Rendiconti del Circolo Matematico di Palermo Series 2
https://doi.org/10.1007/s12215-024-01078-4

Monoidal and submonoidal surfaces, and Cremona transformations

Igor V. Dolgachev[1]

Received: 5 January 2024 / Accepted: 4 June 2024

Abstract
We study irreducible surfaces of degree d in $\mathbb{P}^3$ that contain a line of multiplicity $d-1$ (monoidal surfaces) or $d-2$ (submonoidal surfaces). We relate them to congruences of lines and Cremona transformations. Many of our results are not new and can be found in classical literature, we give them modern proofs. In the last section, we extend some of our results to hypersurfaces of arbitrary dimension. We define two commuting Cremona involutions in the ambient space associated to a linear subspace of multiplicity $d-2$ contained in the hypersurface. Both leave the hypersurface invariant, but one acts as the identity on the hypersurface.

Keywords Monoidal surface · Submonoidal surface · Conic bundles · Cremona transformatiions

Mathematics Subject Classification 14J26 · 14E07

Contents

To the memory of Yuri I. Manin

✉ Igor V. Dolgachev
idolga@umich.edu

[1] Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA

Published online: 18 June 2024

Introduction

An irreducible reduced hypersurface Φ_d in $\mathbb{P}^n$ of degree $d > 1$ is called a *monoidal hypersurface* (resp. a *submonoidal* hypersurface) if it contains a point of multiplicity $d-1$ (resp. $d-2$).

In the present paper, we study monoidal (submonoidal) surfaces Φ_d of degree $d \geq 3$, which contain a line Γ of multiplicity $d-1$ (resp. $d-2$).

In the next section, we will discuss canonical equations of monoidal surfaces. If $d = 3$, these equations give the classical canonical equations of two non-isomorphic ruled cubic surfaces that are not cubic cones. In the case $d = 4$, they are ruled surfaces with a triple line. We find their canonical equations that simplify the equations found in [7] or [21]. A monoidal surface is an example of a rational ruled surface in $\mathbb{P}^3$. We discuss possible normalizations of a monoidal surface, and also a construction of a monoidal surface of degree d as a projection of a minimal ruled surface embedded in $\mathbb{P}^{d+1}$.

In Sect. 2, we study monoidal Cremona transformations defined by homaloidal 3-dimensional linear systems of monoidal surfaces with their singular line contained in the base locus. An example of a monoidal transformation is a quadro-cubic Cremona transformation defined by the linear system of cubic surfaces containing a fixed double line and three isolated points.

Section 3 introduces a condition of the non-degeneracy of a submonoidal surface. This condition is implicitly assumed in the classical literature. A non-degenerate submonoidal surface only has nodes as isolated singularities and its normalization is isomorphic to the proper transform in the blow-up of the singular line. The proper transform admits a structure of a conic bundle, and we study its possible singularities. Using this, we reprove the classical result that a non-degenerate submonoidal surface admits at most $3d-4$ nodes as its isolated singular points. A quartic surface with a double line and eight nodes is the Plücker Complex Surface, which arises in the theory of quadratic line complexes.

In Sect. 4, we extend the classical construction of a birational model of quartic surface with a double line as the blow-up of nine points in the plane. We show that a submonoidal surface of degree d is isomorphic to the image of the plane under a rational map defined by the linear system of plane curves of degree d passing simply through $3d-4$ points $p_1, \ldots, p_{3d-4}$ and with multiplicity $d-2$ through an additional point p_0. The line Γ is the image of a unique curve of degree $d-1$ that passes through $p_1, \ldots, p_{3d-4}$ and has p_0 as a point of multiplicity $d-3$. We use the blow-up model to find special sections of the conic bundle on a submonodidal surface of degree d. These are rational curves of degree $d-2$ that intersect Γ at $d-3$ points. We show that there are 2^{3d-5} such curves and they come in pairs, the union of the curves in a pair is cut out by a hypersurface of degree $d-3$. We associate to a special section S a congruence of lines $\mathcal{C}_S$ of order one and class $d-2$. The surface $\mathcal{C}_S$ is a ruled surface contained in the Grassmannian $G_1(\mathbb{P}^3)$, and we relate sections of the conic bundle on Φ_d to directrices of the ruled surface.

In Sect. 5, using the de Jonquières birational transformation of $\mathbb{P}^2$ associated with the hyperelliptic curve Σ, we construct a canonical Cremona involution of $\mathbb{P}^3$ that leaves the surface invariant and induces a de Jonquères involution of the blow-up plane model of the surface.

One can extend some results of the paper to higher dimensional hypersurfaces in $\mathbb{P}^n$ of degree d that contain a linear subspace Γ with multiplicity $d-2$. We do not pursue this in this paper; however, in the last section, we briefly discuss such hypersurfaces and introduce the satellite polar variety canonically associated with them. We construct a Cremona transformations Θ_Γ of $\mathbb{P}^n$ that leaves the hypersurfaces invariant; it fixes pointwise Γ and the satellite polar variety. As in the case of surfaces, there is another Cremona involution Θ'_Γ that fixes the hypersurface pointwise. In the case of cubic hypersurfaces with zero-dimensional Γ, Yuri Ivanovich Manin was the first who studied the transformations Θ_Γ [17].

We work over an algebraically closed field of characteristic zero, although many results do not need the latter assumption. We leave it to the reader to pinpoint the places where we use this assumption.

A large part of the paper should be considered a modern exposition of facts found in classical literature, for example, in monographs [13, 21, 24]. It would be hard to give precise references.

The author thanks Jérémy Blanc, Ivan Cheltsov, and Alessandro Verra for helpful comments on the first draft of the paper.

1 Monoidal surfaces

1.1 Canonical equations

Let Φ_d be a monoidal surface containing a line Γ with multiplicity $d-1$. We identify the pencil of planes through Γ with the dual line $\Gamma^\perp$ in the dual $\mathbb{P}^3$. If Φ_d contains a singular point x_0 outside Γ, then every line through x_0 that intersects Γ is contained in the surface, and hence, Φ_d is a cone. We will exclude this case.

Intersecting Φ_d with planes from the pencil $\Gamma^\perp$ shows that Φ_d is a ruled surface with the ruling defined by the residual lines.

We choose coordinates such that $\Gamma = V(x_0, x_1)$ and write the equation of Φ_d as

$$F_d = A_d(x_0, x_1) + x_2 B_{d-1}(x_0, x_1) + x_3 C_{d-1}(x_0, x_1) = 0, \tag{1.1}$$

where A_d, B_{d-1} and C_{d-1} are binary forms of degrees indicated by the subscripts. Since Φ_d is irreducible, the binary forms A_d, B_{d-1}, C_{d-1} are mutually coprime. If B_{d-1} and C_{d-1} are proportional, say $B_{d-1} = t_0 C_{d-1}$, then, replacing x_2 with $x_2 + t_0 x_3$, we obtain that Φ_d is a cone with vertex $[0:0:0:1]$. We excluded this case.

It follows from the adjunction formula (4.1) that any automorphism of a monoidal or submonoidal surface comes from a projective automorphism. Since the group of projective automorphisms of $\mathbb{P}^3$ that leaves invariant a line is of dimension 11, the dimension of the moduli space of monoidal surfaces of degree $d \geq 4$ is equal to $3d - 11$.

Example 1.1 Assume $d = 3$. If B_2 and C_2 are coprime binary quadrics, we can find linear forms L_1 and L_2 such that $A_3 = L_1 B_2 + L_2 C_2$. This eliminates A_d and, after reducing C_2 and C_3 to a sum of squares, we reduce the equation to the form

$$x_0^2 x_2 + x_1^2 x_3 = 0. \tag{1.2}$$

If B_2 and C_2 share a root, we may assume that $x_0 | B_2$ and $x_0 | C_2$. We get

$$x_1^3 + x_0 L_1(x_0, x_1) x_2 + x_0 L_2(x_0, x_1) x_3 = 0,$$

and we can further assume, after a linear change of x_2 and x_3, that $L_1 = x_1$ and $L_2 = x_0$. We arrive at the canonical equation

$$x_1^3 + x_0 x_1 x_2 + x_0^2 x_3 = 0. \tag{1.3}$$

Equations (1.2) and (1.3) are well-known canonical equations of non-conical cubic ruled surfaces (see [6, Theorem 9.2.1]).

We could use a further reduction of canonical equations by applying the theory of invariants of pencils of binary forms. A linear change of variables (x_1, x_2) changes a basis in the pencil of binary forms of degree $d-1$ spanned by B_{d-1} and C_{d-1}. If these forms are coprime, they span a base-point-free pencil in the space $V(d-1)$ of binary forms of degree $d-1$. It can be considered as a point in the Grassmannian $G(2, V(d-1))$. The group SL(2) acts faithfully on $G(2, V(d-1))$ via its natural linear representation in $V(d-1)$. It is known that the orbit of the pencil depends only on the ramification scheme of the map $\mathbb{P}^1 \to \mathbb{P}^1$ of degree $d-1$ defined by the pencil [4]. A general orbit corresponds to a map with $2d-4$ simple ramification points; such orbits are parameterized by a variety of dimension $2(d-2)-3 = 2d-7$. After we fix an orbit, we can change the variables

$$(x_1, x_2) \mapsto (cx_1 + l_1(x_0, x_1), x_2 + l_2(x_0, x_1), \tag{1.4}$$

where l_1, l_2 are linear forms. This leaves us with $(2d-7)+(d+1-5) = 3d-11$ parameters that agree with the previous count of constants.

