_2$ intersecting at a point. Each line contains the images of three nodes of $\text{Km}(B)$. The image of R_{2p} intersects each line at these three points. Again this reconstructs the isomorphism classes of the elliptic curves F and F' . \square

4. THE CASE $p = 3$ AND $\mathbb{K} = \mathbb{C}$

4.1. **The Burkhardt quartic and the Coble cubic.** We specialize the discussion from subsection 2.3 to the case $g = 2$ and $n = 3$. In this case we have the theta maps

$$\begin{aligned} \text{Th}^+ : \mathcal{A}(3)^+ &\rightarrow \mathbb{P}_{Bu} \cong \mathbb{P}^4 \\ \text{Th}^- : \mathcal{A}(3)^- &\rightarrow \mathbb{P}_{Ma} \cong \mathbb{P}^3. \end{aligned}$$

According to Proposition 2.6, the first map is an embedding. The second map is an embedding of the open subset of jacobians [Bo].

It is also known that the restriction of the map (1)

$$\widetilde{\text{Th}}^\pm : \mathcal{X}(3)^\pm \rightarrow \mathbb{P}(V_3(2)) \cong \mathbb{P}^8$$

to any fibre (A, Θ, θ) defines a closed embedding

$$\phi_\pm : A \hookrightarrow \mathbb{P}^8 = |\mathfrak{3}\Theta|^*.$$

This embedding is $\mathcal{H}_2(3)$ -equivariant, where $\mathcal{H}_2(3)$ acts on A via an isomorphism $\bar{\theta} : A[3] \rightarrow \mathbb{F}_3^4$ compatible with the symplectic structures. It acts on \mathbb{P}^8 by means of the projectivized Schrödinger representation.

We have the following theorem due to A. Coble (for a modern exposition see, for example, [Hun], 5.3.1).

Theorem 4.1. *Assume $\text{char } \mathbb{K} \neq 2, 3$. Choose the new coordinates in \mathbb{P}^8 as follows.*

$$\begin{aligned} y_0 &= \eta_{00}, \quad 2y_1 = \eta_{01} + \eta_{02}, \quad 2y_2 = \eta_{10} + \eta_{20}, \quad 2y_3 = \eta_{11} + \eta_{22}, \quad 2y_4 = \eta_{12} + \eta_{21}. \\ 2z_1 &= \eta_{01} - \eta_{02}, \quad 2z_2 = \eta_{10} - \eta_{20}, \quad 2z_3 = \eta_{11} - \eta_{22}, \quad 2z_4 = \eta_{12} - \eta_{21}. \end{aligned}$$

Then the image $\phi(A)$ is defined by the equations

$$(5) \quad \begin{pmatrix} y_0^2 & 2(y_1^2 - z_1^2) & 2(y_2^2 - z_2^2) & 2(y_3^2 - z_3^2) & 2(y_4^2 - z_4^2) \\ y_1^2 + z_1^2 & 2y_0y_1 & 2(y_3y_4 - z_3z_4) & 2(y_2y_4 - z_2z_4) & 2(y_2y_3 - z_2z_4) \\ y_2^2 + z_2^2 & 2(y_3y_4 - z_3z_4) & 2y_0y_2 & 2(y_1y_4 + z_1z_4) & 2(y_1y_3 - z_1z_3) \\ y_3^2 + z_3^2 & 2(y_2y_4 + z_2z_4) & 2(y_1y_4 - z_1z_4) & 2y_0y_1 & 2(y_1y_2 + z_1z_2) \\ y_4^2 + z_4^2 & 2(y_2y_3 + z_2z_3) & 2(y_1y_3 + z_1z_3) & 2(y_1y_2 - z_1z_2) & 2y_0y_4 \end{pmatrix} \cdot \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = 0,$$

$$(6) \quad \begin{aligned} z_1\pi_{01} + z_2\pi_{43} + z_3\pi_{24} + z_4\pi_{32} &= 0, \\ z_1\pi_{43} + z_2\pi_{02} + z_3\pi_{14} + z_4\pi_{13} &= 0, \\ z_1\pi_{24} + z_2\pi_{14} + z_3\pi_{03} + z_4\pi_{12} &= 0, \\ z_1\pi_{32} + z_2\pi_{13} + z_3\pi_{12} + z_4\pi_{04} &= 0, \end{aligned}$$

where $\pi_{ij} = \alpha_i y_j - \alpha_j y_i$ and the the vector of the parameters $(\alpha_0, \dots, \alpha_4)$ is a point α on the Burkhardt quartic

$$(7) \quad \mathcal{B}_4 : T_0^4 + 8T_0(T_1^3 + T_2^3 + T_3^3 + T_4^3) + 48T_1T_2T_3T_4 = 0.$$

