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Dual Homogeneous Forms and Varieties of Power Sums

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Abstract. We review the classical definition of the dual homogeneous form of arbitrary even degree which generalizes the well-known notion of the dual quadratic form. Following the ideas of S. Mukai we apply this construction to the study of the varieties parametrizing representations of a homogeneous polynomial as a sum of powers of linear forms.

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1. Introduction

A well-known theorem from linear algebra asserts that a nondegenerate quadratic form F_2 on a complex vector space V of dimension $n+1$ can be written as a sum of $n+1$ squares of linear forms l_i. The linear forms l_i's considered as vectors in the dual space V^* are mutually orthogonal with respect to the dual quadratic form $\check{F}_2$ on the space V^*. For more than hundred years it has been a popular problem for algebraists and geometers to search for a generalization of this construction to homogeneous forms F_d on V of arbitrary degree. It is known as the *Waring problem* or the *canonical forms problem* for homogeneous forms. The main object of the study is the variety of sums of powers $\mathrm{VSP}(F_d; N)^o$ parametrizing all representations of F_d as a sum of powers of N linear forms. According to the traditional

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definition, $\mathrm{VSP}(F_d;N)^o$ is the subset of the projective space $\mathbb{P}S^NV^*$ of homogeneous polynomials of degree N on V equal to the product of linear forms $l_1\cdot\ldots l_N$ such that $F_d=\sum l_i^d$. The zero sets $V(l_i)$ of the forms l_i's are hyperplanes in $\mathbb{P}V$, and the set $V(l_1),\ldots,V(l_N)$ was classically known as the polar polyhedron of F_d. Its faces $V(l_i)$ can be viewed as points in the dual space $\mathbb{P}V^*$, and the variety $\mathrm{VSP}(F_d;N)^o$ is the subvariety of the symmetric power $\mathbb{P}V^{*(N)}$ of $\mathbb{P}V^*$ parametrizing the polar polyhedra of F_d. The varietes $\mathrm{VSP}(F_d;N)^o$ were intensively studied in the classical algebraic geometry and the invariant theory in the works of A. Dixon, F. Palatini, T. Reye, H. Richmond, J. Rosanes, G. Scorza, A. Terracini, and others. However, the lack of techniques of higher dimensional algebraic geometry did not allow them to give any explicit construction of the varieties $\mathrm{VSP}(F_d;N)^o$ or to study a possible compactification $\mathrm{VSP}(F_d;N)$ of $\mathrm{VSP}(F_d;N)^o$ (except in the case $n=1$ and a few cases where $\mathrm{VSP}(F_2,N)^o$ is a finite set of points). The first explicit construction of $\mathrm{VSP}(F_d;N)^o$ was given by S. Mukai in the cases $(n,d,N)=(2,2,3),(2,4,6),(2,6,10)$ for a general polynomial F_d. He also constructed a smooth compactification $\mathrm{VSP}(F_d;N)$ which turned out to be a Fano threefold in the first two cases and a K3 surface in the third case. By a different method, this result was proven later by K. Ranestad and F.-O. Schreyer [12]. Other explicit smooth compactifications of $\mathrm{VSP}(F;N)$ are known only for general cubic polynomials in $n+1\le 6$ variables. If $n=2$, $\mathrm{VSP}(F;4)$ is isomorphic to the projective plane [3], if $n=3$, $\mathrm{VSP}(F;5)$ is the one-point set (this a classical result of Sylvester, see [14]), if $n=4$, $\mathrm{VSP}(F,8)$ is a smooth 5-dimensional Fano variety and, if $n=5$, $\mathrm{VSP}(F,10)$ is a holomorphic symplectic 4-fold (see [8]). The construction of S. Mukai is very beautiful and employs a generalization of the notion of the dual quadratic form to forms of arbitrary even degree $d=2k$. This generalization emerges from the classical theory of apolarity, a nowadays almost forgotten chapter in multilinear algebra. One associates to F_d a quadratic form Ω_F on the symmetric power S^kV of V and, if this form is non-degenerate, the dual quadratic form $\check{\Omega}_F\in S^2(S^kV^*)$. The multiplication map $S^2S^kV^*\to S^{2k}V^*$ defines the dual form $\check{F}$ of degree d on V^*. A representation $F_{2k}=l_1^{2k}+\ldots+l_N^{2k}$ can be viewed as a representation of Ω_F as a sum of squares of linear forms from S^kV^* which happen to be kth powers of linear forms. It follows from this that the minimal N for which $\mathrm{VSP}(F_{2k},N)^o\ne\emptyset$ (which we call the *Waring rank* of F_d) is greater or equal to the rank of Ω_F. The cases considered by Mukai are the cases where the equality takes place.

The main goal of this paper is to fill in some of the details of Mukai's construction sketched in [10]. We make an effort to use the techniques of algebraic geometry as less as possible. A few original results can be found in the last section.

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2. The apolarity pairing

2.1. Apolar forms

Let V be a complex vector space of dimension $n+1$ and V^* be its dual vector space. We have a canonical duality pairing

$$V \times V^* \to \mathbb{C}, \quad \langle v, l \rangle = l(v). \tag{2.1}$$

This can be naturally extended to a canonical perfect bilinear pairings

$$S^k V \times S^d V^* \to S^{d-k} V^*, \quad (\Phi, F) \mapsto \langle \Phi, F \rangle \tag{2.2}$$

between the symmetric powers of V^* and V (by definition, $S^{d-k}V^* = 0$ if $d < k$). The easiest way to do this is via coordinates. Choose coordinates $t_0, \dots, t_n$ in V and the dual coordinates $\xi_0, \dots, \xi_n$ in V^* to identify the symmetric algebra $S^\bullet V^*$ (resp. $S^\bullet V$) with the polynomial algebra $R = \mathbb{C}[t_0, \dots, t_n]$ (resp. $U = \mathbb{C}[\xi_0, \dots, \xi_n]$). Then $S^d V^*$ is identified with the space R_k of degree k homogeneous polynomials and we have a similar identification for $S^k V$. For any homogeneous polynomial $\Phi(\xi_0, \dots, \xi_n) \in U_k$ consider the linear differential operator

$$D_\Phi = \Phi(\partial_0, \dots, \partial_n).$$

by replacing the variable ξ_i with the operator $\partial_i = \frac{\partial}{\partial t_i}$. Now set, for any $\Phi \in S^k V$ and $\Phi \in S^d V^*$,

$$\langle \Phi, F \rangle = D_\Phi(F). \tag{2.3}$$

We call this pairing the *apolarity* paring.

Taking $k = d$, we obtain the duality isomorphisms

$$S^d V^* \cong (S^d V)^*, \quad S^d V \cong (S^k V^*)^*. \tag{2.4}$$

The following properties of the apolarity pairing follow easily from the definition.

(P1) $\langle \Phi\Phi', F \rangle = \langle \Phi, \langle \Phi', F \rangle \rangle$;
(P2) $\langle v^d, F \rangle = F(v)$, for any $v \in V$.

Fixing $F \in S^d V^*$ we get a linear map

$$\mathrm{ap}_F^k : S^k V \to S^{d-k} V^*, \quad \Phi \mapsto D_\Phi(F). \tag{2.5}$$

It is called the *apolarity map*.

Definition 2.1. A homogeneous form $\Phi \in S^k(V)$ is called *apolar to a homogeneous form* $F \in S^d(V^*)$ *if* $D_\Phi(F) = 0$, *or, in other words,* $\Phi \in \mathrm{Ker}(\mathrm{ap}_F^k)$. *The linear space of apolar forms of degree* k *is denoted by* $\mathrm{AP}_k(F)$.