Example 1.2 Assume $d = 4$. A degree 3 map $f : \mathbb{P}^1 \to \mathbb{P}^1$ may have possible ramification schemes: $(2, 2, 2, 2)$, $(2, 2, 3)$, and $(3, 3)$. The orbits of pencils of cubic binary forms with four simple ramification points depend on one parameter. They can be represented by pencils of polar cubics of a general quartic binary form. A canonical equation of a general binary quartic form is $x_0^4 + 2\lambda x_0^2 x_1^2 + x_1^4$. Thus, we may assume that $B_3 = x_0^3 + \lambda x_0 x_1^2$ and $C_3 = x_1^3 + \lambda x_0^2 x_1$. The pencil defines a map $\mathbb{P}^1 \to \mathbb{P}^1$ with four simple ramification points if and only if $\lambda \neq 0, \pm 1, \pm 3$. In this case, we can use (1.4) to transform $A_4(x_0, x_1) = \sum_{i=0}^4 a_i x_0^{4-i} x_1^i$ to $x_0^2 x_1^2$ or to 0 (if $a_2 = \lambda(a_0 + a_4)$). This gives canonical equations of a general monoidal quartic surface

$$x_0^2 x_1^2 + (x_0^3 + \lambda x_0 x_1^2)x_2 + (x_1^3 + \lambda x_0^2 x_1)x_3 = 0, \quad \lambda^2 \neq 0, 1, 9. \tag{1.5}$$

$$(x_0^3 + \lambda x_0 x_1^2)x_2 + (x_1^3 + \lambda x_0^2 x_1)x_3 = 0, \quad \lambda^2 \neq 0, 1, 9. \tag{1.6}$$

If we take $\lambda = \pm 3$ (replacing x_0 with ix_0, we may assume $\lambda = 3$), the pencil has the ramification scheme $(2, 2, 3)$. There is a unique orbit with this ramification scheme, so we may take the pencil of polars of $x_0^4 + 6x_0^2 x_1^2 + x_1^4$ as its representative. This gives the normal forms

$$x_0^2 x_1^2 + (x_0^3 + 3x_0 x_1^2)x_2 + (x_1^3 + 3x_0^2 x_1)x_3 = 0, \tag{1.7}$$

$$(x_0^3 + 3x_0 x_1^2)x_2 + (x_1^3 + 3x_0^2 x_1)x_3 = 0. \tag{1.8}$$

If $\lambda = \pm 1$, we obtain a pencil with two base points (may be equal)) that gives canonical equations:

$$x_0^2 x_1^2 + (x_0^2 + x_1^2)(x_0 x_2 + x_1 x_3) = 0, \tag{1.9}$$

$$x_0^2 x_1^2 + (x_0 + x_1)^2 (x_0 x_2 + x_1 x_3) = 0, \tag{1.10}$$

There is one more case of pencils with one base point. It gives canonical equation

$$x_0^2 x_1^2 + (x_0 + x_1)(x_0^2 x_2 + x_1^2 x_3) = 0. \tag{1.11}$$

All normal forms reveal a non-trivial projective symmetry $(x_0, x_1, x_2, x_3) \mapsto (x_1, x_0, x_3, x_2)$. Our equations agree with Salmon's equations from [21, Chapter XVI, Art.

546–549]. Equations (1.5) and (1.6) (resp. equations (1.7) and (1.8)) are Salmon's type I (resp, II). Equation (1.11) (resp. (1.9), resp. (1.10)) is Salmon's type III (resp. IV, resp. V).

Salmon also finds the equations of the dual surfaces, some of them are self-dual.

1.2 Monoidal surfaces and rational ruled surfaces in $\mathbb{P}^3$

A ruled surface F in $\mathbb{P}^N$ is the image in $\mathbb{P}^N$ of the tautological $\mathbb{P}^1$-bundle over an irreducible curve $\mathrm{Gen}(F)$ in the Grassmannian $G_1(\mathbb{P}^N)$ of lines in $\mathbb{P}^N$. The curve $\mathrm{Gen}(F)$ is the *generatrix* of F. The lines of the ruling are *generators* of F. The pull-back of the $\mathbb{P}^1$-bundle to the normalization $\mathrm{Gen}(F)^{\mathrm{norm}}$ is a minimal ruled surface over $\mathrm{Gen}(F)^{\mathrm{norm}}$. When F is rational, and only those we will encounter in this paper, $\mathrm{Gen}(F)^{\mathrm{norm}} \cong \mathbb{P}^1$ and the minimal ruled surface is isomorphic to $\mathbf{F}_n := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n))$ for some $n \geq 0$. The map $\mathbf{F}_n \to F$ is the normalization map unless F is a cone. We assumed that F is not a cone.

Another invariant of a ruled surface besides its normalization is the smallest degree of its *directrix*, an irreducible curve on F that intersects each generator at one point and is not contained in $\mathrm{Sing}(F)$. If F is smooth, the directrix $D_{\min}$ of smallest degree coincides with the image of the exceptional section $\mathfrak{e}$ of $\mathbf{F}_n$ (or of any generator from the other ruling on $\mathbf{F}_0$).

A non-degenerate rational ruled surface F of degree d in $\mathbb{P}^{d+1}$ is an example of a non-degenerate surface of minimal degree. In particular, it must be a normal ruled surface. Conversely, a normal ruled surface is isomorphic to a non-degenerate surface of minimal degree.

Take $d-1$ general points $x_1, \ldots, x_{d-1}$ on F that span a projective subspace L^{d-2} of dimension $d-2$. The projection to $\mathbb{P}^2$ with the center at L^{d-2} defines a rational map $F \dashrightarrow \mathbb{P}^2$. It can be regularized by passing to the blow-up $\mathrm{Bl}_\Sigma(F)$ of the set of points $\Sigma = \{x_1, \ldots, x_{d-1}\}$. The birational morphism $\mathrm{Bl}_\Sigma(F) \to \mathbb{P}^2$ blows down the proper transforms g_i' of generators g_i of F passing through x_i to points p_i in the plane. The span of L^{n-2} and a generator g skew to it spans a hyperplane that cuts out F along g and a curve $C_{d-1}(g)$ of degree $d-1$. When we let g vary in a pencil, we obtain a pencil of hyperplanes. It corresponds to the pencil of lines in the plane passing through a point O. Taking $g = g_i$, we get a hyperplane section that consists of g_i and a curve of degree $d-1$ containing $x_1, \ldots, x_{d-1}$. This means that O is a $(d-1)$-multiple point of the projections of hyperplane sections. Also, it shows that F admits a rational parameterization $\phi : \mathbb{P}^2 \dashrightarrow F$ given by the complete linear system

$$|L| = C^d(p_0^{d-1}, p_1, \ldots, p_{d-1}) \tag{1.12}$$

of curves of degree d passing through points $p_1, \ldots, p_{d-1}$ and containing a point p_0 with multiplicity $\geq d-1$ (see [3, 10.3.6]). The map ϕ regularizes on the blow-up $X = \mathrm{Bl}_{\{p_0, p_1, \ldots, p_{d-1}\}}(\mathbb{P}^2)$. In the standard geometric basis $(e_0, e_1, \ldots, e_d)$ of X, where e_d is the divisor class of the exceptional curve E_d over p_0, the proper transform of $|L|$ is given by

$$|M| = |de_0 - e_1 - \cdots - e_{d-1} - (d-1)e_d|.$$

The pencil $|e_0 - e_d|$ is mapped to the ruling of F. A directrix D_m of degree m is the image of a curve C on X such that $C \cdot e_i \leq 1, i = 1, \ldots, d-1, C \cdot (e_0 - e_d) = 1$, and $C \cdot M = m$. It follows that

$$C \in |(m + \#I - d + 1)e_0 - \sum_{i \in I} e_i - (m + \#I - d)e_d|,$$

where $I = \{i \in \{1, \ldots, d-1\}, x_i \notin D_m\}$.

For example, the image of a general line in the plane is a directrix of degree d that contains all the points x_i. We have

$$C^2 = 2m - 2d + 1 + \#I, \quad -C \cdot K_X = 2m - 2d + 3 + \#I,$$

so we expect that

$$\dim |C| = 2m + \#I - 2d + 2.$$

This suggests that the smallest degree of a directrix is equal to $k = \lceil \frac{d-1}{2} \rceil$, and in this case $\#I = d - 1$. It represents a curve $C^k(p_0^{k-1}, p_1, \dots, p_{d-1})$ in the plane. The curve is unique if d is even, and vary in a pencil if d is odd.

In fact, we have the following (see [6, 10.4.6]).