The vector α depends only the choice of a 3-level structure on A and its coordinates can be identified with explicit modular forms of weight 2 with respect to $\Gamma_2(3)$ [FSM2], p. 253. As we will review below, the coordinates T_i may be naturally identified with the coordinates y_i in \mathbb{P}_{Bu} . One easily

notice that the 9 quadratic forms are the partials of a unique cubic form (surprisingly it was missed by Coble). It defines a cubic hypersurface \mathcal{C}_3 in \mathbb{P}^8 , called by the first author, the *Coble cubic*. It has a beautiful moduli interpretation in terms of rank 3 vector bundles on the genus 2 curve C (see [Mi], [Or]). Thus, the previous theorem expresses the fact that $\phi(A)$ is the singular locus of the Coble hypersurface \mathcal{C}_3 .

The negation involution $[-1]_A$ acts on $\phi_+(A)$ via the projective transformation $\eta_\sigma \mapsto \eta_{-\sigma}$. In the new coordinates, it is given by $y_i \mapsto y_i$, $z_j \mapsto -z_j$. Its fixed locus in \mathbb{P}^8 is the union of two subspaces

$$\mathbb{P}_{Ma} = \{y_0 = \dots = y_4 = 0\}, \quad \mathbb{P}_{Bu} = \{z_1 = \dots = z_4 = 0\}.$$

Intersecting $\phi_-(A)$ with \mathbb{P}_{Ma} we find 6 points in $A[2]$ lying on Θ . One of them is the origin of A . The remaining 10 points in $A[2]$ is the intersection of $\phi_+(A)$ with \mathbb{P}_{Bu} . Let us compute this intersection: Plugging $y_i = 0$ in the equations in Theorem 4.1, we obtain that the parameters $(\alpha_0, \dots, \alpha_4)$ satisfy the equations

$$(8) \quad \begin{pmatrix} 0 & -2z_1^2 & -2z_2^2 & -2z_3^2 & -2z_4^2 \\ z_1^2 & 0 & -2z_3z_4 & -2z_2z_4 & -2z_2z_4 \\ z_2^2 & -2z_3z_4 & 0 & 2z_1z_4 & -2z_1z_3 \\ z_3^2 & 2z_2z_4 & -2z_1z_4 & 0 & 2z_1z_2 \\ z_4^2 & 2z_2z_3 & 2z_1z_3 & -2z_1z_2 & 0 \end{pmatrix} \cdot \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = 0,$$

As is well-known the coordinates of a non-trivial solution of a skew-symmetric matrix of corank 1 can be taken to be the pfaffians of the principal matrices. This gives a rational map

$$(9) \quad \begin{aligned} c_- : \mathbb{P}_{Ma} &\rightarrow \mathcal{B}_4, \\ \alpha_0 &= 6z_1z_2z_3z_4 \\ \alpha_1 &= z_1(z_2^3 + z_3^3 - z_4^3) \\ \alpha_2 &= -z_2(z_1^3 + z_3^3 + z_4^3) \\ \alpha_3 &= z_3(-z_1^3 - z_2^3 + z_4^3) \\ \alpha_4 &= z_4(z_1^3 + z_2^3 - z_3^3) \end{aligned}$$

We now go back to compute the intersection $\phi_-(A) \cap \mathbb{P}_{Bu}$: Plugging $z_i = 0$ in (5) we obtain that α satisfies the equations

$$(10) \quad \begin{pmatrix} y_0^2 & 2y_1^2 & 2y_2^2 & 2y_3^2 & 2y_4^2 \\ y_1^2 & 2y_0y_1 & 2y_3y_4 & 2y_2y_4 & 2y_2y_3 \\ y_2^2 & 2y_3y_4 & 2y_0y_2 & 2y_1y_4 & 2y_1y_3 \\ y_3^2 & 2y_2y_4 & 2y_1y_4 & 2y_0y_1 & 2y_1y_2 \\ y_4^2 & 2y_2y_3 & 2y_1y_3 & 2y_1y_2 & 2y_0y_4 \end{pmatrix} \cdot \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = 0,$$