Lemma 2.1. *Let us identify* $S^k V$ *with* $(S^k V^*)^*$ *by means of the apolarity paring. Then*

$$\mathrm{ap}_F^k(S^k V)^\perp = \mathrm{AP}_{d-k}(F).$$

Proof. By Property (P1), for any $\Phi_k \in S^k V$ and any $\Phi'_{d-k} \in S^{d-k} V$, we have

$$\begin{aligned} \langle \Phi'_{d-k}, \mathrm{ap}_F^k(\Phi_k) \rangle &= \langle \Phi'_{d-k}, \langle \Phi_k, F \rangle \rangle = \langle \Phi'_{d-k} \Phi_k, F \rangle \\ &= \langle \Phi_k, \langle \Phi'_{d-k}, F \rangle \rangle = \mathrm{ap}_F^{d-k}(\Phi'_{d-k})(\Phi_k). \end{aligned}$$

Thus, if $\langle \Phi'_{d-k}, \mathrm{ap}_F^k(\Phi_k) \rangle = 0$ for all Φ_k we get $\mathrm{ap}_F^{d-k}(\Phi'_{d-k})(\Phi_k) = 0$ for all Φ_k. By nondegeneracy of the apolarity pairing we get $\mathrm{ap}_F^{d-k}(\Phi'_{d-k}) = 0$, i.e. $\Phi'_{d-k} \in \mathrm{AP}_{d-k}(F)$. The converse is proven in the same way. □

2.2. Quadratic forms

If $d = 2$, the space $S^2 V^*$ is the space of quadratic forms on V. In coordinates, $Q \in S^2 V^*$ is given by a symmetric matrix $A = (a_{ij})$

$$Q(t_0, \ldots, t_n) = \sum_{i,j=0}^{n} a_{ij} t_i t_j.$$

The apolarity map is

$$\mathrm{ap}_Q^1 : V \to V^*, \quad v \mapsto D_v(Q) = 2 \sum_{0 \le i \le j = n} a_{ij} v_i t_j = \sum_{j=0}^{n} \frac{\partial Q}{\partial t_j}(v) t_j. \tag{2.6}$$

Considered as a bilinear form $V \times V \to \mathbb{C}$, this is the polar symmetric bilinear form b_Q associated to Q. We shall identify ap_Q^1 with b_Q via the equality

$$b_Q(v, w) = \langle w, \mathrm{ap}_Q^1(v) \rangle. \tag{2.7}$$

The quadratic form Q is nondegenerate if and only if ap_Q^1 is an invertible linear map. In this case, there exists a unique quadratic form $\check{Q}$ on V^* such that

$$b_{\check{Q}} = b_Q^{-1}.$$

This quadratic form is called the *dual quadratic form* of Q. By solving the linear system of equations

$$\sum_{i=0}^{n} a_{ij}x_i = y_j, \quad j = 0, \dots, n,$$

we obtain that

$$\check{Q} = \sum_{i,j=0}^{n} a^*_{ij}\xi_i\xi_j,$$

where $(a^*_{ij}) = \mathrm{adj}(A)$ is the matrix of complementary minors of A.

Applying (2.7), we get

$$b_{\check{Q}}(b_Q(v), b_Q(v)) = \langle b_Q(v), b_{\check{Q}}(b_Q(v)\rangle = \langle b_Q(v), v\rangle = b_Q(v, v).$$

This shows that

$$Q(v) = 0 \Longleftrightarrow \check{Q}(b_Q(v)).$$

Thus the map $b_Q : V \to V^*$ sends the zero locus of the quadric Q to the zero locus of the dual quadric. It follows from (2.6) that, for any nonzero $v \in Q^{-1}(0)$ the hyperplane $b_Q(v)^\perp$ is the tangent hyperplane of $Q^{-1}(0)$ at v. Thus the dual quadric $\check{Q} \subset \mathbb{P}V^*$ is the locus of tangent hyperplanes of the quadric $V(Q) \subset \mathbb{P}V$.

2.3. Dual homogeneous forms of higher degree

We would like to generalize the notion of the dual quadratic form to the case $d > 2$. Such a generalization is known for even $d = 2k$. Let $F \in S^{2k}V^*$ and

$$\mathrm{ap}^k_F : S^kV \to S^kV^*$$

be the apolarity map (2.5). Using (2.4), we can view this map as a symmetric bilinear form

$$\Omega_F : S^kV \times S^kV \to \mathbb{C}, \quad \Omega_F(\Phi_1, \Phi_2) = \mathrm{ap}^k_F(\Phi_1)(\Phi_2) = \langle \Phi_2, \mathrm{ap}^k_F(\Phi_1)\rangle. \tag{2.8}$$

Let us identify Ω_F with the associated quadratic form on S^kV (the restriction of Ω_F to the diagonal). This defines a linear map

$$\Omega : S^{2k}V^* \to S^2S^kV^*, \quad F \mapsto \Omega_F.$$

There is also a natural left inverse map of Ω

$$P : S^2S^kV^* \to S^{2k}V^*$$

defined by multiplication $S^kV^* \times S^kV^* \to S^{2k}V^*$. All these maps are $\mathrm{GL}(V)$-equivariant and realize the linear representation $S^{2k}V^*$ as a direct summand in the representation $S^2S^kV^*$.

To see the expression of Ω_F in terms of coordinates, let us write F in the form

$$F = \sum_{|\mathbf{i}|=d} \frac{d!}{\mathbf{i}!} a_{\mathbf{i}} t^{\mathbf{i}}.$$

Consider a basis in $S^k(V)$ formed by monomials $\xi^{\mathbf{i}}$ and a basis in S^kV^* formed by monomials $t^{\mathbf{i}}$ ordered lexigraphically. The matrix of ap_F^k with respect to these bases is called the kth *catalecticant matrix* of F and is denoted by $\mathrm{Cat}_k(F)$. Its entries $c_{\mathbf{uv}}$ are parametrized by pairs $(\mathbf{u}, \mathbf{v}) \in \mathbb{N}^k \times \mathbb{N}^k$ with $|\mathbf{u}| = |\mathbf{v}| = k$. We have

$$c_{\mathbf{uv}} = a_{\mathbf{u}+\mathbf{v}}.$$

Considering $a_{\mathbf{i}}$ as independent variables $T_{\mathbf{i}}$, we obtain the definition of a general catalecticant matrix $\mathrm{Cat}_k(n+1)$. This is a square matrix of size $\dim S^kV = \binom{n+k}{k}$. For $n = 1$ this matrix is known as a *Hankel matrix*. Similarly one can define a non-square catalecticant matrix which defines the apolarity map $S^kV \to S^{d-k}V^*$.

As in the case of quadratic forms, the quadratic form Ω_F on the space S^kV is defined by the symmetric matrix $\mathrm{Cat}_k(F)$. In particular, Ω_F is a non-degenerate quadratic form if and only if $\det \mathrm{Cat}_k(F) \neq 0$.

Definition 2.2. A homogeneous form $F \in S^{2k}V^*$ is called *nondegenerate* if Ω_F is a nondegenerate quadratic form on S^kV.

Theorem 2.3. *Assume that $F \in S^{2k}V^*$ is nondegenerate. There exists a unique homogeneous form $\check{F} \in S^{2k}V$ such that*

$$\Omega_{\check{F}} = \check{\Omega}_F.$$

Proof. We know that $\check{\Omega}(F)$ is defined by the cofactor matrix $\mathrm{adj}(\mathrm{Cat}_k(F)) = (c^*_{\mathbf{uv}})$ so that

$$\check{\Omega}_F = \sum c^*_{\mathbf{uv}} \xi^{\mathbf{u}} \xi^{\mathbf{v}}.$$

Let

$$\check{F} = \sum_{|\mathbf{u}+\mathbf{v}|=2k} \tfrac{d!}{(\mathbf{u}+\mathbf{v})!} c^*_{\mathbf{uv}} \xi^{\mathbf{u}+\mathbf{v}}.$$

Recall that the entries $c_{\mathbf{uv}}$ of the catalecticant matrix depend only on the sum of the indices. Thus the entries of the cofactor matrix $\mathrm{adj}(\mathrm{Cat}_k(F)) = (c^*_{\mathbf{uv}})$ depend only on the sum of the indices. For any $t^{\mathbf{i}} \in S^k V^*$, we have

$$D_{t^{\mathbf{i}}}(\check{F}) = \sum_{\mathbf{u},\mathbf{v},\mathbf{u}+\mathbf{v}\geq\mathbf{i}} \tfrac{d!}{(\mathbf{u}+\mathbf{v})!} c^*_{\mathbf{uv}} \tfrac{(\mathbf{u}+\mathbf{v})!}{(\mathbf{u}+\mathbf{v}-\mathbf{i})!} \xi^{\mathbf{u}+\mathbf{v}-\mathbf{i}} = \sum_{\mathbf{j},|\mathbf{j}|=k} \tfrac{d!}{\mathbf{j}!} c^*_{\mathbf{ij}} \xi^{\mathbf{j}}$$

This checks that the matrix of the linear map $S^k V^* \to S^k V$ defined by $\Omega_{\check{F}}$ is equal to the matrix $\mathrm{adj}(\mathrm{Cat}_k(F))$. Thus the quadratic form $\Omega_{\check{F}}$ is the dual of the quadratic form Ω_F. □

Recall that the locus of zeroes of the quadric $\check{Q}$ in V^* is equal to the set of linear functions of the form $l = b_Q(v)$ such that $\langle v, l\rangle = 0$. The same is true for the dual form $\check{F}$. Its locus of zeroes consists of linear forms l such that $\Omega_F^{-1}(l^k) \in S^k V$ vanishes on l. The degree k homogeneous form $\Omega_F^{-1}(l^k)$ is classically known as the *anti-polar* of l (with respect to F).

3. Sums of powers

3.1. The variety VSP$(F; N)$ of sums of powers

For any $F \in S^m V^*$ we denote by $V(F)$ the zero locus of F in the projective space $\mathbb{P}V \cong \mathbb{P}^n$. For any nonzero $v \in V$ we denote by $[v]$ the corresponding point $\mathbb{C}v$ in $\mathbb{P}V$. The same notation is used for the dual space $\mathbb{P}V^*$. Thus any linear form $l \in V^*$ defines a hyperplane $V(l)$ in $\mathbb{P}V$ and at the same time a point $[l]$ in $\mathbb{P}V^*$.