Lemma 1.3 *Let $F \cong \mathbf{F}_e \hookrightarrow \mathbb{P}^n$ be an embedding of a minimal rational ruled surface $\mathbf{F}_e$ of degree $n-1$ by a complete linear system $|a\mathfrak{f} + \mathfrak{e}|$, where $\mathfrak{e}^2 = -e$, $2a - e = n - 1$. Then, the degree* $\deg(D)$ *of a directrix D on X satisfies*

$$\deg(D) \geq a - e = \frac{1}{2}(e + \deg(F)) - e,$$

and the equality takes place if and only if D is the image of the exceptional section from $|\mathfrak{e}|$.

Our ruled surface Φ_d is the image of some $\mathbf{F}_e$ under a linear projection to $\mathbb{P}^3$. The projection of a directrix is a directrix on Φ_d of the same degree. Thus, the smallest degree $d_{\min}$ of a directrix on Φ_d is equal to $\frac{1}{2}(e + d) - e \leq \lceil \frac{d-1}{2} \rceil$. The equality takes place if and only if $(d, e) = (2m + 1, 0)$, or $(d, e) = (2m, 1)$).

So, we expect that the normalization F^{norm} of a general F is isomorphic to $\mathbf{F}_1$ (resp. $\mathbf{F}_0$) if d is even (resp. d is odd). For example, if $A_d = 0$ in its Eq. (1.17), then the line $V(x_2, x_3)$ is its directrix of degree 1.

If F is not general, then a directrix D of degree m smaller than $\lceil \frac{d}{2} \rceil$ necessarily passes through some points x_i, and it has negative self-intersection. Since it coincides with the exceptional section of F^{norm}, we get

$$F \cong \mathbf{F}_{d-2m}. \tag{1.13}$$

Note that the blow-up of $\mathbb{P}^2$ at the point p_0 is a minimal ruled surface isomorphic to $\mathbf{F}_1$. The surface $X = \mathrm{Bl}_{p_0,\dots,p_{d-1}}(\mathbb{P}^2)$ is obtained from this surface by blowing up $d-1$ points not lying on the exceptional section of $\mathbf{F}_1$. The map $X \to F$ blows down the proper transforms of the generators passing through these points. We see that F is obtained from $\mathbf{F}_1$ by a composition of $d-1$ elementary birational transformations elm_{x_i} (see [6, 7.4.2]).

The normalization Φ_d^{norm} of Φ_d is isomorphic to the proper transform of Φ_d in the blow-up $\sigma : \mathrm{Bl}_\Gamma(\mathbb{P}^3) \to \mathbb{P}^3$. It is well-known that

$$\mathrm{Bl}_\Gamma(\mathbb{P}^3) \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(1)). \tag{1.14}$$

The projection $\pi : \mathbb{P} \to \mathbb{P}^1$ is given by the pencil of planes through Γ, so we may identify the base of the projective $\mathbb{P}^2$-bundle $\mathbb{P}$ with $\Gamma^\perp$.

We have

$$\mathrm{Pic}(\mathbb{P}) = \mathbb{Z}H + \mathbb{Z}P, \tag{1.15}$$

where $H = c_1(\mathcal{O}_{\mathbb{P}}(1)) = \sigma^* c_1(\mathcal{O}_{\mathbb{P}^3}(1))$ and $P = \pi^* c_1(\mathcal{O}_{\mathbb{P}^1}(1))$. The exceptional divisor E of the blow-up is linearly equivalent to $H - P$ and

$$\Phi_d^{\text{norm}} \sim dH - (d-1)E \sim H + (d-1)P. \tag{1.16}$$

We know from (1.13) that

$$\Phi_d^{\text{norm}} \cong \mathbf{F}_n, \quad n = d - 2m,$$

where m is the smallest degree of a directrix of Φ_d.

We use that

$$\mathbb{P} = (\mathbb{A}^3 \setminus \{0\}) \times (\mathbb{A}^2 \setminus \{0\})/\mathbb{G}_m^2,$$

where the action is defined by

$$(\lambda, \mu) : ((y_0, y_1, y_2)), (t_0, t_1)) \mapsto ((\lambda^{-1} y_0, \mu y_1, \mu y_2), (\lambda t_0, \lambda t_1)).$$

Moreover, it comes with the projections

$$\sigma : \mathbb{P} \to \mathbb{P}^3, \ ([y_0, y_1, y_2], [t_0, t_1]) \mapsto [t_0 y_0, t_1 y_0, y_1, y_2]$$

and

$$\sigma : \mathbb{P} \to \mathbb{P}^1 = \Gamma^\perp, \ ([y_0, y_1, y_2), (t_0, t_1)] \mapsto [t_0, t_1].$$

The proper transform of Φ_d in $\mathbb{P}$ is now given by equation

$$A_d(t_0, t_1)y_0 + B_{d-1}(t_0, t_1)y_1 + C_{d-1}(t_0, t_1)y_2 = 0. \tag{1.17}$$

It can be also considered as the equation of a residual line. It shows that each point on Γ is contained in $d - 1$ residual lines.

The curve Σ contains a unique irreducible component Σ_0 that defines a section of $\sigma : \Gamma^\perp \to \mathbb{P}$. All other components are fibers of the projection $\sigma|E : E \to \Gamma^\perp$.

The equation of the intersection $\Sigma := \tilde{\Phi}_d \cap E$ becomes:

$$B_{d-1}(t_0, t_1)y_1 + C_{d-1}(t_0, t_1)y_2 = 0.$$

It exhibits Σ as a curve in $E \cong \Gamma \times \Gamma^\perp$ of bidegree $(1, d - 1)$. The projection $\Sigma \to \Gamma$ is a finite map of degree $d - 1$. These are the points $x \in \Gamma$ where a plane in $\Gamma^\perp$ is tangent to Φ_d along a line passing through x. If Σ is smooth, the branch points are called the *pinch points*. We expect $2(d - 2)$ pinch points.

Let $\Phi_d^{\text{norm}} \cong \mathbf{F}_{d-2m} \hookrightarrow \mathbb{P}^{d+1}$ be the embedding of Φ_d^{norm} by the complete linear system $|(d - m)\mathfrak{f} + \mathfrak{e}|$.

The surface Φ_d is a projection of $\Phi_d^{\text{norm}} \subset \mathbb{P}^{d+1}$ to $\mathbb{P}^3$ from a subspace L^{d-3} of codimension 4. The image of the curve Σ spans a subspace M^{d-1} of codimension 2. The projection map $p : \Sigma \to \Gamma$ is of degree $d - 1$, hence L^{d-3} must be contained in M^{d-1}. The projection p is defined by the pencil of hyperplanes in M^{d-1} containing L^{d-3}.

The image of the exceptional section is a directrix $D_{\min}$ of degree m that spans a linear subspace M^{m-1} of dimension $m-1$ disjoint from the center of the projection. Since $\mathfrak{e} \cdot [\Sigma] = m - 1$, the directrix $D_{\min}$ intersects Γ at $m - 1$ points.

Example 1.4 Assume $d = 3$ and keep the notation from Example 1.1. It follows from the previous formulas that the smallest degree of a directrix on Φ_3 is equal to 1. In fact, this can be easily derived from the equations. If $A_3 = 0$, then $V(x_2, x_3)$ is a line directrix. If $A_3 \neq 0$, the line $V(x_1, x_3)$ is a line directrix. Thus, $\Phi_d^{\text{norm}} \cong \mathbf{F}_1$ embedded in $\mathbb{P}^4$ by the linear system $|2\mathfrak{f} + \mathfrak{e}|$. It is a smooth cubic ruled surface. The projection from a point in $\mathbb{P}^4 \setminus \Phi_d^{\text{norm}}$ maps it to Φ_d. There are two possibilities: the curve $\Sigma \in |\mathfrak{f} + \mathfrak{e}|$ is irreducible, or Σ is the union of the exceptional section identified with $\mathfrak{e}$ and a fiber $F \in |\mathfrak{f}|$. In the first case, we get a surface with canonical Eq. (1.2). The double curve is a non-exceptional section. The image of the exceptional section is the line $x_2 = x_3 = 0$. In the second case, the pre-image of Γ consists of the exceptional section and a line from the ruling.

Example 1.5 Assume $d = 4$. In this case, the minimal degree m of a directrix is equal to 2 (this is the general case) or $m = 1$. In the first case, $|2\mathfrak{f}_1 + \mathfrak{f}_2|$ maps $\Phi_d^{\mathrm{norm}} \cong \mathbf{F}_0$ to a smooth quartic ruled surface in $\mathbb{P}^5$, and in the second case, $|3\mathfrak{f} + \mathfrak{e}|$ embeds $\Phi_d^{\mathrm{norm}} \cong \mathbf{F}_2$ in $\mathbb{P}^5$, also as a smooth quartic surface. The center of the projection is a line disjoint from $\tilde{\Phi}_d$. In the classification of quartic ruled surfaces in $\mathbb{P}^3$ (see [6, 10.4.4], [7]), the first case corresponds to Types II (A) and III (A) in Edge's notation. His equations of the surfaces can be simplified and correspond to our canonical Eqs. (1.5),(1.7) for Type II (A), and (1.11) for Type III (A).