Recall that the Hessian hypersurface $\text{Hess}(V(F)) \subset \mathbb{K}^m$ of a hypersurface $V(F)$ is defined by the determinant of the matrix of the second partials of F . It is equal to the locus of points x such that the polar quadric $P_{x^{m-2}}(V(F))$ of $V(F)$ is singular. The locus of singular points of the polar quadrics is the Steinerian hypersurface $\text{St}(V(F))$. It coincides with the locus of points x

such that the first polar $P_x(V(F))$ is singular. One immediately recognizes that the matrix of the coefficients in (10), after multiplying the last four rows by 2, coincides with the matrix of the second partials of a polynomial defining the Burkhardt quartic (7). Thus α is a point on the Steinerian hypersurface of the Burkhardt quartic. On the other hand, we know that it lies on the Burkhardt quartic. This makes \mathcal{B}_4 a very exceptional hypersurface: it coincides with its own Steinerian. This beautiful fact was first discovered by A. Coble [Co].

The first polar of \mathcal{B}_4 at a nonsingular point is a cubic hypersurface with 10 nodes at nonsingular points of the Hessian hypersurface. Any such cubic hypersurface is projectively isomorphic to the *Segre cubic primal* \mathcal{S}_3 given by the equations in \mathbb{P}^5 exhibiting the S_6 -symmetry:

$$Z_0^3 + \cdots + Z_5 = Z_0 + \cdots + Z_5 = 0.$$

The map from the nonsingular locus of $\text{Hess}(\mathcal{B}_4)$ to $\mathcal{B}_4 = \text{St}(\mathcal{B}_4)$ which assigns to a point x the singular point α of the polar quadric $P_{x^2}(\mathcal{B}_4)$ is of degree 10. Its fibres are the sets of singular points of the first polars. We will give its moduli-theoretical interpretation in the next section.

Let

$$c_+ : \text{Hess}(\mathcal{B}_4)^{nsg} \rightarrow \mathcal{B}_4, (y_0, \dots, y_4) \mapsto \alpha$$

be the map given by the cofactors of any column of the matrix of coefficients in (10).

Theorem 4.2. *The image of the map Th^+ is equal to $\text{Hess}(\mathcal{B}_4)^{nsg}$ and the composition of this map with the map c_+ is equal to the forgetful map $\mathcal{A}_2(3)^+ \xrightarrow{10:1} \mathcal{A}_2(3)$. The composition of the map Th^- with c_- is the forgetful map $\mathcal{A}_2(3)^- \xrightarrow{6:1} \mathcal{A}_2(3)$.*

The first assertion is proved in [vdG] (see also [FSM2]). The second assertion is proved in [Bo] (see also [FSM1]).

4.2. The 3-canonical map of a genus 2 curve. Let (A, Θ, θ) be a member of the universal family $\mathcal{X}_2(3)^-$. We assume that the divisor Θ is irreducible, i.e. $A \cong \text{Jac}(C)$ for some smooth genus 2 curve $C \cong \Theta$. By the adjunction formula, the restriction of the map $\phi_- : A \rightarrow \mathbb{P}^8$ to Θ is the 3-canonical map

$$\phi_{3K_C} : C \rightarrow |3K_C|^* \subset \mathbb{P}^8.$$

Here the identification of $|3K_C|^*$ with the subspace of $\mathbb{P}^8 = |3\Theta|^*$ is by means of the canonical exact sequence

$$0 \rightarrow \mathcal{O}_A(2\Theta) \rightarrow \mathcal{O}_A(3\Theta) \rightarrow \mathcal{O}_\Theta(3\Theta) \rightarrow 0.$$

Denote the subspace $|3K_C|^* \cong \mathbb{P}^4$ by \mathbb{P}_Θ^4 . The hyperelliptic involution ι_C acts naturally on \mathbb{P}_Θ^4 and its fixed locus set consists of the union of a hyperplane H_0 and an isolated point x_0 . The dual of H_0 is the divisor

$W = w_1 + \cdots + w_6$, where w_i are the Weierstrass points. It coincides with $\phi_-(A) \cap \mathbb{P}_{Ma}$ and hence

$$H_0 = \mathbb{P}_{Ma}.$$

The dual of x_0 is the hyperplane spanned by the image of the Veronese map $|K_C| \rightarrow |3K_C|$. The projection map $C \rightarrow H_0$ from the point x_0 is a degree 2 map onto a rational normal curve R_3 of degree 3 in H_0 . It is ramified at the Weierstrass points.