For any finite set of points $p_1, \ldots, p_N$ in $\mathbb{P}V$ we denote by $L_d(\mathbb{P}V, p_1, \ldots, p_N)$ the linear space of homogeneous forms F of degree d in $\mathbb{P}V$ such that $V(F)$ contains the points $p_1, \ldots, p_N$.

Definition 3.1. An unordered set of points $[l_1], \ldots, [l_N]$ in $\mathbb{P}V^*$ is called a *polar N-polyhedron* of $F \in S^d V^*$ if there exist some nonzero scalars $\lambda_1, \ldots, \lambda_N$ such that

$$F = \lambda_1 l_1^d + \ldots + \lambda_N l_N^d,$$

and moreover the powers l_i^d are linearly independent in $S^d V^*$.

Lemma 3.1. *A set $\mathcal{P} = \{[l_1], \ldots, [l_N]\}$ is a polar polyhedron of F if and only if*

$$L_d(\mathbb{P}V^*, [l_1], \ldots, [l_N]) \subset \mathrm{AP}_d(F) \tag{3.1}$$

and the above inclusion is not true anymore if we delete any $[l_i]$ from $\mathcal{P}$.

Proof. Observe that vanishing of $\Phi \in S^dV$ at $l_i \in V^*$ is equivalent to $\langle \Phi, l_i^d \rangle = 0$. Therefore

$$\langle l_1^d, \dots, l_N^d \rangle^\perp = L_d(\mathbb{P}V^*, [l_1], \dots, [l_N]).$$

The forms l_i^d are linearly independent in S^dV^* if and only if

$$\dim \langle l_1^d, \dots, l_N^d \rangle^\perp = \operatorname{codim}(\langle l_1^d, \dots, l_N^d \rangle, S^dV^*) = \dim S^dV^* - N.$$

Since passing through a point impose one linear condition on the coefficients of a form, we see that the forms l_i^d are linearly independent if and only if the points $[l_i]$ impose independent conditions on hypersurfaces of degree d.

Suppose the conditions of the lemma are satisfied. Then

$$F \in \mathrm{AP}_d(F)^\perp \subset L_d(\mathbb{P}V^*, [l_1], \dots, [l_N])^\perp = \langle l_1^d, \dots, l_N^d \rangle.$$

Thus F is a linear combination of the powers l_i^d. If $l_1^d, \dots, l_N^d$ are linearly dependent, $\langle \mathcal{P} \rangle = \langle \mathcal{P}' \rangle$ for some proper subset $\mathcal{P}'$ of $\mathcal{P}$. Then

$$\mathrm{AP}_d(F)^\perp \subset L_d(\mathbb{P}V^*, [l_1], \dots, [l_N])^\perp = \langle \mathcal{P}' \rangle$$

and hence $\langle \mathcal{P}' \rangle \subset \mathrm{AP}_d(F)$ contradicting the assumption. Thus $\mathcal{P}$ is a polar polyhedron of F.

Conversely, suppose $\mathcal{P}$ is a polar polyhedron of F. Then $F \in \langle \mathcal{P} \rangle$ and hence $\langle \mathcal{P} \rangle^\perp \subset \langle F \rangle^\perp = \mathrm{AP}_d(F)$. We have seen already that the second condition follows from the linear independence of the powers l_i^d. □

We denote by $\mathrm{VSP}(F; N)^o$ the subset of the symmetric power

$$(\mathbb{P}V^*)^{(N)} = (\mathbb{P}V^*)^N / S_N$$

which consists of polar N-polyhedra of F. By a standard argument from algebraic geometry, Lemma 3.1 implies that $\mathrm{VSP}(F; N)^o$ is a locally closed subset of $(\mathbb{P}V^*)^{(N)}$.

One can consider different compactifications of the set $\mathrm{VSP}(F; N)^o$. For example, we can view a polar polyhedron $\mathcal{P}$ as a reducible hypersurface $V(l_1 \cdots l_N)$ of degree N. Then $\mathrm{VSP}(F; N)^o$ becomes a locally closed subset of $\mathbb{P}(S^NV^*)$ and we define $\mathrm{VSP}(F; N)$ to be its closure in the projective space. Another approach is to view $(\mathbb{P}V^*)^{(N)}$ as the locus of reduced subschemes in the Hilbert scheme $(\mathbb{P}V^*)^{[N]}$ of 0-dimensional closed subschemes of $\mathbb{P}V^*$ of length N. Then we define $\mathrm{VSP}(F; N)$ to be the closure of $\mathrm{VSP}(F; N)^o$ in the Hilbert scheme. Points in the complementary set $\mathrm{VSP}(F; N) \setminus \mathrm{VSP}(F; N)^o$ can be called *generalized polar polyhedra* of F. In the case $n = 1$, the two compactifications coincide and generalized polar polyhedra admit the following interpretation. Let $P \in S^NV^*$ be a binary

form such that $V(P) \in \mathrm{VSP}(F; N)$. Write P as the product of linear forms $P = l_1^{k_1} \cdots l_s^{k_s}$ with $k_1 + \ldots + k_s = N$. Then

$$F = A_1 l_1^{d-k_1+1} + \cdots + A_s l_s^{d-k_s+1}$$

for some binary forms A_i of degree $k_i - 1$.

In the case when F is a sufficiently general form, the assertion of Lemma 3.1 can be extended to the case of generalized polar polyhedra: a closed subscheme $Z \in (\mathbb{P}V^*)^{[N]}$ belongs to $\mathrm{VSP}(F; N)$ if and only if the linear space $L_d(\mathbb{P}V^*, Z)$ of degree d forms vanishing on Z belongs to $\mathrm{AP}_d(F)$ and this is not true anymore for any proper closed subscheme of Z. This can be easily deduced from the fact that the limit ideal of the homogeneous ideal defining Z is saturated (see [7], 5.2).

Proposition 3.2. *Assume $n = 2$. For general $F \in S^dV^*$ the variety $\mathrm{VSP}(F; N)$ is either empty or a smooth irreducible variety of dimension $N(n+1) - \binom{n+d}{d}$.*

Proof. We consider the closure of $\mathrm{VSP}(F; N)^o$ in the Hilbert scheme $(\mathbb{P}V^*)^{[N]}$. Recall that $Z \in (\mathbb{P}V)^{[N]}$ is a generalized polar polyhedron of F if and only if $F \in L_d(\mathbb{P}V^*, Z)^\perp$ but this is not true for any proper closed subscheme Z' of Z. Consider the incidence variety

$$X = \{(Z, F) \in (\mathbb{P}V^*)^{[N]} \times S^dV^* : Z \in \mathrm{VSP}(F; N)\}.$$

It is known that the Hilbert scheme of 0-cycles of a nonsingular surface is nonsingular ([5]). Thus $(\mathbb{P}V^*)^{[N]}$ is nonsingular. Let U be the open subset of the first factor such that for any point $Z \in U$, $\dim L_d(\mathbb{P}V^*, Z) = \dim S^dV - N$. The fibre of the first projection over $Z \in U$ is an open Zariski subset of the linear space $L_d(\mathbb{P}V^*, Z)^\perp$. This shows that X is irreducible and nonsingular. The fibres of the second projection are the varieties $\mathrm{VSP}(F; N)$. Thus for an open Zariski subset of S^dV^* the varieties $\mathrm{VSP}(F; N)$ are nonsingular or empty.

□

3.2. The Waring rank of a homogeneous form

Since any quadratic form Q can be reduced to a sum of squares, one can characterize its rank as the smallest number r such that

$$Q = l_1^2 + \ldots + l_r^2$$

for some linear forms $l_1, \ldots, l_r$.

Definition 3.3. Let $F \in S^dV^*$. Its *Waring rank* $\mathrm{wrk}(F)$ is the smallest number r such that

$$F = l_1^d + \ldots + l_r^d \tag{3.2}$$

for some linear forms $l_1, \ldots, l_r$.