The second case corresponds to Type IV (A) and V (B). Edge's equations are our Eqs. (1.6), (1.6) for Type IV (A) and (1.9),(1.10) for Type V (B). The double curve is irreducible if the surface is of Type IV (A) in Edge's notation.

2 Monoidal Cremona transformations

According to modern terminology, a *monoidal birational transformation* (or a σ-process) is the blow-up of a smooth subvariety of a projective space [10, Chapter IV, §3]. The following is the classical definition of a monoidal transformation [12].

Definition 2.1 A birational transformation $T : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is called a *monoidal transformation* if it is given by a linear system whose general member is a monoidal hypersurface of degree d whose singular locus Γ of multiplicity $d-1$ is contained in the base locus.

The two definitions are obviously related, in order to resolve indeterminacy points of a monoidal transformation one has to blow up its base locus.

Here, we will discuss a particular case when $n = 3$ and monoidal surfaces are ruled surfaces of degree d with a $(d-1)$-multiple line.

Lemma 2.2 *Let Φ_d and Φ'_d be two monoidal surfaces of degree d containing a line Γ with multiplicity $d-1$. Then,*

$$\Phi_d \cap \Phi'_d = (d-1)^2\Gamma + C_{2d-1},$$

where C_{2d-1} is a curve of degree $d^2 - (d-1)^2 = 2d-1$. Assume that C_{2d-1} intersects transversally Γ. Then, the number of intersection points is equal to $2d-2$.

Proof Take a general plane Π in $\Gamma^\perp$. Its residual intersections with Φ_d and Φ'_d are lines. They intersect at one point. This shows that Π intersects C_{2d-1} along $C_{2d-1} \cap \Gamma$ and one point. Hence, $\#C_{2d-1} \cap \Gamma = 2d-1-1 = 2(d-1)$. □

The linear system of monoidal surfaces of degree d with a line Γ of multiplicity $d-1$ is of dimension $(d+1) + 2d - 1 = 3d$. Choose α general lines $\ell_1, \ldots, \ell_\alpha$ intersecting Γ and β general points $P_1, \ldots, P_\beta$ in $\mathbb{P}^3$. Since passing through ℓ_i imposes 2 conditions on monoidal surfaces, we obtain that the dimension of the linear system

$$|L| = |\mathcal{O}_{\mathbb{P}^3}(d) - (d-1)\Gamma - \sum_{i=1}^{\alpha} \ell_i - \sum_{i=1}^{\beta} P_i|$$

is equal to $3d - 2\alpha - \beta$. It is a web (i.e dim $|L| = 3$) if

$$2\alpha + \beta = 3(d-1). \tag{2.1}$$

Suppose that the web $|L|$ is homaloidal, that is, it defines a birational map $T : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$. Its restriction to a general plane Π is given by the monoidal linear system of curves of degree d with $(d-1)$-multiple point $p_0 = \Pi \cap \Gamma$ and α simple points $p_i = \ell_i \cap \Pi$. The image of Π under the restriction of T to Π is a monoidal surface of degree

$$d' = 2d - 1 - \alpha.$$

Varying Π, we obtain a homaloidal web of monoidal surfaces of degree d'. It defines the inverse birational transformations T^{-1}.

Replacing T with T^{-1}, we get the following equalities:

$$2\beta + \alpha = 3(d'-1), \quad \beta = 2d' - d - 1 \tag{2.2}$$

(see [12, p. 216]). The image of each line in the plane through p_0 is a line. The union of lines $\langle p_i, p_0\rangle$ and any line through p_0 form a subpencil of cones with vertex at p_0 in the linear system. This shows that the image of the point p_0 is a line Γ' taken with multiplicity $d-1$

Consider the projective space of monoidal surfaces of degree $d-1$ with a line Γ of multiplicity $d-2$. The dimension of this space is equal to $3d-3$. Using (2.1), we see that there exists a unique surface J_{d-1} from this space that contains α lines ℓ_i and β points P_i. The union $J_{d-1}+\Pi$, where $\Pi \in \Gamma^\perp$, belongs to $|L|$. This shows that the image of the surface J_{d-1} under T is a line Γ'. In other words, J_{d-1} is an exceptional surface of T. Other exceptional surfaces are the planes $\Pi_i = \langle \Gamma, \ell_i\rangle, i = 1, \ldots, \alpha$, and $\Pi'_i = \langle \Gamma, P_i\rangle, i = 1, \ldots, \beta$. The degree of the jacobian of the affine transformation $\mathbb{A}^4 \to \mathbb{A}^4$ defined by T is equal to $4(d-1)$. Each plane Π_i is blown down to a point, hence it must be taken with multiplicity 2 in the jacobian. We get $4(d-1) - (d-1) - 2\alpha - \beta = 3(d-1) - 2\alpha - \beta = 0$. This shows that all exceptional surfaces are accounted for.

The line Γ' is a common singular line of monoidal surfaces of degree d' from the homaloidal web defining the inverse transformation T^{-1}. The exceptional surface $J'_{d'-1}$ of T^{-1} is equal to the image of the exceptional divisor E under the composition $\mathrm{Bl}_\Gamma(\mathbb{P}^3) \to \mathbb{P}^3 \overset{T}{\dashrightarrow} \mathbb{P}^3$.

Example 2.3 Let us take $d = 2, \alpha = 0, \beta = 3$. This gives $d' = 3$. The transformation T is an example of a quadro-cubic transformation. Its base locus consists of a line Γ and three isolated points P_i outside Γ. The inverse transformation T^{-1} is given by monoidal cubic surfaces with singular line Γ' that contains three skew lines ℓ_1, ℓ_2, ℓ_3 intersecting Γ'. The exceptional surfaces of T are the three planes $\langle \Gamma, P_i\rangle$ and the plane $\langle P_1, P_2, P_3\rangle$. The exceptional surfaces of T^{-1} are the three planes $\langle \Gamma', \ell_i\rangle$ and the unique quadric surface containing the lines $\Gamma', \ell_1, \ell_2, \ell_3$.

If $d = 3$, we can solve for possible α, β and obtain Cremona transformations T with T^{-1} of possible algebraic degrees $d' = 2, 3, 4, 5$ (see [5, 12]).

Example 2.4 Let us take $\alpha = \beta = d - 1$. In this case, the algebraic degree of T^{-1} is also equal to d. After we compose the transformation with a projective transformation, we will be able to identify the source and the target $\mathbb{P}^3$ and assume that $\Gamma = \Gamma'$. The restriction of T to each non-exceptional plane $\Pi \in \Gamma^\perp$ is a projective transformation that sends Π to a plane $\Pi' \in \Gamma^\perp$. One can compose T with another projective transformation that leaves Γ invariant to obtain a Cremona involution [19].

3 Non-degenerate submonoidal surfaces

3.1 Conic bundle structure

Let Φ_d be a submonoida surface of degree d that contains a line Γ of multiplicity $d-2$. For any general point $x \in \mathbb{P}^3$, the plane $\Pi = \langle x, \Gamma\rangle$ intersects Φ_d along the union of Γ, taken with multiplicity $d-2$, and a conic. We assume, as before, that Γ is given by equations $x_0 = x_1 = 0$. Then, the equation of Φ_d is of the form

$$\begin{aligned} &A_d(x_0,x_1) + 2B_{d-1}(x_0,x_1)x_2 + 2C_{d-1}(x_0,x_1)x_3 \\ &\quad + D_{d-2}(x_0,x_1)x_2^2 + 2E_{d-2}(x_0,x_1)x_2x_3 + F_{d-2}(x_0,x_1)x_3^2 = 0. \end{aligned} \tag{3.1}$$

We will assume that Γ does not contain points of multiplicity $d-1$. Otherwise, the surface is monoidal. The equation depends on $(d+1)+2d+3(d-1)-1 = 6d-3$ parameters. The group of projective automorphisms of $\mathbb{P}^3$ that fix a line is of dimension 11. Thus, we expect that the dimension of the moduli space of submonoidal surfaces is equal to $6d-14$.

As in the previous section, we consider the projective bundle $\pi : \mathbb{P} = \mathrm{Bl}_\Gamma(\mathbb{P}^3) \to \Gamma^\perp$ and denote by Φ_d^{bp} the proper transform of Φ_d under the blow-up map $\sigma : \mathrm{Bl}_\Gamma(\mathbb{P}^3) \to \mathbb{P}^3$.

As in the case of monoidal surfaces, we use the coordinates $(y_0, y_1, y_2, t_0, t_1)$ in $\mathbb{P}$ to obtain the following equation of Φ_d^{bp}:

$$\begin{aligned} &A_d(t_0,t_1)y_0^2 + 2y_0(B_{d-1}(t_0,t_1)y_1 + C_{d-1}(t_0,t_1)y_2) \\ &\quad + D_{d-2}(t_0,t_1)y_1^2 + 2E_{d-2}(t_0,t_1)y_1y_2 + F_{d-2}(t_0,t_1)y_2^2 = 0. \end{aligned} \tag{3.2}$$

The line $\Gamma^\perp$ has coordinates (t_0, t_1). The fibers of the projection π are isomorphic to the residual conics $C_{[t_0,t_1]}$ in the plane $\Pi_{[t_0,t_1]} \in \Gamma^\perp$ (the common zeros of $D_{d-2}, E_{d-2}, F_{d-2}$ correspond to conics that contain Γ as an irreducible component).