Since Θ is an odd theta divisor, the image of $\Gamma(\mathcal{O}_A(2\Theta))$ in $\Gamma(\mathcal{O}_A(3\Theta))$ is contained in $V_3(2)^- \cong \mathbb{C}^5$. Thus the image of $V_3(2)^-$ in $\Gamma(\mathcal{O}_\Theta(3\Theta))$ is the one-dimensional subspace corresponding to the point x_0 . The projectivization of the image of $V_3(2)^+$ in $\Gamma(\mathcal{O}_\Theta(3\Theta))$ is the subspace H_0 .

Observe that

$$\{x_0\} = \mathbb{P}_\Theta^4 \cap \mathbb{P}_{Bu}.$$

It is known that the subspace \mathbb{P}_Θ^4 is contained in the Coble cubic \mathcal{C}_3 and $\mathbb{P}_{Bu} \cap \mathcal{C}_3$ is equal to the polar cubic $P_\alpha(\mathcal{B}_4)$ (see [Mi], Proposition 4.3 and section 5.3). A natural guess is that $x_0 = \alpha$. This turns out to be true.

Lemma 4.3. *Let $\alpha = c_-(W) \in \mathcal{B}_4$. Then, considering \mathcal{B}_4 as a subset of \mathbb{P}_{Bu} , we have*

$$x_0 = \alpha.$$

Proof. For simplicity of the notation let us denote $\phi_-(A)$ by A . Let $I_A(2)$ be the subspace of $S^2V_3(2)$ that consists of quadrics containing A . As we know, it is spanned by the partial derivatives of the Coble cubic $V(F_3)$. Let $I_\Theta(2)$ be the space of quadrics in \mathbb{P}_Θ^4 vanishing on Θ . The polar map $v \mapsto P_v(F_3)$ defines a $\mathcal{H}_3(2) \times \langle D_{-1} \rangle$ -equivariant isomorphism $V_3(2) \rightarrow I_A(2)$.

Consider the restriction map

$$r : I_A(2) \rightarrow I_\Theta(2).$$

By [Mi], Proposition 4.7, the map is surjective. By Riemann-Roch, its kernel L is of dimension 5. We know that $I_A(2) = I_A(2)^+ \oplus I_A(2)^- = \mathbb{C}^5 \oplus \mathbb{C}^4$ with the obvious notation. The subspace $I_A(2)^+$ is spanned by the four quadrics from (6). Obviously they vanish on $\mathbb{P}_{Ma} \subset \mathbb{P}_\Theta^4$. Since they also contain a non-degenerate curve Θ they vanish on the whole space \mathbb{P}_Θ^4 . Thus $L = I_A(2)^+ \oplus L^-$, where $L^- = L \cap I_A(2)^-$ is of dimension 1. In other words, there exists a unique point $x \in \mathbb{P}_{Bu}$ such that the polar quadric $P_x(\mathcal{C}_3)$ vanishes on \mathbb{P}_Θ^4 . It remains to prove that x_0 and α both play the role of the x .

Recall the important property of the polar is

$$P_x(\mathcal{C}_3) \cap \mathcal{C}_3 = \{c \in \mathcal{C}_3 : x \in \mathbb{T}_c(\mathcal{C}_3)\},$$

where $\mathbb{T}_c(\mathcal{C}_3)$ denotes the embedded Zariski tangent space. Since \mathbb{P}_Θ^4 is contained in \mathcal{C}_3 , for any $c \in \mathbb{P}_\Theta^4$ we have $\mathbb{P}_\Theta^4 \subset \mathbb{T}_c(\mathcal{C}_3)$. But x_0 belongs to \mathbb{P}_Θ^4 , therefore $c \in P_{x_0}(\mathcal{C}_3)$. This proves that $\mathbb{P}_\Theta^4 \subset P_{x_0}(\mathcal{C}_3)$.