Proposition 3.4. *Let Ω_F be the quadratic form on S^kV associated to $F \in S^{2k}V^*$. Then the Waring rank of F is greater or equal than the rank of Ω_F.*

Proof. Suppose (3.2) holds with $d = 2k$. Since Ω_F is linear with respect to F, we have $\Omega_F = \sum \Omega_{l_i^{2k}}$. If we choose coordinates such that l_i is a coordinate function t_0, we easily compute the catalecticant matrix of l_i^{2k}. It is equal to the matrix with 1 at the upper left corner and zero elsewhere. The corresponding quadratic form is equal to $(t_0^k)^2$. Thus $\Omega_{l_i^{2k}} = (l_i^k)^2$ and we obtain

$$\Omega_F = \sum_{i=1}^{r} \Omega_{l_i^{2k}} = \sum_{i=1}^{r} (l_i^k)^2.$$

Thus the Waring rank of F is greater or equal than the rank of Ω_F. □

Corollary 3.5. *Suppose F is a nondegenerate form of even degree $2k$, then*

$$\mathrm{wrk}(F) \geq \tbinom{k+n}{n}.$$

A naive way to compute the Waring rank is by counting constants. Consider the map

$$s : (V^*)^r \to \mathbb{C}^{\binom{d+n}{n}}, \quad (l_1, \ldots, l_r) \mapsto \sum l_i^d. \tag{3.3}$$

If $r(n+1) \geq \binom{d+n}{n}$ one expects that this map is surjective and hence $\mathrm{wrk}(F) \leq r$ for general F. Here "general" means that the coefficients of F belong to an open Zariski subset of the affine space $\mathbb{C}^{\binom{d+n}{n}}$. It is a remarkable result of J. Alexander and A. Hirschowitz [1] that this is "almost always" true. The only exceptional cases when it is false and the map s fails to be surjective are the following cases:

- $n = 2, d = 2, r = 2, \mathrm{wrk}(F) = 3$;
- $n = 2, d = 4, r = 5, \mathrm{wrk}(F) = 6$;
- $n = 3, d = 4, r = 9, \mathrm{wrk}(F) = 10$;
- $n = 4, d = 3, r = 7, \mathrm{wrk}(F) = 8$;
- $n = 4, d = 4, r = 14, \mathrm{wrk}(F) = 15$;

Here we assume that F is a general form.

Proposition 3.6. *Let F be a general homogeneous form of even degree $2k$. Then*

$$\mathrm{wrk}(F) > \text{rank } \Omega_F$$

except in the following cases, where the equality takes place.

- $k = 1$;
- $n = 1$;
- $n = 2, k \leq 4$;
- $n = 3, k = 2$.

Proof. The first case is obvious. It follows from considering the map (3.3) that $\mathrm{wrk}(F) \geq \binom{n+2k}{n}/(n+1)$. On the other hand the rank of Ω_F for general F is equal to $\dim S^k V = \binom{n+k}{n}$.

We know that the case $n = 1$ is not exceptional so that we can compute the Waring rank of F by counting constants and get $\mathrm{wrk}(F) = k + 1 = \text{rank } \Omega_F$.

If $n = 2$, we get $\mathrm{wrk}(F) \geq (2k + 2)(2k + 1)/6 = (k + 1)(2k + 1)/3$ and rank $\Omega_F = \binom{k+2}{2} = (k + 2)(k + 1)/2$. We have $(k + 1)(2k + 1)/3 > (k + 2)(k + 1)/2$ if $k > 4$. By Alexander-Hirschowitz's result,

$$\mathrm{wrk}(F) = \begin{cases} 6 & \text{if k = 2,} \\ 10 & \text{if k = 3,} \\ 15 & \text{if k = 3.} \end{cases}$$

This shows that $\mathrm{wrk}(F) = \text{rank } \Omega_F$ in all these cases.

If $n = 3$, we get

$$\mathrm{wrk}(F) \geq (2k + 3)(2k + 2)(2k + 1)/24 > \binom{k+3}{3} = (k + 3)(k + 2)(k + 1)/6$$

unless $k = 2$.

Finally, it is easy to see that for $n > 3$

$$\mathrm{wrk}(F) \geq \binom{2k+n}{n}/(n + 1) > \binom{k+n}{n}$$

for $k > 1$. □

Definition 3.7. Two linear forms $l, m \in V^*$ are called *conjugate* with respect to a nondegenerate form $F \in S^{2k}V^*$ if

$$\Omega_{\check{F}}(l^k, m^k) = \check{F}(l^k m^k) = 0.$$

The next result can be found in [9] (certainly used by classics, see, for example, [13]).

Proposition 3.8. *Suppose F is given by* (3.2)*, where the powers l_i^k are linearly independent in S^kV^*. Then each pair l_i, l_j is conjugate with respect to F.*

Proof. It follows from computation of Ω_F in the proof of Proposition 3.4 that it suffices to check the assertion for quadratic forms. Choose a coordinate system such that $l_i = t_0, l_j = t_1$ and $F = t_0^2 + t_2^2 + \ldots + t_n^2$. Then $\check{F} = \xi_0^2 + \ldots + \xi_n^2$, where $\xi_0, \ldots, \xi_n$ are dual coordinates. Now the assertion is easily checked. □

3.3. Mukai's skew-symmetric form

Let $\omega \in \Lambda^2 V$ be a skew-symmetric bilinear form on V^*. It admits a unique extension to a Poisson bracket $\{,\}_\omega$ on $S^\bullet V^*$ which restricts to a skew-symmetric bilinear form

$$\{,\}_\omega : S^{k+1}V^* \times S^{k+1}V^* \to S^{2k}V^*.$$

Recall that a Poisson bracket on a commutative algebra A is a skew-symmetric bilinear map $A \times A \to A, (f, g) \mapsto \{f, g\}$ such that its left and right partial maps $A \to A$ are derivations.

Let $F \in S^{2k}V^*$ be a nondegenerate form and $\check{F} \in S^{2k}V = (S^{2k}V^*)^*$ be its dual form. For each ω as above define $\sigma_{\omega,F} \in \Lambda^2(S^{k+1}V)^*$ by

$$\sigma_{\omega,F}(f, g) = \check{F}(\{f, g\}_\omega).$$

Theorem 3.9 (S. Mukai). *Let F be a nondegenerate form in $S^{2k}V^*$.For any $\mathcal{P} = \{[l_1], \ldots, [l_N]\} \in \mathrm{VSP}(F; N)^o$ let $E(\mathcal{P})$ be the linear span of the powers l_i^{k+1} in $S^{k+1}V^*$. Then*

(i) *$E(\mathcal{P})$ is isotropic with respect to each form $\sigma_{\omega,F}$;*
(ii) *$\mathrm{ap}_F^{k-1}(S^{k-1}V) \subset E(\mathcal{P})$;*
(iii) *$\mathrm{ap}_F^{k-1}(S^{k-1}V)$ is contained in the radical of each $\sigma_{\omega,F}$.*

Proof. To prove the first assertion it is enough to check that $\sigma_{\omega,F}(l_i^{k+1}, l_j^{k+1}) = 0$ for all i, j. We have

$$\sigma_{\omega,F}(l_i^{k+1}, l_j^{k+1}) = \check{F}(\{l_i^k, l_j^k\}_\omega) = \check{F}(l_i^k l_j^k)\omega(l_i, l_j).$$

By Proposition 3.8, $\check{F}(l_i^k l_j^k) = \Omega_{\check{F}}(l_i^k, l_j^k) = 0$. This checks the first assertion.

For any $\Phi \in S^{k-1}V$,

$$D_\Phi(F) = D_\Phi(\sum_{i=1}^N l_i^{2k}) = \sum D_\Phi(l_i^{2k}) = \tfrac{(2k)!}{(k+1)!} \sum_{i=1}^N D_\Phi(l^{k-1}) l_i^{k+1}.$$

This shows that $\mathrm{ap}_F^{k-1}(S^{k-1}V)$ is contained in $E(\mathcal{P})$. It remains to check that for any $\Phi \in S^{k-1}V, G \in S^{k+1}V^*$ and any $\omega \in \Lambda^2 V$, one has $\sigma_{\omega,F}(D_\Phi(F), G) = 0$. Choose coordinates $t_0, \dots, t_n$ in V and the dual coordinates $\xi_0, \dots, \xi_n$ in V^*. The space $\Lambda^2 V$ is spanned by the forms $\omega_{ij} = \xi_i \wedge \xi_j$. We have

$$\{D_\Phi(F), G\}_{\omega_{ij}} = D_{\xi_i}(D_\Phi(F))D_{\xi_j}(G) - D_{\xi_j}(D_\Phi(F))D_{\xi_i}(G)$$

$$= D_{\xi_i\Phi}(F)D_{\xi_j}(G) - D_{\xi_j\Phi}(F)D_{\xi_i}(G) = D_{\Phi\xi_i}(F)D_{\xi_j}(G) - D_{\Phi\xi_j}(F)D_{\xi_i}(G).$$

For any $A, B \in S^k V^*$,

$$\check{F}(AB) = \Omega_{\check{F}}(A, B) = \langle \Omega_F^{-1}(A), B\rangle.$$

Thus

$$\sigma_{\omega_{ij},F}(D_\Phi(F), G) = \check{F}(D_{\Phi\xi_i}(F)D_{\xi_j}(G) - D_{\Phi\xi_j}(F)D_{\xi_i}(G))$$

$$= \langle \Phi\xi_i, D_{\xi_j}(G)\rangle - \langle \Phi\xi_j, D_{\xi_i}(G)\rangle = D_\Phi(D_{\xi_i\xi_j}(G) - D_{\xi_j\xi_i}(G)) = D_\Phi(0) = 0.$$

□

3.4. The Mukai map

Let $F \in S^{2k}V^*$ be a nondegenerate form. We assume that (k, n) is one of the exceptional cases from Proposition 3.6.