Computing the discriminant of (3.2), we find that it is equal to

$$P(t_0,t_1) = \det\begin{pmatrix} A_d & B_{d-1} & C_{d-1} \\ B_{d-1} & D_{d-2} & E_{d-2} \\ C_{d-1} & E_{d-2} & F_{d-2} \end{pmatrix}. \tag{3.3}$$

It is a binary form of degree $3d-4$. A point $[a,b]$ is a zero of $P(t_0,t_1)$ if and only if the fiber of the conic bundle is singular. For a general Φ_d, we expect that the polynomial is reduced, and we get $3d-4$ singular fibers. We will clarify this later.

Let $\mathcal{D}_1 = V(P(t_0,t_1))$, considered as a closed subscheme of $\mathbb{P}^1 = \Gamma^\perp$. We call it the *discriminant* of the conic bundle.

The intersection of Φ_d^{bp} with the exceptional divisor $E \cong \Gamma \times \Gamma^\perp \cong \mathbb{P}^1 \times \mathbb{P}^1$ is a curve Σ of bidegree $(d-2, 2)$ given by equation

$$D_{d-2}(t_0,t_1)y_1^2 + 2E_{d-2}(t_0,t_1)y_1y_2 + F_{d-2}(t_0,t_1)y_2^2 = 0. \tag{3.4}$$

The projections $\sigma : E \to \Gamma$ and $\pi : E \to \Gamma^\perp$ define two projections

$$p : \Sigma \to \Gamma, \quad q : \Sigma \to \Gamma^\perp$$

of degrees $d-2$ and 2, respectively.

If Σ is smooth, then its genus is equal to $d-3$, and, applying the Hurwitz formula, we expect that the cover $p : \Sigma \to \Gamma$ has $4(d-3)$ branch points. They are called the *pinch points*. They are the points $x \in \Gamma$ at which a general plane section passing through x has less than $d-2$ local branches at x.

Let

$$R(t_0, t_1) := E_{d-2}(t_0, t_1)^2 - D_{d-2}(t_0, t_1)F_{d-2}(t_0, t_1).$$

It is a binary form of degree $2(d-2)$. A point $[a, b]$ is a root of $R(t_0, t_1)$ if and only if it is a branch point of the projection $q : \Sigma \to \Gamma^\perp$. Equivalently, $R(a, b) = 0$ if and only if the residual conic C_t in the plane Π_t intersects Γ at one point.

Let $\mathcal{D}_2 = V(R(t_0, t_1))$ considered as a closed subscheme of $\Gamma^\perp$. We call it the *small discriminant* of the conic bundle.

Lemma 3.1 *A point $x = ([a, b], [c, d]) \in \Sigma$ is a singular point of Σ if and only if $[a, b]$ is a multiple root of $R(t_0, t_1)$.*

Proof Here, and in the sequel, we use that a quadratic form over a discrete valuation ring of characteristic $\neq 2$ can be reduced to a sum of squares [14, Satz (15.1)]. By localizing, we may assume that the quadratic form (3.4) over $A = \mathcal{O}_{\mathbb{P}^1,[a,b]}$ with local parameter t is given by $t^n x^2 + t^{n+k} y^2$. If $n > 0$, then $[a, b] \in \mathrm{Sing}(\mathcal{D}_2)$ and Σ contains the point $([a, b], [c, d])$, where $c^2 + t^k d^2 = 0$. It is a singular point of Σ. If $n = 0$ and $k > 1$, then $[a, b] \in \mathrm{Sing}(\mathcal{D}_2)$ and the point $t = x = 0$, $y = 1$ is a singular point.

Conversely, if $[a, b] \in \mathrm{Sing}(\mathcal{D}_2)$, then $2n + k \geq 2$, and we can reverse the argument to show that Σ is singular over $t = 0$ if and only if $2n + k \geq 2$. □

All singular points of Φ_d outside Γ define singular points of the conic bundle Φ_d^{bp}. Other singular points of Φ_d^{bp} may lie on the curve Σ.

Taking partials of (3.2) and setting $y_0 = 0$, we obtain the following.

Lemma 3.2 *The following assertions are equivalent:*

(i) *The rank of the matrix*

$$\begin{pmatrix} B_{d-2}(a,b) & D_{d-2}(a,b) & E_{d-2}(a,b) \\ C_{d-2}(a,b) & E_{d-2}(a,b) & F_{d-2}(a,b) \end{pmatrix} \tag{3.5}$$

at the point $[a, b]$ is equal to one.

(ii) *The residual conic $C_{a,b}$ over a point $(a, b) \in \mathcal{D}_2$ has a singular point $[0, 0, -E_{d-2}(a, b), D_{d-2}(a, b)] = [0, 0, -F_{d-2}(a, b), E_{d-2}(a, b)] \in \Gamma$.*

(iii) *A point on Σ over $[a, b]$ is a singular point of a fiber of the conic bundle.*

If $d \geq 4$, then these conditions are also equivalent to the condinition

(iv) *The surface Φ_d^{bp} has a singular point on Σ over $[a, b] \in \mathcal{D}_2$.*

Proof We only prove that (i) implies (ii) and leave to prove other implications to the reader. Suppose (i) holds. then $D_{d-2}(a, b)F_{d-2}(a, b) - E_{d-2}(a, b)^2 = 0$, hence $[a, b] \in \mathcal{D}_2$. Let $(0, y_1, y_2)$ be the coordinates of the intersection point of the fiber of conic bundle $K_{a,b}$ over $[a, b]$ with Σ. Then, $y_0 B_{d-2}(a, b) = y_0 C_{d-2}(a, b) = 0$. If $B_{d-2}(a, b) \neq 0$ or $C_{d-2}(a, b) \neq 0$, then $y_0 = 0$, and hence, $[0, y_1, y_2]$ is a singular point of $K_{a,b}$. If $B_{d-2}(a, b) = C_{d-2}(a, b) = 0$, then $P(a, b) = A_d(a, b)(D_{d-2}(a, b)F_{d-2}(a, b) - E_{d-2}(a, b)^2) = 0$ vanishes again. Thus, the conic has equation $y_0^2 A_d(a, b) + (E_{d-2}(a, b)y_1 + F_{d-2}(a, b)y_2)^2 = 0$, hence again the point $(0, y_1, y_2)$ is its singular point.

If $d \geq 3$, taking the partials in y_0, y_1, y_2, t_0, t_1 and using that D_{d-2}, E_{d-2} and F_{d-2} are binary forms of degree ≥ 2, we get condition (iv). □

By analogy with the case of cubic surfaces, a singular point of a residual conic lying on Γ may be called an *Eckardt point* of Φ_d.

Proposition 3.3 *Φ_d^{bp} is a normal surface if and only if $P(t_0,t_1)\neq 0$ and Φ_d has no singular lines intersecting Γ (besides Γ).*

Proof Since Φ_d^{bp} is a hypersurface in a smooth variety, Φ_d^{bp} is normal if and only if it has only isolated singular points.

Suppose $\mathcal{D}_1\neq\Gamma^{\perp}$, i.e. $P(t_0,t_1)\neq 0$. Thus, a general residual conic is smooth. Taking a general plane section of Φ_d, we obtain that $\mathrm{Sing}(\Phi_d\setminus\Gamma)$ consists of isolated singular points or lines intersecting Γ. By the assumption, Φ_d does not have such lines. So, if Φ_d^{bp} is non-normal, the curve Σ must be singular along its irreducible component. By Lemma 3.2, the rank of matrix (3.5) is identically equal to 1. This implies that the rank of matrix (3.3) is equal to 2, hence $P(t_0,t_1)=0$, a contradiction.

Conversely, suppose Φ_d^{bp} is normal. Then, a general residual conic is irreducible, hence its proper transform in Φ_d^{bp} is irreducible, hence smooth. Therefore, $P(t_0,t_1)\neq 0$. The second assumption is obviously satisfied. □

Proposition 3.4 *Φ_d^{bp} is smooth if and only if the discriminant polynomial $P(t_0,t_1)$ has no multiple roots.*

Proof This can be deduced from the well-know properties of conic bundles over any regular base (see [22]). In our situation, when the base is one-dimensional, the proof is very easy and we will present it here. After localizing at a point on the base, we may assume that $X\subset\mathbb{P}^2_R$, where R is a discrete valuation ring with local parameter t. We use again that a quadratic form over R can be reduced to a sum of squares. This shows that Φ_d can be given by equation

$$u^2+\epsilon_1 t^n v^2+\epsilon_2 t^m w^2=0, \tag{3.6}$$

where ϵ_i are units and $n\leq m$. Taking the partials, we see that $(t;u,v,w)=[0,0,0,1]$ is a non-regular point of Φ_d if $m>1$. Thus, if Φ_d is regular, the discriminant $D=\epsilon_1\epsilon_2 t^{n+m}$ is of order ≤ 2. Conversely, if D is of order 1, then $(n,m)=(0,1)$, then Φ_d is regular and the fiber of $t=0$ is a line-pair. If the order of D is equal to 2, then $(n,m)=(1,1)$ or $(0,2)$. In the first case, Φ_d is non-regular at $[0,0,v_0,w_0]$, where $\epsilon_1(0)u_0^2+\epsilon_2(0)v_0^2=0$ and the fiber over $t=0$ is a double line $u^2=0$. In the second case, $x=[0,0,0,1]$ is a non-regular point of Φ_d, the fiber over $t=0$ is a line-pair with a singular point at x. □

Remark 3.5 In the classical terminology, the image in Φ_d of a non-reduced fiber of the conic bundle Φ_d^{norm} is a *torsal line*. In general, a torsal line is a line on a surface with the same tangent plane at each smooth point of the surface lying on the line.