Now consider the polar quadric $P_\alpha(\mathcal{C}_3)$. It is defined by the quadratic form

$$(\alpha_0, \dots, \alpha_4) \cdot M(y, z) \cdot \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = 0,$$

where $M(y, z)$ is the matrix from (5). Restricting the quadric to the subspace \mathbb{P}_{Ma} we see that it is equal to ${}^t\alpha \cdot M(0, z) \cdot \alpha$, where $M(0, z)$ is the skew-symmetric matrix from (8). Therefore, it is identically zero on the Maschke subspace, and, as above, since it also contains Θ , it must contain the whole \mathbb{P}_Θ^4 . \square

4.3. The invariant theory on the Maschke space. Recall that a general point in the Maschke space \mathbb{P}_{Ma} represents the isomorphism class of a genus 2 curve together with an odd theta structure. The group $\mathrm{P}\mathrm{Sp}(4, \mathbb{F}_3)$ of order 25920 acts projectively in \mathbb{P}_{Ma} with the quotient map representing the forgetful map $\mathcal{A}_2(3)^- \rightarrow \mathcal{A}_2^-$, where \mathcal{A}_2^- is the moduli space of principally polarized abelian surfaces together with an odd theta divisor. If C is a genus 2 curve, then the odd theta structure on $\mathrm{Jac}(C)$ is a choice of a Weierstrass point on C .

It is known that the projective representation of $\mathrm{P}\mathrm{Sp}(4, \mathbb{F}_3)$ can be lifted to a 5-dimensional linear representation of its central extension $G = \mathbb{Z}/3\mathbb{Z} \times \mathrm{Sp}(4, \mathbb{F}_3)$. The group G act in \mathbb{C}^5 as a complex reflection group with generators of order 3 (Number 32 in Shepherd-Todd's list). It was first described by Maschke in [Ma].

We use the coordinates (z_1, z_2, z_3, z_4) in \mathbb{C}^4 corresponding to the projective coordinates in the Maschke space introduced in 4.1. Let $x \cdot y$ be the standard hermitian dot-product in \mathbb{C}^4 . In these coordinates G is generated by complex reflections

$$z \mapsto z - \frac{(1-w)z \cdot r}{r \cdot r} r,$$

where $\eta = e^{2\pi i/3}$ and r is one of the vectors

$$(\sqrt{-3}, 0, 0, 0), (0, \sqrt{-3}, 0, 0), (0, \eta^a, \eta^b, \eta^c), (\eta^a, 0, \eta^b, -\eta^c),$$

or one of the vectors obtained from these by permutations of the last three coordinates or multiplied by $\pm\eta$. There are 40 different hyperplanes of fixed points of the complex reflections. Their union is given by the equation

$$\Phi_{40} = z_1 z_2 z_3 z_4 ABCD = 0,$$

where

$$\begin{aligned} A &= (z_2^3 + z_3^3 + z_4^3)^3 - 27z_2^3 z_3^3 z_4^3, \\ B &= (z_1^3 - z_2^3 + z_3^3)^3 + 27z_1^3 z_2^3 z_3^3, \\ C &= (z_1^3 + z_2^3 - z_3^3)^3 + 27z_1^3 z_2^3 z_3^3, \\ D &= (z_1^3 + z_3^3 - z_4^3)^3 + 27z_1^3 z_3^3 z_4^3. \end{aligned}$$

Proposition 4.4. *A point $p = (z_1 : z_2; z_3 : z_4) \in \mathbb{P}_{Ma}$ is equal to the value of the theta map Th^- at some $(\text{Jac}(C), \Theta)$ with nonsingular C if and only if $\Phi_{40}(z_1, z_2, z_3, z_4) \neq 0$.*

Proof. Consider the rational map $c_- : \mathbb{P}_{Ma} \rightarrow \mathbb{P}_{Bu}$ from (9). Its image is the Burkhardt quartic \mathcal{B}_4 . The restriction of c_- to any reflection hyperplane defines a rational map $\mathbb{P}^2 \rightarrow \mathbb{P}_{Bu}$ whose image is a plane contained in \mathbb{P}_{Bu} . It is one of the j -planes described in [Hum]. It is shown in loc.cit., Lemma 5.7.3, that $\mathcal{B}_4^0 \setminus \{40 \text{ } j\text{-planes}\}$ is equal to the image of the jacobian locus of $\mathcal{A}_3(3)$ under the map $c_+ : \mathcal{A}_3(3)^+ \rightarrow \mathcal{B}_4$. One checks that the union of j -planes is equal to the intersection $\mathcal{B}_4 \cap \text{Hess}(\mathcal{B}_4)$. Since the Hessian is of degree 10 and the map c_- is given by quartics, its pre-image in \mathbb{P}_{Ma} is a hypersurface of degree 40. Therefore it coincides with the union of the reflection hyperplanes.