Let

$$N_k = \text{rank } \Omega_F = \binom{n+k}{n}. \tag{3.4}$$

We know that $\mathrm{VSP}(F; N_k)^o \neq \emptyset$ for general enough F.

Let $\mathcal{P} \in \mathrm{VSP}(F; N_k)^o$ and

$$\bar{E}(\mathcal{P}) = \langle l_1^{k+1}, \dots, l_{N_k}^{k+1}\rangle / \mathrm{ap}_F^{k-1}(S^{k-1}V).$$

It is a subspace of $W = S^{k+1}V^*/\mathrm{ap}_F^{k-1}(S^{k-1}V)$. By Lemma 2.1,

$$W^* = \mathrm{ap}_F^{k-1}(S^{k-1}V)^\perp \cong \mathrm{AP}_{k+1}(F),$$

hence

$$W \cong \mathrm{AP}_{k+1}(F)^*$$

and

$$\bar{E}(\mathcal{P}) \subset \mathrm{AP}_{k+1}(F)^*.$$

Lemma 3.2. (i) $\dim \mathrm{AP}_{k+1}(F)^* = \binom{n+k}{n-1} + \binom{n+k-1}{n-1}$;
(ii) $\dim E(\mathcal{P}) = N_k$;
(iii) $\dim \bar{E}(\mathcal{P}) = \binom{n+k-1}{n-1}$.

Proof. (i) Since F is nondegenerate, $\mathrm{Ker}(\mathrm{ap}_F^k) = \{0\}$, hence $\mathrm{Ker}(\mathrm{ap}_F^{k-1}) = \{0\}$ and

$$\dim \mathrm{ap}_F^{k-1}(S^{k-1}V) = \dim S^{k-1}V^* = \tbinom{k-1+n}{n}.$$

This gives

$$\begin{aligned}\dim \mathrm{AP}_{k+1}(F)^* &= \dim S^{k+1}V - \dim \mathrm{ap}_F^{k-1}(S^{k-1}V) = \tbinom{n+k+1}{n} - \tbinom{n+k-1}{n} \\ &= \tbinom{n+k}{n-1} + \tbinom{n+k-1}{n-1}.\end{aligned}$$

(ii) Let $\mathcal{P} = \{[l_1], \dots, [l_{N_k}]\}$. We have to show that the forms $l_1^{k+1}, \dots, l_{N_k}^{k+1}$ are linearly independent. This is equivalent to the equality

$$\dim L_{k+1}(\mathbb{P}V^*, [l_1], \dots, [l_{N_k}]) = \tbinom{k+1+n}{n} - N_k = \tbinom{k+n}{n-1}.$$

This is obvious in the case $n = 1$. If $k = 1$, the space $L_2(\mathbb{P}V^*, [l_1], \dots, [l_{N_k}])$ is of expected dimension unless $l_1^2, \dots, l_{N_k}^2$ are linearly dependent. In our case this is impossible because F is nodegenerate.

Assume $n = 2, k = 2$ and $\dim L_3(\mathbb{P}V^*, [l_1], \dots, [l_6]) > 4$. Since $\mathrm{AP}_2(F) = \{0\}$, no conic passes through the 6 points. In particular, no 4 points are collinear. Take a conic $C = V(Q)$ through 5 points $[l_1], \dots, [l_5]$ and 2 points x_1, x_2 such that each component of C contains ≥ 4 points (if C is reducible we take x_1 to be the intersection point of the two line components). Since

$$\dim L_3(\mathbb{P}V^*, [l_1], \dots, [l_6], x_1, x_2) > 2,$$

there exist 3 linearly independent cubic forms Φ_i such that $V(\Phi)$ has 7 common points with C. Applying Bézout's Theorem, we see that the cubics contain C, hence we get a 2-dimensional space of lines passing through $[l_6]$. This is impossible.

Assume $n = 2, k = 3$ and $\dim L_4(\mathbb{P}V^*, [l_1], \dots, [l_{10}]) > 5$. Again there are no cubics through the ten points. In particular, no 5 points are collinear, and no 8 points are on a conic. We argue as before. Choose a cubic C through the points $[l_i], i \neq 10$, and 3 points x_1, x_2, x_3 such that at least 5 points lie on each irreducible component of C. Since $\dim L_4(\mathbb{P}V^*, [l_1], \dots, [l_{10}], x_1, x_2, x_3) > 2$, there exist 3 linearly independent quartic forms Φ_i such that $V(\Phi)$ has 13 common points with C. Applying Bézout's Theorem, we see that the quartics contain C, hence we get a 2-dimensional space of lines passing through $[l_{10}]$. This is impossible.

A similar argument checks the assertion in the case $n = 2, k = 4$.

The last case $n = 3, k = 2$ needs to apply more techniques from algebraic geometry which we allow ourselves to use here. So, we skip the proof.

(iii) We have

$$\dim \bar{E}(\mathcal{P}) = N_k - \dim \mathrm{ap}_F^{k-1}(S^{k-1}V) = \tbinom{n+k}{n} - \tbinom{n+k-1}{k} = \tbinom{n+k-1}{n-1}. \quad (3.5)$$

□

We denote by $G(a, E)$ the Grassmannian of a-dimensional subspaces of a linear space E. Consider the regular map (we call it the *Mukai map*)

$$\mathrm{Mu} : \mathrm{VSP}(F; N_k)^o \to G(\tbinom{n+k-1}{k}, \mathrm{AP}_{k+1}(F)^*), \quad \mathcal{P} \mapsto \bar{E}(\mathcal{P}). \quad (3.6)$$

Proposition 3.10. *The Mukai map is injective.*

Proof. Suppose $\mathrm{Mu}([l_1], \dots, [l_{N_k}]) = \mathrm{Mu}([m_1], \dots, [m_{N_k}])$. Then

$$\langle l_1^{k+1}, \dots, l_{N_k}^{k+1} \rangle = \langle m_1^{k+1}, \dots, m_{N_k}^{k+1} \rangle,$$

hence

$$L_{k+1}(\mathbb{P}V^*, [l_1], \dots, [l_{N_k}]) = L_{k+1}(\mathbb{P}V^*, [m_1], \dots, [m_{N_k}]). \quad (3.7)$$

Without loss of generality we may assume that $[l_1]$ is not equal to any $[m_i]$. Since $\dim L_k(\mathbb{P}V^*, [l_2], \dots, [l_{N_k}]) > 0$ we can find a degree k form Φ vanishing at the last $N_k - 1$ points. Let $L \in V$ be a linear form on V^* which vanishes at $[l_1]$ but does not contain any of the points $[m_1], \dots, [m_{N_k}]$. Since $L\Phi \in L_{k+1}(\mathbb{P}V^*, [l_1], \dots, [l_{N_k}])$, we obtain that $L\Phi$ vanishes at any $[m_i]$, hence Φ vanishes at any $[m_i]$. Thus $\mathrm{AP}_k(F) \neq \{0\}$ which contradicts the nondegeneracy of F.

Therefore we may assume that $[l_1] = [m_1]$. Now if $[l_2] \neq [m_i]$ for any $i \geq 2$, we repeat the argument replacing l_1 with l_2 and obtain a contradiction. Proceeding in this way we show that the sets $\mathcal{P}$ and $\mathcal{P}'$ coincide. □

Remark 3.11. The same proof works for generalized polar polyhedra and general polynomial F. The analog of equality (3.7) is the equality $L_{k+1}(\mathbb{P}V^*, Z) = L_{k+1}(\mathbb{P}V^*, Z')$ of the spaces of polynomials of degree $k+1$ vanishing on Z and Z'. Suppose $Z \neq Z'$. Choose a subscheme Z_0 of Z of length $N_k - 1$ which is not a subscheme of Z'. Since $\dim L_k(\mathbb{P}V^*, Z_0) \geq \binom{n+k}{k} - N_k + 1 > 0$, we can find a nonzero $\Phi \in L_k(\mathbb{P}V^*, Z_0)$. The sheaf $\mathcal{I}_Z/\mathcal{I}_{Z_0}$ is concentrated at one point x and is annihilated by the maximal ideal $\mathfrak{m}_x$. Thus $\mathfrak{m}_x\mathcal{I}_{Z_0} \subset \mathcal{I}_Z$. Let L be a linear form vanishing at x but not vanishing at any subscheme of Z'. This implies that $L\Phi \in L_{k+1}(\mathbb{P}V^*, Z) = L_{k+1}(\mathbb{P}V^*, Z')$ and hence $\Phi \in L_k(\mathbb{P}V^*, Z')$ contradicting the nondegeneracy of F. Using more techniques from algebraic geometry one can also prove that the Mukai map is a closed embedding.