Using local Eq. (3.6), we can analyze possible singularities of Φ_d.

Suppose $n=0$, then the singular point is formally isomorphic to the singularity of $xy+z^m=0$ at the origin. It is a rational double point of type A_{m-1}. In this case, the point is a singular point of a reduced conic fiber.

Suppose $n>0$. Then, the fiber over $t=0$ is a double line. If $n=m=1$ (and hence, $[a,b]$ is a double root of $P(t_0,t_1)$), we have two ordinary double points on it. If $n>1$, $[a,b]$ is a root of multiplicity >2 of $P(t_0,t_1)$ and the surface is non-normal. If $1\leq n<m$, the singularity is formally isomorphic to the singularity of $x^2+t^ny^2+t^{n+k}=0$, $k>0$, at the origin. If k is odd, the singularity is quasi-homogeneous with weights $(q_1,q_2,q_3)=$

$(n+k, 2, k)$ and degree $d = 2n + 2k$. It is known that it is a rational double point if and only if $d - q_1 - q_2 - q_3 = n - 2 < 0$. The case $n = 0$ was considered before, if $n = 1$, it is a rational double point of type D_{n+k+1}.

Definition 3.6 We say that Φ_d is non-degenerate if the following conditions are satisfied:

(i) $\Phi_d^{\mathrm{bp}} = \Phi_d^{\mathrm{norm}}$;
(ii) $P(t_0, t_1)$ has no roots of multiplicity larger than 2;
(iii) $R(t_0, t_1)$ has no multiple roots.

The following proposition follows from the previous analysis of possible singularities of Φ_d.

Proposition 3.7 *Suppose Φ_d is non-degenerate. Then,*

(i) *the curve Σ is a smooth curve of genus $d-3$;*
(ii) *no singular point of Φ^{norm} lies on Σ;*
(iii) *all isolated singular points of Φ_d are ordinary nodes;*
(iv) *every singular point of Φ_d^{norm} is a singular point of some fiber of the conic bundle;*
(v) *each double fiber of the conic bundle on Φ_d^{norm} has two singular points.*

Conversely, properties (i), (ii) and (iii) imply that Φ_d is non-degenerate.

Proof Suppose that Φ_d is non-degenerate. Property (i) follows from the definition and Lemma 3.2. If x is a singular point of Φ_d^{norm} lying on Σ, then the exceptional divisor E of the blow-up of Γ intersects Σ at a singular point of Φ_d, hence, this point must be a singular point of Σ contradicting (i).

(iii) Since $P(t_0, t_1)$ has no roots of multiplicity larger than 2, the analysis of possible singular points of Φ_d in the proof of Proposition 3.4 shows that all singular points of Φ^{norm} are ordinary nodes. This shows that (iii) follows from (ii).

(iv) Follows from the analysis of singularities of Φ_d in the proof of Proposition 3.4.

Let us prove the converse. Property (i) implies that Σ is reduced, and hence, $\Phi_d^{\mathrm{bp}} = \Phi_d^{\mathrm{norm}}$. It also implies that $R(t_0, t_1)$ has no multiple roots. Properties (ii) and (iii), together with the analysis of singularities of Φ_d from Proposition 3.4, imply that the polynomial $P(t_0, t_1)$ has no roots of multiplicity higher than 2. ◻

Remark 3.8 One can show that, only assuming that $\Phi_d^{\mathrm{bp}} = \Phi_d^{\mathrm{norm}}$, all singularities of Φ_d are rational double points.

Corollary 3.9 *Suppose $d > 3$. The number of isolated singular points on a non-degenerate surface Φ_d is at most $3d-4$.*

Proof Every singular point lies over a singular point of the discriminant $\mathcal{D}_1$, and two singular points may lie over the same point if the corresponding fiber is not reduced. This gives the asserted bound. ◻

3.2 4-Nodal cubic surfaces

A normal cubic surface Φ_3 with a fixed line Γ on it is a monoidal surface.

Each reducible residual conic is cut out by a tritangent plane containing the line Γ. The surface is smooth if and only if there are five different tritangent planes corresponding to the roots of the binary form $P(t_0,t_1)$ of degree $3d-4=5$. The roots of the binary form $R(t_0,t_1)$ of degree 2 correspond to the residual conics intersecting Γ at one point. The common roots of $P(t_0,t_1)$ and $R(t_0,t_1)$ correspond to Eckardt points on Γ.

The surface Φ_3 is non-degenerate if and only if all its singular points are ordinary nodes and they do not lie on Γ. It is isomorphic to the blow-up of the set Σ of six points in the plane. Since $s_1+2s_2+2s_3=5$, we have the following possibilities $(s_1,s_2,s_3)=(5,0,0),(3,1,0),(3,0,1),(1,2,0),(1,0,2),(1,1,1)$. The number of nodes is equal to 0, 1, 2, 2, 4, 3, respectively. It is known that a normal cubic surface contains at most four nodes, and the surface with four nodes is unique, up to isomorphism. A monoidal cubic surface with $(s_1,s_2,s_3)=(1,0,2)$ gives a birational model of this surface as the blow-up of six points p_0,p_1,p_2,p_3,p_4,p_5 with two infinitely near points $p_3\succ p_2$ and $p_5\succ p_4$ lying on the lines $\langle p_1,p_2\rangle$ and $\langle p_1,p_4\rangle$. The Cremona transformation defined by the homaloidal linear system of cubic curves $C^3(p_0^2,p_2,p_3,p_4,p_5)$ passing simply through $p_1,\ldots,p_5$ and passing through p_0 with multiplicity two defines a biregular model of Φ_d as the blow-up of the vertices of a complete quadrilateral. This is a familiar model of a minimal resolution of a 4-nodal cubic surface.

3.3 Plücker complex surface

Non-degenerate quartic surfaces with a double line are discussed in Jessop's book [13, Chapter VI] and in Salmon's book [21, Chapter XVI]. Let us see how some of the classical results follow from our discussion. Note that Jessop and Salmon only implicitly assumed that the surfaces Φ_4 are non-degenerate.

There are $s_2+2s_3=8-s_1-s_2$ ordinary nodes on Φ_4. The maximal number is equal to 8. It is realized for the *Plücker's Complex Surface*,[1] a submonoidal quartic surface with 8 nodes (see [13, Section 83]). The nodes lie on four torsal lines. So, the discriminant polynomial $P(t_0,t_1)$ is a square. The four planes Π_t corresponding to the roots of $P(t_0,t_1)$ intersect Φ_4 along a torsal line and the double line.

Let P_i,P_i', $i=1,\ldots,4$, be the four pairs of nodes lying on torsor lines. Projecting from one of them, say P_1, we find that the branch sextic splits into the union of a double line ℓ (the projection of Γ) and a complete quadrilateral T whose vertices are the projections of the points. The image of the exceptional curve over P_1 is a *contact-conic*, a conic tangent to all sides of the complete quadrilateral and to the double line. The tangency point on the double line is the image of the torsal line containing P_1. The sides of T are the images of eight tropes intersecting each torsal line at one of the nodes. The configuration of 8 tropes and 8 nodes is a symmetric configuration (8_4) (i.e., each node is contained in four tropes and each trope contains four nodes). For example, we can find this configuration by taking eight vertices of a cube with 6 faces and two diagonal planes.

By a well-known Cayley's characterization of quartic symmetroids as double planes branched along the union of two plane cubic curves with a contact-conic, we obtain that a Plücker Complex Surface admits an equation given by a determinant of a symmetric matrix with linear forms as its entries. One of these equations can be found in [11]:

$$\det\begin{pmatrix} 0 & x_0-x_1+x_2 & x_0-x_1+x_2 & x_0\\ x_0-x_1+x_2 & 0 & x_3 & x_1\\ x_0-x_1+x_2 & x_3 & 0 & x_2\\ x_0 & x_1 & x_2 & 0\end{pmatrix}=0, \tag{3.7}$$

[1] Here Complex refers to a quadratic complex of lines in $\mathbb{P}^3$, see Remark 3.10.