Note that the $x \notin \mathcal{B}_4 \cap \text{Hess}(\mathcal{B}_4)$ if and only if the polar quadric $P_x^2(\mathcal{B}_4)$ is singular. The intersection of the cubic polar and the quadric polar is birationally isomorphic to the Kummer surface of a genus 2 curve C with $\text{Jac}(C)$ corresponding to the point x [Co]. \square

The algebra of G -invariant polynomials z_1, z_2, z_3, z_4 was computed by Maschke [Ma]. It is freely generated by polynomials $F_{12}, F_{18}, F_{24}, F_{30}$ of degrees indicated by the subscripts. This shows that

$$\mathbb{P}^3/G \cong \mathbb{P}(12, 18, 24, 30) \cong \mathbb{P}(2, 3, 4, 5).$$

It follows from Proposition 4.4 that the moduli space of genus 2 curves together with a choice of a Weierstrass point is isomorphic to $\mathbb{P}(2, 3, 4, 5) \setminus \{P_{20} = 0\}$, where P_{20} is the polynomial of degree 20 corresponding to the invariant Φ_{40}^3 . This shows that the moduli space of genus 2 curves together with a choice of a Weierstrass point is isomorphic to $\mathbb{P}(2, 3, 4, 5) \setminus V(P_{20})$ for some explicit weighted homogeneous polynomial P_{20} of degree 20.

A genus 2 curve together with a choice of a Weierstrass point can be represented by the equation $y^2 + f_5(x) = 0$ for some polynomial of degree 5 without multiple roots. The above discussion suggests that the quotient of the open subset of the projectivized space of binary forms of degree 5 without multiple roots by the affine group $\mathbb{C} \rtimes \mathbb{C}^*$ must be isomorphic to $\mathbb{P}(2, 3, 4, 5) \setminus V(P_{20})$. It is easy to see directly. Using translations we may choose a representative of an orbit of the form

$$(11) \quad f_5 = x^5 + 10ax^3 + 10bx^2 + 5cx + d.$$

The group \mathbb{C}^* acts by weighted scaling $(a, b, c, d) \mapsto (t^2a, t^3b, t^4c, t^5d)$. This shows that the orbits of nonsingular quintic forms are parametrized by an open subset of $\mathbb{P}(2, 3, 4, 5)$. According to G. Salmon [Sa], p. 230, the discriminant of f_5 is equal to

$$D = d^4 - 120abd^3 + 160ac^2d^2 + 360b^2cd^2 - 640bc^3d + 256c^5 - 1440b^3cd^2 + 2640a^2b^2d^2 + 4480a^2bc^2d - 2560a^2c^4 - 10080ab^3cd + 5760ab^3c^3 + 3456b^5d$$

$$+3456a^5d^2 - 2160b^4c^2 - 11520a^4bcd + 6400a^4c^3 + 5120a^3c^3d - 3200a^3b^2c^2.$$

If we weight a, b, c, d with 2, 3, 4, 5, respectively, we obtain that $D = P_{20}$ (up to a scalar factor).

5. AN EXPLICIT ALGORITHM

Let C be a genus 2 curve and F be a maximal isotropic subspace in $\text{Jac}(C)[3]$. We would like to find explicitly a stable genus 2 curve C' such that $\text{Jac}(C') \cong \text{Jac}(C)/F$.

5.1. $\mathbb{K} = \mathbb{C}$. We start with the complex case. Unfortunately, we do not know how to input explicitly the pair (C, F) . Instead we consider C with an odd theta structure. It follows from Proposition 4.4 that the isomorphism class of such a structure $(\text{Jac}(C), \Theta, \bar{\theta})$ is defined by a point p in \mathbb{P}_{Ma} not lying on the union of the reflection hyperplanes. The theta structure defines a maximal isotropic subspace F in $\text{Jac}(C)[2]$. Two points p and q define the same maximal isotropic subspace if and only if they lie in the same orbit with the respect to the stabilizer of the maximal isotropic subspace F_0 of the symplectic space \mathbb{F}_3^4 that consists of vectors with the first two coordinates equal to zero. It is a maximal subgroup of $\text{PSp}(4, \mathbb{F}_3)$ of index 40 isomorphic to $(\mathbb{Z}/3\mathbb{Z})^3 \rtimes S_4$.