Recall that we have a linear map

$$\Lambda^2 V \to \Lambda^2 S^{k+1} V = (\Lambda^2 S^{k+1} V^*)^*, \quad \omega \mapsto \sigma_{\omega,F}.$$

Since, $\mathrm{ap}_F^{k-1}(S^{k-1})$ is contained in the radical of each $\sigma_{\omega,F}$ this map defines an injective map

$$\Lambda^2 V \to \Lambda^2 \mathrm{AP}_{k+1}(F). \tag{3.8}$$

Let $\mathcal{N} \subset \Lambda^2 \mathrm{AP}_{k+1}(F)$ be the image of this map. This is a linear subspace of the space of 2-forms on $\mathrm{AP}_{k+1}(F)^*$. For each $\mathcal{P} \in \mathrm{VSP}(F; N_k)$, the subspace $\bar{E}(\mathcal{P})$ is isotropic with respect to any $\sigma_{\omega,F}$. Let

$$G(\tbinom{n+k-1}{k}, \mathrm{AP}_{k+1}(F)^*)_{\mathcal{N}} \subset G(\tbinom{n+k-1}{k}, \mathrm{AP}_{k+1}(F)^*)$$

be the subvariety of the Grassmannian which consists of isotropic subspaces with respect to all 2-forms from $\mathcal{N}$. Recall that a Grassmannian $G = G(s, E)$ admits a tautological exact sequence of vector bundles

$$0 \to \mathcal{S}_G \to E_G \to \mathcal{Q}_G \to 0, \tag{3.9}$$

where E_G is the trivial bundle with fibre E, and $\mathcal{S}_G$ is the universal subbundle whose fibre at a point $x \in G$ is the linear s-dimensional subspace of E corresponding to x. A 2-form α on E defines, by restriction, a section σ_α of of $\Lambda^2 \mathcal{S}_G$. Its set of zeroes $Z(\sigma_\alpha)$ is a closed subvariety of G of codimension $\leq \binom{s}{2}$ and the equality holds for a Zariski open subset (maybe empty) of sections (see [6]). It follows that for any t-dimensional subspace L of sections, the set of common zeroes is of codimension $\leq t\binom{s}{2}$, and the equality holds for a Zariski open subset of the Grassmann variety of t-dimensional spaces of sections.

Returning to our case, we see that

$$\mathrm{Mu}(\mathrm{VSP}(F; N_k)) \subset G(\tbinom{n+k-1}{k}, \mathrm{AP}_{k+1}(F)^*)_{\mathcal{N}}.$$

The dimension of the left-hand side is equal to

$$\dim \mathrm{VSP}(F; N_k) = (n+1)\tbinom{n+k}{n} - \tbinom{n+2k}{n}$$

$$= \begin{cases} 1 & \text{if } n = 1, \\ \binom{n+1}{2} & \text{if } k = 1, \\ 3 & \text{if } n = 2, k = 2, \\ 2 & \text{if } n = 2, k = 3, \\ 0 & \text{if } n = 2, k = 4, \\ 5 & \text{if } n = 3, k = 2. \end{cases}$$

It is known that dim $G(a,E) = a(\dim E - a)$. In our case

$$\dim G(\tbinom{n+k-1}{k}, \mathrm{AP}_{k+1}(F)^*) = \tbinom{n+k-1}{n-1}\tbinom{n+k}{n-1}.$$

It follows from above that the expected dimension of $G(\binom{n+k-1}{k}, \mathrm{AP}^*_{k+1})_{\mathcal{N}}$ is equal to dim $VSP(F;N_k)$ in the cases $n=1$ and $n=2$. In all other cases, it is strictly less than dim $VSP(F;N_k)$.

We already know that in the case $n=1$,

$$\mathrm{VSP}(F,k+1) = \mathbb{P}^1 \cong G(1,\mathrm{AP}_{k+1}(F))_{\mathcal{N}} = G(1,\mathrm{AP}_{k+1}(F)).$$

The following theorem of Mukai says that the same is true for $n=2, k\le 4$.

Theorem 3.12. *(S. Mukai) Let $F_d \in S^dV^*$ be a general polynomial of degree $d=2k$. Assume that $n=2, k\le 4$. Then*

$$\mathrm{VSP}(F;\tbinom{k+2}{2}) \cong G(k+1,\mathrm{AP}_{k+1}(F)^*)_{\mathcal{N}}.$$

(i) *The variety* $\mathrm{VSP}(F_2;3)$ *is a smooth Fano 3-fold (of genus* 21 *and index 2);*
(ii) *The variety* $\mathrm{VSP}(F_4;6)$ *is a smooth Fano 3-fold (of genus 12 and index 1);*
(iii) *The variety* $\mathrm{VSP}(F_6;10)$ *is a smooth K3 surface;*
(iv) *The variety* $\mathrm{VSP}(F_8;15)$ *is a set of 16 points.*

Proof. By Proposition 3.2 the varieties $\mathrm{VSP}(F;\binom{k+2}{2})$ are smooth of the asserted dimension. Let $\mathcal{S}_G$ be the tautological vector bundle of rank $k+1$ on

$$G(k+1,\mathrm{AP}_{k+1}(F)^*) \cong \begin{cases} G(2,5) & \text{if } k=1\ , \\ G(3,7) & \text{if } k=2\ , \\ G(4,9) & \text{if } k=3\ , \\ G(5,11) & \text{if } k=4\ . \end{cases}$$

A basis of a 3-dimensional space $\mathcal{N}$ of sections of $\mathcal{E} = \Lambda^2\mathcal{S}^*_G$ defines a section s of the vector bundle $\mathcal{E}^{\oplus 3}$. It is known that the vector bundle $\mathcal{E}$ is generated by global sections. By Bertini's Theorem for sections of vector bundles (see [11]) the zero locus $Z(s)$ of its general section s is a smooth and of the expected codimension equal to $3\binom{k+1}{2}$. One can also show, using vanishing of cohomology of exterior powers of $\mathcal{S}^*_G$ that $Z(s)$ is connected if it is of positive dimension. Assume that $k<4$ so that $\dim Z(s) > 0$. It is known that the determinant of the tangent bundle of the Grassmannian $G = G(s,n)$ is equal to $c_1(G) = nc_1(S^*_G)$. The determinant of $\Lambda^2\mathcal{S}^*_G$ is equal

to $(s-1)c_1(\mathcal{S}_G^*)$. The normal bundle of $Z(s)$ in G is isomorphic to $\mathcal{E}^{\oplus 3}$. Thus, in our case,

$$c_1(\mathcal{E}^{\oplus 3}) = 3kc_1(\mathcal{S}_G^*),\ c_1(G) = (2k+3)c_1(\mathcal{S}_G^*).$$

By adjunction formula,

$$c_1(Z(s)) = (3-k)c_1\big(\mathcal{O}_{Z(s)}(1)\big),$$

where $\mathcal{O}_{Z(s)}(1)$ is equal to the restriction of $\det \mathcal{S}_G^*$ to $Z(s)$. Its sections are hyperplane sections of $Z(s)$ in the Plücker embedding of the Grassmannian. Thus $Z(s)$ is a Fano 3-fold if $k = 1$ or 2 and a K3 surface if $k = 3$. If $k = 4$, $\dim Z(s) = 0$. The rank of $\mathcal{E}^{\oplus 3}$ is equal to $30 = \dim G(5, 11)$. Its general section vanishes at the number of points equal to the Chern number $c_{30}(\mathcal{E}^{\oplus 3})$. Easy computation shows that this number is equal to 16.

We know that, for general F, the image of $\mathrm{VSP}(F; N_k)$ in $G(k+1, \mathrm{AP}_{k+1}\ (F)^*)$ is smooth, irreducible, and of codimension $3\binom{k+1}{2}$. It is also contained in $Z(s)$, where s is a section of $\mathcal{E}^{\oplus 3}$. We finish the proof if we can show that for a general F the equality takes place. Unfortunately, I don't know how to do this for $k = 3, 4$ (a proof using sysygies can be found in [12]). So let us assume that $k = 1$ or 2.