The double line is $V(x_0-x_1, x_2)$. The four torsal lines are cut out by planes $V(x_0-x_1+tx_2)$, where $t = \frac{1+\sqrt{-3}}{2}, \frac{1-\sqrt{-3}}{2}, 1, \infty$. They are the intersections of these planes with the planes $V(x_0+x_3)$, $V(x_0+x_3)$, $V(x_0+x_2-x_3)$, $V(x_2-x_3)$, respectively. The eight nodes lie by pairs in the torsal lines

$$(P_1, P_1') = ([0, e, 1, 0], [-1, 0, \bar{e}, 1]), \quad (P_2.P_2') = ([0, \bar{e}, 1, 0], [-1, 0, e, 1]),$$
$$(P_3.P_3') = ([0, 1, 1, 1], [1, 0, -1, 0]), \quad (P_4, P_4') = ([1, 0, 0, 0], [0, 1, 0, 1]).$$

where $e = \frac{1+\sqrt{-3}}{2}, \bar{e} = \frac{1-\sqrt{-3}}{2}$.

The eight tropes are

$$\begin{aligned}
V(x_0) &= \langle P_1, P_2, P_3, P_4' \rangle,\\
V(x_1) &= \langle P_1', P_2', P_3', P_4 \rangle,\\
V(x_3) &= \langle P_1, P_2, P_3', P_4 \rangle,\\
V(x_0+x_3-x_1) &= \langle P_1', P_2', P_3, P_4' \rangle,\\
V(ex_2+\bar{e}x_3-x_1) &= \langle P_1, P_2', P_3, P_4 \rangle,\\
V(\bar{e}x_2+ex_3-x_1) &= \langle P_1', P_2, P_3, P_4 \rangle,\\
V((x_0+x_2)-e(x_2-x_3)) &= \langle P_1, P_2', P_3', P_4' \rangle,\\
V((x_0+x_2)-\bar{e}(x_2-x_3)) &= \langle P_1', P_2, P_3', P_4 \rangle.
\end{aligned}$$

Remark 3.10 The Plücker's Complex Surface occurs in the geometry of a quadratic line complex $\mathfrak{C}$ in the Grassmann variety $G_1(\mathbb{P}^3)$ of lines in $\mathbb{P}^3$ (see [13, Art 86], [21, Art 455]). Fix a general line Γ in $\mathbb{P}^3$. The set of rays in $\mathfrak{C}$ contained in a general plane Π is a conic K_Π in $G_1(\mathbb{P}^3)$. A pencil of lines $\Omega(x, \Pi)$ contained in Π and passing through a point x is line in $G_1(\mathbb{P}^3)$. A Plücker Complex Surface Φ_4 of $\mathfrak{C}$ is defined to be the locus of points $x \in \mathbb{P}^3$ such that the line $\Omega(x, \langle x, \Gamma \rangle)$ is tangent to the conic $K_{\langle x, \Gamma \rangle}$. There will be four planes $\Pi_1, \ldots, \Pi_4$ in $\Gamma^\perp$ such that the conic K_Π is reducible. They correspond to the intersection points of the pencil $\Gamma^\perp$ with the dual of the Kummer surface associated with $\mathfrak{C}$. The lines corresponding to their singular points are the torsal lines, and the irreducible components of $K(\Pi_i)$ define two singular points on the trope.

4 A blow-up model of a submonoidal surface

4.1 Blowing down to a minimal ruled surface

From now on, we will assume that Φ_d is a nondegenerate submonoidal surface.

Let $\Phi_d^{\mathrm{norm}} = \Phi_d^{\mathrm{bp}} \subset \mathrm{Bl}_\Gamma(\mathbb{P}^3)$ be the normalization of a submonoidal surface Φ_d. Let $f : \tilde{\Phi} \to \Phi_d^{\mathrm{norm}} \to \Phi_d$ be a minimal resolution of singularities. The composition $\tilde{\pi} = \pi \circ f$ defines a structure of a conic bundle on $\tilde{\Phi}$. There are three kinds of singular fibers of the conic bundle.

A fiber of the *first kind* is the union of two different components which intersect at a nonsingular point of $\tilde{\Phi}_d$. A fiber of the *second kind* is the pre-image of a reduced residual conic with singular point at a node of Φ. It is a divisor $\bar{F}_1 + \bar{F}_2 + R$, where $\bar{F}_i$ are the proper transforms of the irreducible components of F and R is the exceptional curve. The curves $\bar{F}_i$ are (-1)-curves and the curve R is a (-2)-curve. Here, we use the standard abbreviation for a smooth rational curve on a smooth projective surface with negative self-intersection $-n$.

A fiber of the *third kind* is the pre-image of a non-reduced fiber of π. Each fiber of the third kind is a divisor $R_1 + R_2 + 2\bar{F}$, where R_1 and R_2 are the exceptional fibers over the singular points lying on the fiber and $2\bar{F}$ is the proper transform of the fiber.

Let s_1 (resp. s_2, resp. s_3) be the number of fibers of the first (resp. second, resp. third) kind.

We have

$$3d - 4 = s_1 + 2s_2 + 2s_3,$$

where s_1 is the number of fibers F with the Euler-Poincaré characteristic $e(F) = 3$ and $s_2 + s_3$ is number of fibers with $e(F) = 4$. The formula for the Euler-Poincaré characteristic of a fibered surface gives

$$e(\tilde{\Phi}) = 4 + s_1 + 2(s_2 + s_3) = 3d.$$

By Noether Formula,

$$K_{\tilde{\Phi}}^2 = 12 - 3d.$$

Since all singular points of Φ_d are rational double points, the adjunction formula for a surface in $\mathbb{P}^3$ [25, Appendix to Chapter III] gives

$$K_{\tilde{\Phi}} = (d-4)H - (d-3)\Sigma. \tag{4.1}$$

A straightforward computation shows that

$$\Sigma^2 = d - 4.$$

Let $\phi : \tilde{\Phi} \to \mathbf{F}_n$ be a birational map to a minimal ruled surface $\mathbf{F}_n$. It blows down one irreducible component $R_1^{(1)}, \ldots, R_{s_1}^{(1)}$ in each fiber $R_i^{(1)} + R_i'^{(1)}$ of the first kind, two (-1)-curves $R_i^{(2)} + R_i'^{(2)}, i = 1, \ldots, s_2$, in each fiber $R_i^{(2)} + R_i'^{(2)} + Q_i^{(2)}$ of the second kind, and two components $R_i^{(3)} + R_i'^{(3)}, i = 1, \ldots, s_3$, in fibers $R_i^{(3)} + Q_i{}^{(3)} + Q_i'^{(3)}$ of the third kind, where $R_i^{(3)}$ is a (-1)-curve.

We check that $e(\mathbf{F}_n) = 3d - s_1 - 2s_2 - 2s_3 = 4$, as it should be.

Let

$$\mathcal{R} = \sum_{i=1}^{s_1} R_i^{(1)} + \sum_{i=1}^{s_2} (R_i^{(2)} + R_i'^{(2)}) + 2\sum_{i=1}^{s_3} R_i^{(3)},$$

and

$$A = d\Sigma - (d-1)H + \mathcal{R}.$$

It follows from Proposition 3.7 that Σ does not pass through singular points of Φ_d^{norm}; hence, Σ' and H intersect each irreducible component of R with the same multiplicity equal to 1. This immediately implies that $A \cdot \mathcal{R} = 0$. We also get

$$\begin{aligned} A \cdot H &= -(d-1)d - d(d-2) + s_1 + 2s_2 + 2s_3 = 2d - 4, \\ A \cdot \Sigma &= -(d-1)(d-2) + d(d-4) + (3d-4) = 2d - 6. \end{aligned}$$

It follows that

$$A^2 = (d-1)A \cdot H - (d-1)A \cdot \Sigma = (d-1)(4-2d) - d(-2d+6) = -4.$$

Let $\sigma(A)$ be the image of A on $\mathbf{F}_n$ and $[\sigma(A)] = a\mathfrak{f} + b\mathfrak{e}$. Since $A \cdot \mathfrak{f} = 2d - 2(d-1) = 2$, we get $b = 2$. Since $A^2 = \sigma(A)^2 = -4$, we get $4a + 4n^2 = -4$, hence $a = 0, n = -1$. This shows that

$$n = 1, \quad \sigma(A) = 2\mathfrak{e},$$

where we identify $\mathfrak{e}$ with the exceptional section.

Let $\tilde{\Phi} \to \mathbf{F}_1 \to \mathbb{P}^2$ be the composition of ϕ with the blowing down of the exceptional section $\mathfrak{e}$. The surface Φ_d^{norm} becomes isomorphic to the blow-up of a set of $1 + (s_1 + 2s_2 + 2s_3) = 3d - 3$ points

$$\mathcal{P} = (p_0, p_1, \ldots, p_{s_1}, q_1, q_1', \ldots, q_{s_2}, q_{s_2}', r_1, r_1', \ldots, r_{s_3}, r_{s_3}')$$

in the plane, where

- p_0 is the image of the exceptional section $\mathfrak{e}$,
- $r_i' \succ r_i$ is infinitely near to r_i,
- p_0, r_i, r_i' are collinear,
- p_0, q_i, q_i' are collinear.