Step 1: Evaluating the Maschke fundamental invariants $F_{12}, F_{18}, F_{24}, F_{30}$ at p we can find an equation $y^2 + f_5(x) = 0$ of C , where f_5 is as in (11).

Step 2: Next we consider the point $\alpha = c_-(p) \in \mathcal{B}_4$. From Lemma 4.3 we know that the span of \mathbb{P}_{Ma} and α in \mathbb{P}^8 is the space \mathbb{P}_Θ^4 where the tri-canonical image of C lies. We know from the proof of Theorem 3.3 that there are two pairs of points $(x, y), (x', y')$ on C such that $x - y = x' - y' = e$. The group \mathbb{F}_3^4 acts on \mathbb{P}^8 via the Schrödinger representation. Take a point $e \in F_0$ that corresponds to some point in F under the theta structure. Intersecting \mathbb{P}_Θ^4 with its image $\mathbb{P}_\Theta^4 + e$ under e , we get a plane Π_e . It is clear that \mathbb{P}_Θ^4 intersects $\mathbb{P}_\Theta^4 + e$ along the plane Π_e spanned by the secant lines $\overline{x, y}$ and $\overline{x', y'}$. The hyperelliptic involution acting in \mathbb{P}_Θ^4 switches the two lines. In particular, the lines intersect at a point $p_e \in H_0 = \mathbb{P}_{Ma}$ (cf. [Bo], Lemma 5.1.4). Note that replacing e with $-e$ we get the same pair of secants. In this way we obtain 8 secant lines $\overline{x_e, y_e}, \overline{x'_e, y'_e}$, each pair corresponds to the pair $(e, -e)$ of 3-torsion points from F . Thus the plane Π_e intersects C at 4 points x_e, y_e, x'_e, y'_e . They define a pair of concurrent secants.

Step 3: Using α we can write down the equation of the abelian surface $A = \text{Jac}(C)$ in \mathbb{P}^8 given in (5). Intersecting A with \mathbb{P}_Θ^4 we find the equations of tri-canonical model of C in \mathbb{P}_Θ^4 . They are given by the restriction of the quadrics containing A to \mathbb{P}_Θ^4 ([Mi], Proposition 4.7).

Step 4: Next we project C from α to \mathbb{P}_{Ma} . The image is a rational normal curve R_3 . The images of the 4 pairs of concurrent secants l_1, \dots, l_4 is a set of 4 secants of R_3 .

Step 5: Now we need to locate the 6-tuple of points $\{x_1, \dots, x_6\}$ on R_3 corresponding to the Weierstrass points of C . This is the branch locus of the map $C \rightarrow R_3$, which is computed explicitly by choosing a rational coordinate on R_3 (note that we do not need the coordinate of any of the x_i 's by itself).

Step 6: Identifying R_3 with \mathbb{P}^1 let us consider the Veronese map $R_3 \rightarrow R_6 \subset \mathbb{P}^5$. Let y_1, \dots, y_6 be the images of the six points x_1, \dots, x_6 and ℓ_1, \dots, ℓ_4 be the secants defined by the images of the 4 pairs of points defining the secants l_1, \dots, l_4 of R_3 . The points y_1, \dots, y_6 span a hyperplane H in \mathbb{P}^5 .

Step 7: This is our final step. Following the proof of Theorem 3.3, we intersect the four secants ℓ_1, \dots, ℓ_4 with H . The four intersection points span a plane π . We project H from π to \mathbb{P}^2 . The images of the points y_1, \dots, y_6 lie on a conic and determine a stable genus 2 curve C' with $\text{Jac}(C') \cong \text{Jac}(C)/F$.

5.2. The case $\text{char } \mathbb{K} = 3$. Recall (see the discussion before Proposition 2.4) that $F = \text{Jac}(C)[3]_{\text{red}} \cong (\mathbb{Z}/3\mathbb{Z})^2$. The algorithm we present below gives the curve C' such that $\text{Jac}(C') = \text{Jac}(C)/F$. This construction is known to be the inverse of the Frobenius map on $\mathcal{A}_2(\mathbb{K})$ (this is seen by considering the quotient of $\text{Jac}(C)$ by the Weil-pairing dual of the group $\text{Jac}(C)[3]_{\text{red}}$, which is the group scheme isomorphic to $\mu_{3, \mathbb{K}}^2$).