Fix a linear space E of dimension $2k+3$. Suppose we have a 3-dimensional linear subspace L of $\Lambda^2 E$. We have a natural linear map

$$\gamma : S^{k+1}L \to \Lambda^{2k+2}E,\ (v_1, \dots, v_{k+1}) \mapsto v_1 \wedge \dots \wedge v_{k+1}.$$

Let K be the kernel of γ. For a general $L \in G(3, \Lambda^2 E)$, the dimension of K is equal to $\frac{1}{2}(k+3)(k+2) - (2k+3) = \frac{1}{2}k(k+1)$. Let

$$\alpha : V \to L, \quad \beta : S^{k-1}V \to K$$

be some linear isomorphisms such that the injection homomorphism $K \hookrightarrow S^{k+1}L$ is isomorphic to the injective homomorphism $\iota : S^{k-1}V \hookrightarrow S^{k+1}V^*$. Suppose we can choose α and β such that this inclusion corresponds to a totally symmetric tensor

$$t \in S^{k-1}V^* \otimes S^{k+1}V^*.$$

Let $F \in S^{2k}V^*$ be the corresponding homogeneous form on V. It is easy to see that the homomorphsism ι coincides with the homomorphism ap_F^{k-1}. Then the space $\Lambda^{2k+2}E \cong E^*$ (after a choice of a volume form) and can be identified with the space $\mathrm{AP}_{k+1}(F)^* = S^{k+1}V^*/\mathrm{ap}_F^{k-1}(S^{k-1}V)$. Moreover, the subspace L of $\Lambda^2 E$ is identified with the subspace $\mathcal{N}$ defined by the Mukai 2-forms. This can be directly verified by choosing appropriate bases.

Thus $G(k+1, \mathrm{AP}_{k+1}(F)^*)_{\mathcal{N}}$ is isomorphic to a general subscheme $Z(s)$ and hence the assertion is verified for F, hence for any general F.

It remains to find α and β such that the tensor t is symmetric. This can be done by counting constants. The pairs (α, β) are parametrized by $\mathrm{GL}(3) \times \mathrm{GL}(\frac{1}{2}k(k+1))$, so depend on $a = 9 + k^2(k+1)^2/4$ parameters. The condition that a tensor t is symmetric is given by

$$b = \dim S^{k-1}V^* \otimes S^{k+1}V^* - \dim S^{2k}V^*$$
$$= \frac{(k+3)(k+2)(k+1)k}{4} - (k+1)(2k+1)$$

linear equations. Thus if $a > b$ we can always choose (α, β). If $k = 1$, there are no conditions at all. If $k = 2$ we have $a = 18, b = 15$ and it works too. If $k > 3$, $a < b$ and we are in trouble.

□

4. Reconstructing a polar polyhedron from its side

4.1. The variety of conjugate pairs

Let $F \in S^{2k}V$ be a nondegenerate form. Consider the *variety of conjugate pairs*

$$\mathrm{CP}(F) = \{([l], [m]) \in \mathbb{P}V^* \times \mathbb{P}V^* : \Omega_{\check{F}}(l^k, m^k) = 0\}.$$

For any $([l], [m]) \in \mathrm{CP}(F)$ we can find a unique (up to a scalar factor) pair $(\Phi, \Phi') \in S^kV \times S^kV$ of anti-polars, i.e.,

$$D_\Phi(F) = l^k, D_{\Phi'}(F) = m^k. \tag{4.1}$$

It follows from (2.8) that

$$\Phi(m) = \Phi'(l) = 0.$$

The equation $\Omega_{\check{F}}(l^k, m^k) = 0$ is bi-homogeneous of degree (k, k) in coefficients of l, m. For F general enough, it is a smooth $(2n-1)$-fold Y. By adjunction formula, its canonical class K_Y is a divisor class corresponding to a hypersurface of bi-degree $(k-n-1, k-n-1)$. So, Y is a *Fano variety* (i.e., the minus of the canonical class is ample) for $k \leq n$, it is a *Calabi-Yau variety* if $k = n+1$ (i.e. the canonical class is trivial) and a variety of general type (i.e. the canonical class is ample) if $k > 3$.

Consider the projections to the first and the second factor

$$q : \mathrm{CP}(F) \to \mathbb{P}V^*, \quad q' : \mathrm{CP}(F) \to \mathbb{P}V^*.$$

For any $[l] \in \mathbb{P}V^*$ the fibre $q^{-1}([l])$ is a hypersurface of degree k given by the equation $V(\Phi) = 0$, where $D_\Phi(F) = l^k$. The set of points $[l]$ such that $([l], [l]) \in \mathrm{CP}(F)$ is a hypersurface $\mathcal{D}$ of degree $2k$, it is given by the equation $V(\check{F}) = 0$.

Let

$$\widetilde{\mathrm{VSP}}(F; N) = \{([l], \mathcal{P}) \in \mathbb{P}V^* \times \mathrm{VSP}(F; N) : [l] \in \mathcal{P}\}.$$

The projection to the second factors is a finite map of degree N ramified over the complement of $\mathrm{VSP}(F; N)^o$. If we choose the Hilbert scheme compactification $\mathrm{VSP}(F; N)$, then $\widetilde{\mathrm{VSP}}(F; N)$ is the restriction of the universal scheme over $\mathrm{VSP}(F; N)$.

Consider the first projection

$$p : \widetilde{\mathrm{VSP}}(F; N) \to \mathbb{P}V^*.$$

Its fibre over a line $V(l)$ is the set of (generalized) polar polyhedra with the line $V(l)$ as its side.

4.2. The case $(n, d, N) = (2, 2, 3)$.

Let $([l], [m]) \in \mathrm{CP}(F), [l] \neq [m]$, and (Φ, Φ') be the corresponding pair of anti-polars (4.1). Let $[n]$ be the the intersection point of the corresponding lines $V(\Phi)$ and $V(\Phi')$. Obvioulsy $[n] \neq [l], [n] \neq [m]$ since $[l]$ and $[m]$ are not self-conjugate. For any linear forms L_1, L_2 such that $L_1(l) = L_2(m) = 0$, the conic $L_1\Phi_1 + L_2\Phi_2$ vanishes at the points $[l], [m], [n]$ and obviously belongs to $\mathrm{AP}_2(F)$. The dimension of the space of such conics is equal to 3. By Lemma 3.1, we obtain that $\{[l], [m], [n]\}$ is a polar triangle of F. Thus we obtain a regular map

$$f : \mathrm{CP}(F)^o \to \widetilde{\mathrm{VSP}}(F; 3), \ ([l], [m]) \mapsto ([n], \{[l], [m], [n]\})$$

where

$$\mathrm{CP}(F)^o = \mathrm{CP}(F) \setminus (\mathrm{CP}(F) \cap \text{diagonal}).$$

One can show that this map extends to the blow-up $\widehat{\mathrm{CP}(F)}$ of $\mathrm{CP}(F)$ along its intersection with the diagonal. It defines a regular map of degree 3

$$\widehat{\mathrm{CP}(F)} \to \widetilde{\mathrm{VSP}}(F; 3).$$

The involution $\iota : ([l], [m]) \mapsto ([m], [l])$ extends to $\widehat{\mathrm{CP}(F)}$ and the quotient $\widehat{\mathrm{CP}(F)}/\iota$ is a nonsingular hypersurface in the Hilbert scheme $(\mathbb{P}V)^{[2]}$. The

composition of the map f with the projection $\widetilde{\mathrm{VSP}}(F;3) \to \mathrm{VSP}(F;3)$ defines a regular map of degree 2

$$\widehat{\mathrm{CP}(F)}/\iota \to \mathrm{VSP}(F;3). \tag{4.2}$$

It is unramified over $\mathrm{VSP}(F;3)^o$. In particular, we see that any polar triangle of F is uniquely reconstructed from its two sides, and any line occurs as a side of one of polar triangles.

Remark 4.1. It is easy to see that the universal cover of $\mathrm{VSP}(F,3)^o$ is isomorphic to the algebraic group $\mathrm{SO}(3)$. The fundamental group is isomorphic to its Weyl group $G = 2^2 \rtimes S_3$ of order 24. The quotient by the normal subgroup $2^2 = (\mathbb{Z}/2)^2$ is the variety of ordered polar triangles of F. The quotient by the group $2^2 \rtimes 2$ is isomorphic to $\mathrm{CP}(F)^o$.

4.3. The case $(n,d,N) = (2,4,6)$.

Let $([l],[m]) \in \mathrm{CP}(F)^o$ and Φ, Φ' be their anti-polars of degree 2. Assume that Φ, Φ' have no common irreducible components. Consider the linear subspace L of S^4V formed by forms $A\Phi + A'\Phi'$, where $A, A' \in S^2V$ and $A(l) = 0, A'(m) = 0$. Since

$$D_{A\Phi+A'\Phi'}(F) = D_A(D_\Phi(F)) + D_{A'}(D_{\Phi'}(F)) = D_A(l^2) + D_{A'}(m^2) = 0$$

we see that

$$L \subset \mathrm{AP}_4(F).$$

Consider the map $(A, A') \mapsto A\Phi + A'\Phi'$. The kernel of this map consists of pairs (A, A') such that $A\Phi = -A'\Phi'$. Since no irreducible component of Φ' divides Φ, we obtain that Φ' divides A and hence $A = c\Phi', A = -c\Phi$. Thus the kernel of the map is one-dimensional and is spanned by $(\Phi', -\Phi)$. From this we obtain that $\dim L = 9$ and hence coincides with $\dim L_4(\mathbb{P}V^*, Z)$, where Z is the closed subscheme formed by the points $[l], [m]$ and the common zeroes of the conics $V(\Phi), V(\Phi')$. Thus, by Lemma 3.1, Z is a polar hexagon of F (a generalized one if $V(\Phi)$ does not intersect transversally $V(\Phi')$).

Thus, as in the previous case we obtain a rational map.

$$\widehat{\mathrm{CP}(F)}/\iota - \to \mathrm{VSP}(F;6). \tag{4.3}$$

It is of degree $15 = \binom{6}{2}$. Thus any polar hexagon of F is uniquely reconstructed from its two sides, and any line occurs as a side of one of polar hexagons.

Remark 4.2. Under the regular map $f : \mathrm{CP}(F)^o \to \widetilde{\mathrm{VSP}}(F;N)$ the fibres of any of the two projections $\mathrm{CP}(F)^o \to \mathbb{P}V^*$ are mapped isomorphically to the fibres of the projection $\widetilde{\mathrm{VSP}}(F;6) \to \mathbb{P}V^*$. This implies that a general point of $\mathrm{VSP}(F;6)$ lies on 6 rational curves. It can be shown that these curves are conics in the anti-canonical embedding $\mathrm{VSP}(F;6) \subset G(3,7) \subset \mathbb{P}^{20}$ (see [9]). It is known that the variety of conics on the Fano variety $\mathrm{VSP}(F;6)$ is isomorphic to the projective plane. According to [10], this plane can be identified with $\mathbb{P}V^*$ and the six conics through a point x represent the sides of the polar hexagon corresponding to the point x.

4.4. The case $(n,d,N) = (2,6,10)$.

The situation here is a little different from the previous cases.

Theorem 4.3. *Let F be a general homogeneous polynomial of degree 6 in 3 variables. Any line in $\mathbb{P}^2$ can be realized as a side of a polar polyhedron of F. For a general line, there are at most two polyhedra with the same side. For a general pair of lines there are at most one polyhedron which has these lines as its sides.*

Proof. We have to show that the projection $\widetilde{\mathrm{VSP}}(F;10) \to \mathbb{P}^2$ is surjective and of degree 2. Note that it cannot be of degree 1 since $\mathrm{VSP}(F;10)$ is a K3 surface and hence is not unirational. The assertion follows from the following fact which can be found in [2]: There are at most two polar polyhedra of F which share a common side $[l]$. Since I do not understand Dixon's proof I suggest my own proof of the statement. Suppose

$$F = l^6 + l_1^6 + \ldots + l_9^6 = \lambda l^6 + m_1^6 + \ldots + m_9^6.$$

Choose $\Phi, \Phi' \in S^3V$ such that $V(\Phi)$ vanishes at the points $[l_i]$ and $V(\Phi')$ vanishes at the points $[m_i]$. We can normalize them to assume that $\Phi(l) = \lambda\Phi'(l)$. Then

$$D_\Phi(F) = D_{\Phi'}(F) = 120\Phi(l)l^3,$$

hence $D_{\Phi-\Phi'}(F) = 0$. Since F is nondegenerate, $\mathrm{AP}_3(F) = \{0\}$, and thus $\Phi = \Phi'$. The equality

$$(\lambda - 1)l^6 = (l_1^6 + \ldots + l_9^6) - (m_1^6 + \ldots + m_9^6),$$

implies that $(\lambda - 1)D_\Phi(l^6) = 0$. If $\lambda \neq 1$ we get $D_\Phi(l) = 0$ and hence $D_\Phi(F) = 0$ contradicting the nondegeneracy of F. Thus $\lambda = 1$, and

$$l_1^6 + \ldots + l_9^6 = m_1^6 + \ldots + m_9^6.$$

Since $l_1^6, \ldots, l_9^6, m_1^6, \ldots, m_9^6$ span a subspace $L \subset S^6V^*$ of dimension ≤ 17, the dimension of the orthogonal complement of L in S^6V is of dimension

$\geq 28 - 17 = 11$. The space of forms divisible by Φ is of dimension 10. Thus there exists a form $G \in S^6V$ not vanishing on the cubic curve $V(\Phi)$ but vanishing at the 18 points $[l_i], [m_i]$ on $V(\Phi)$. Assume there is a third polar polyhedron of F with a side $[l]$. We have

$$F = l^6 + n_1^6 + \ldots + n_9^6 = l^6 + l_1^6 + \ldots + l_9^6 = l^6 + m_1^6 + \ldots + m_9^6,$$

and all 27 points $[l_i], [m_i], [n_i]$'s lie on the cubic $V(\Phi)$. Similar to above we get a form G' of degree 6 vanishing at $[m_i], [n_i]$'s and a form G'' of degree 6 vanishing at $[l_i], [n_i]$'s. This easily implies that the divisors $\sum[l_i]$, $\sum[m_i]$, $\sum[n_i]$ of degree 9 on the cubic $V(\Phi)$ are linear equivalent. Also we see that the divisor $\sum 2[l_i]$ is cut out by a sextic C. It can be chosen to have at least double points at $[l_i]$'s (see [4], Lemma 4.4). The pencil of sextics formed by C and $V(\Phi^2)$ is an Halphen pencil of elliptic sextics. Choose a member $V(K)$ of this pencil passing through the point $[l]$. Then we obtain that $D_K(F) = 0$, hence $\mathrm{AP}_6(F)$ contains a sextic with 9 double points.

To summarize, we obtain that for a general point $[l] \in \mathbb{P}V^*$, the variety $\mathrm{VSP}(F; 10)$ contains a polar polyhedron $[l], [l_1], \ldots, [l_9]$ such that the 9 points $[l_i]$ form the set of base points of an Halphen pencil of curves of degree 6. We call such a polyhedron an Halphen polyhedron. It is easy to see that the locus of such sets in $(\mathbb{P}V^*)^{(9)}$ is of codimension 1. Thus, for general F, the surface $\mathrm{VSP}(F; 10)$ intersects this divisor along a variety of dimension ≤ 1 (an easy count of constants shows that it cannot be contained in the divisor). Its pre-image in $\widetilde{\mathrm{VSP}}(F; 10)$ is of dimension ≤ 1. Its image under the projection $q : \widetilde{\mathrm{VSP}}(F; 10) \to \mathbb{P}V^*$ is of dimension ≤ 1. Thus for a general $[l]$, the fibre $q^{-1}([l])$ cannot not contain an Halphen polyhedron. This contradiction proves the first assertion.

Let us prove the second assertion. Assume that

$$F = l^6 + m^6 + l_1^6 + \ldots + l_8^6 = \lambda l^6 + \mu m^6 + m_1^6 + \ldots + m_8^6.$$

Let Φ be the cubic anti-polar of l, i.e. $D_\Phi(F) = l^3$ and similarly let Φ be the cubic anti-polar of m. By the nondegeneracy of F, the anti-polars are unique, hence $V(\Phi)$ is the cubic containing the points $[m], [l_1], \ldots, [l_8]$ and $V(\Phi')$ is the cubic containing the points $[l], [l_1], \ldots, [l_8]$. Thus the points $[l_1], \ldots, [l_8]$ are among 9 intersection points of the cubics $V(\Phi)$ and $V(\Phi')$. Using the second polar polyhedron, we obtain that the points $[m_1], \ldots, [m_8]$ are in the same set of 9 points. This implies that either the sets $[l_1], \ldots, [l_8]$ and $[m_1], \ldots, [m_8]$ coincide or have a common set of 7 points. In the first

case we are done. In the second case we may assume that

$$F = l^6 + m^6 + l_1^6 + \ldots + l_7^6 + l_8^6 = \lambda l^6 + \mu m^6 + \lambda_1 l_1^6 + \ldots + \lambda_7 l_7^6 + m_8^6.$$

Let $V(G)$ be a cubic passing through the points $[l], [m], [l_1], \ldots, [l_7]$. We have $D_G(F) = 120G(l_8)l_8^3 = 120G(m_8)m_8^3$. This implies $[l_8] = [m_8]$ and proves the assertion. □

Remark 4.4. Let F be a general form of degree 6 in 3 variables. It follows from the proof of the previous theorem that the variety $\widetilde{\mathrm{VSP}}(F, 10)$ is birationally isomorphic to a double cover of projective plane. It would be interesting to find the degree of the branch curve.

4.5. The case $(n, d, N) = (3, 4, 10)$.

Let F be a general quartic form in 4 variables. We have already remarked that this is one of the cases where the Waring rank reaches its minimum equal to 10. It is not even known the birational type of the variety $\mathrm{VSP}(F; 10)$. The arguments similar to ones used in the previous section show that a polar polyhedron of F can be reconstructed in at most two ways from a pair of its two faces (i.e. a pair of conjugate planes). Using the method of syzygies from [12], Michael Sagraloff [15] proves that it is exactly two ways. Note that this result was claimed by A. Dixon [2] and Reye [13] erroneously claimed that the reconstruction can be done in one way.

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