First let us remind an explicit algorithm for finding 3-torsion points on $\text{Jac}(C)$ [CF]. Let w_1, \dots, w_6 be the Weierstrass points of C . Fix one of them, say $w = w_1$, i.e. choose to define C by equation $y^2t^3 - f_5(x, t) = 0$ in \mathbb{P}^2 . The plane quintic model C_0 has a triple singular point and an infinitely near cusp at $(t, x, y) = (0, 0, 1)$. It is the projection of the quintic curve C in \mathbb{P}^3 embedded by the linear system $|2K_C + w|$ from any point, not on C , lying on the ruling of the unique quadric containing C which cuts out the divisor $3w$ on C . The pencil of lines through the singular point of C_0 cuts out the linear system $|K_C| + 3w$.

A plane cubic with equation $yt^2 - f_3(x, t) = 0$ intersects C_0 at 6 nonsingular points p_1, \dots, p_6 and at the point $(0, 0, 1)$ with multiplicity 9. This implies that $p_1 + \dots + p_6$ is linearly equivalent to $3K_C$. Using the well-known description of $\text{Jac}(C)$ in terms of the symmetric square of C , we see that $[p_1 + p_2] \oplus [p_3 + p_4] = -[p_5 + p_6]$ in the group law on $\text{Jac}(C)$, where $[p + q]$ is the divisor class of the divisor $p + q - K_C$. Here $-[p + q]$ is equal to $[p' + q']$, where $p \mapsto p'$ is the hyperelliptic involution $(t, x, y) \mapsto (t, x, -y)$.

Choosing coordinates so that the affine piece of C_0 is given by $y^2 = x^5 + \sum_{i=0}^4 b_i x^i$. Replacing x with $x + b_4/5$ we may assume that $b_4 = 0$. It is known that $\text{Jac}(C)$ is an ordinary abelian variety if and only if the Cartier-Manin matrix

$$A = \begin{pmatrix} b_2 & b_1 \\ 1 & b_4 \end{pmatrix}$$

is nonsingular [Yui], Theorem 3. Since we assumed that $b_4 = 0$, this is equivalent to

$$b_1 \neq 0.$$

We have to find coefficients $a, d_0, d_1, d_2, c_0, c_1$ which solve the equation

$$(x^3 + d_2x^2 + d_1x + d_0)^2 - a(x^5 + \sum_{i=0}^3 b_ix^i) - (x^2 + c_1x + c_0)^3 = 0.$$

Equating coefficients at powers of x^i we find

$$\begin{aligned} i = 5 : \quad d_2 &= -a, \\ i = 4 : \quad d_1 &= a^2, \\ i = 3 : \quad c_1^3 &= -ab_3 + a^3 - d_0, \\ i = 2 : \quad d_0 &= b_2 - a^3, \\ i = 1 : \quad b_1 &= -ad_0, \\ i = 0 : \quad c_0^3 &= d_0^2 - ab_0. \end{aligned}$$

The first, the second, and the fourth equations eliminate d_0, d_1, d_2 . The fifth equation gives a quartic equation for a

$$(12) \quad X^4 - b_2X - b_1 = 0.$$

Note that equation is separable if and only if $b_1 \neq 0$.

It is easy to find the roots of this equation since the splitting field of the resolvent polynomial is an Artin-Schrier extension of \mathbb{K} . Finally, the third and the last equations give

$$c_1^3 = a^3 - ab_3 + \frac{b_1}{a}, \quad c_0^3 = \frac{b_1^2}{a^2} - ab_0.$$

Each solution a of (12) defines a quadratic equation $x^2 + c_1x + c_0$ and a pair $\{x_a, x'_a\}$ of its roots. These are the x -coordinates (or, equivalently, the orbits with respect to the hyperelliptic involution ι) of a pair of points p_1, p_2 such that $p_1 + p_2 - H$ is a 3-torsion divisor class in $\text{Jac}(C)$. Since $\text{Jac}(C)[3]_{\text{red}} \cong (\mathbb{Z}/3\mathbb{Z})^2$, we have four distinct pairs of roots. Consider the pairs of roots as points on the image R_6 of C under the map given by the linear system $|6K_C|^\iota$. They define the four secants from the proof of Theorem 3.3. Now we finish as in Step 7 from the case $\mathbb{K} = \mathbb{C}$.

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