Let

$$(e_0, e_1, \ldots, e_{3d-4}, e_{3d-3})$$

be the standard geometric basis of $\mathrm{Pic}(\tilde{\Phi})$ corresponding to the blow-up of the ordered set of points $\mathcal{P}$. Here, we denote by e_{3d-3} the divisor class of the exceptional curve over p_0. We find that the conic bundle on $\tilde{\Phi}$ is given by the linear system $|e_0 - e_{3d-3}|$, its members are the pre-images of lines passing through the point p_0.

We also find that

$$[H] = de_0 - \sum_{i=1}^{3d-4} e_i - (d-2)e_{3d-3}, \tag{4.2}$$

$$[\Sigma] = (d-1)e_0 - \sum_{i=1}^{3d-4} e_i - (d-3)e_{3d-3}. \tag{4.3}$$

Lemma 4.1

$$|(d-1)e_0 - \sum_{i=1}^{3d-4} e_i - (d-3)e_{3d-3}| = \{\Sigma\}.$$

Proof Suppose the linear system contains a pencil. Since

$$de_0 - \sum_{i=1}^{3d-4} e_i - (d-2)e_{3d-3} = ((d-1)e_0 - \sum_{i=1}^{3d-4} e_i - (d-3)e_{3d-3}) + (e_0 - e_{3d-3}),$$

the linear system defines a map from $\tilde{\Phi}$ to $\mathbb{P}^1 \times \mathbb{P}^1$ embedded into $\mathbb{P}^3$ as a nonsingular quadric. This contradiction proves the lemma. □

The following theorem sums up what we have found:

Theorem 4.2 *Let $\tilde{\Phi}_d$ be a minimal resolution of the nodes on the normalization Φ_d^{norm} of a non-degenerate submonoidal surface Φ_d of degree d. Then, $\tilde{\Phi}_d$ is isomorphic to the blow-up*

$\mathrm{Bl}_{\mathcal{P}}(\mathbb{P}^2)$ *of a set* $\mathcal{P}$ *of* $3d-3$ *points in* $\mathbb{P}^2$. *The linear system* $|de_0 - \sum_{i=1}^{3d-4} e_i - (d-2)e_{3d-3}|$ *defines a birational morphism* $\phi : \tilde{\Phi}_d \cong \mathrm{Bl}_{\mathcal{P}}(\mathbb{P}^2) \to \Phi_d$. *The singular line* Γ *of* Φ_d *is the image of the unique curve* $\Sigma \in |(d-1)e_0 - \sum_{i=1}^{3d-4} e_i - (d-3)e_{3d-3}|$. *The residual conics on* Φ_d *are the images of the members of the pencil* $|e_0 - e_{3d-3}|$. *The proper transform of a line* $\langle p_0, q_i, q_i'\rangle$ *is mapped to a node* P_i *on* Φ_d *equal to the singular point of the residual conic in* $\Phi_d \cap \langle \Gamma, P_i\rangle$. *Other nodes lie in pairs on the torsal lines on* Φ_d *equal to the images of the exceptional curves over* r_i'.

Note that the blow-up model of a submonoidal surface depends on a choice of $3d-3$ points in the plane modulo projective equivalence. It depends on $2(3d-3)-8 = 6d-14$ parameters. This agrees with the number of parameters of submonoidal surfaces.

4.2 Special sections of the conic bundle

The blow-up model allows us to give simple proofs of many facts about submonoidal surfaces which can be found in [23, §131]. For example, we get the following.

Theorem 4.3 *A submonoidal surface* Φ_d *contains a smooth rational curve* S *of degree* $d-2$ *that intersects* Γ *at* $d-3$ *points and defines a section* E *of the conic bundle* $\tilde{\Phi}_d \to \Gamma^\perp$. *If* Φ_d^{norm} *is smooth, there are* 2^{3d-5} *such curves.*

Proof Suppose such a curve exists. Let us see what would be its pullback E in $\tilde{\Phi}_d = \mathrm{Bl}_{\mathcal{P}}(\mathbb{P}^2)$. Using (4.1), we get $E \cdot K_{\tilde{\Phi}} = E \cdot ((d-4)H - (d-3)\Sigma) = (d-4)(d-2) - (d-3)^2 = -1$. Since E is rational, $E^2 = -1$, and E is a (-1)-curve on $\tilde{\Phi}$. We must also have $E \cdot H = d-2$ and $E \cdot \Sigma = d-3$. This implies that $E \cdot (e_0 - e_{3d-3}) = E \cdot (H - \Sigma) = 1$. This shows that the image of E is a section of the conic bundle. Also, it suggests a construction of all possible curves S from the assertion of the theorem as the images of (-1)-curves E on $\tilde{\Phi}$ satisfying $E \cdot H = d-2$ and $E \cdot \Sigma = d-3$. If we write the divisor class of E as $ne_0 - \sum_{i=1}^{3d-4} n_i e_i - ke_{3d-3}$, we obtain:

$$n^2 - \sum_{i=1}^{3d-4} n_i^2 - k^2 = -1,$$

$$3n - \sum_{i=1}^{3d-4} n_i - k = 1,$$

$$dn - \sum_{i=1}^{3d-4} n_i - (d-2)k = d-2,$$

$$(d-1)n - \sum_{i=1}^{3d-4} n_i - (d-3)k = d-3. \tag{4.4}$$

Subtracting the fourth equation from the third one gives $n = k+1$. Substituting this in the first and the second equations, we get

$$2k = \sum_{i=1}^{3d-4} n_i^2 = \sum_{i=1}^{3d-4} n_i.$$

This implies that $n_i \in \{0, 1\}$. Let $I = \{i \in [1, 3d-4], n_i = 1\}$. We get $2k = \#I$, and the third equation gives $n = k+1$. From this, we deduce that

$$E \in |ne_0 - \sum_{i \in I} e_i - (n-1)e_{3d-3}|, \tag{4.5}$$

where $\#I = 2n$. The divisor class e on a rational surface X with $e^2 = E \cdot K_X = -1$ is not necessary represented by a (-1)-curve. However, since $E \cdot (e_0 - e_{3d-3}) = 1$, if E is reducible, then one of its irreducible components E' is a section, and other irreducible components are blown down to singular points of Φ_d. So, if Φ_d^{bp} is smooth, all the divisor class es E define special sections, and their number is equal to the number of subsets of $[1, \ldots, 3d-4]$ of even cardinality. This is the claimed number. □

We call S a *special section* of the conic bundle. Its degree (with respect to the blowing down $\tilde{\Phi}_d \to \mathbb{P}^2$) is equal to $E \cdot e_0 = n$.

We know that Φ_d^{bp} is not smooth only if either some paints p_0, p_a, p_b in $\mathcal{P}$ are collinear, or there is an infinitely near point $p_{i+1} \succ p_i$. The divisor class $e_0 - e_a - e_b - e_{3d-3}$, (resp. $e_0 - e_a - e_{a+1} - e_{3d-3}$) is the class of a (-2)-curve $R_{a,b}$ (resp. R_a) that is blown down to a singular point. Let $E = E' + R_1 + \cdots + R_k$, where E' is a section and R_i are disjoint (-2)-curves blown down to singular points (recall that, by assumption, Φ_d is non-degenerate, and hence, all singular points are ordinary nodes). Since

$$E \cdot R_i = (E' + R_1 + \cdots + R_k) \cdot R_i = -1,$$

we find that $R_i = R_{a,b}$ (resp. $R = R_a$) if and only if $a, b \in I$ (resp. $a, a+1 \in I$). The divisor class of E' is still given by formula (4.5) only with maybe smaller n.

Since the torsal line corresponds to a double fiber of the conic bundle, a special section S passes through one of the singular points of Φ_d lying on it. It may or may not pass through other singular points of Φ_d.

Assume $d = 2m$. Let S be a special section of Φ_d equal to the image of an exceptional curve E from (4.5). Let

$$[E^\dagger] \in |(m-1)H - (m-2)\Sigma - E| = |(3m-2-n)e_0 - \sum_{i \in I} e_i - (3m-n-3)e_{3d-3}|. \tag{4.6}$$

We check that $E^\dagger$ satisfies (4.4), and hence, its image in Φ_d is a special section $S^\dagger$.

We call it the *dual special section*. It follows from the definition that $S^\dagger$ is the residual curve cut out by a monoidal surface Φ'_{m-1} that contains S. Counting parameters, we obtain that such a surface is unique. For example, if $m = 2$, the union of a special section and its dual section is a plane section of Φ_4.

Example 4.4 Suppose $d = 4$. The surface $\tilde{\Phi}_4$ is isomorphic to the blow-up of a set $\mathcal{P}$ of 9 points $p_0, \ldots, p_8$ in $\mathbb{P}^2$. The linear system $|H|$ is given by curves $C^4(p_0^2, p_1, \ldots, p_8)$ and the curve Σ' is a cubic $C^3(p_0, p_1, \ldots, p_8)$. If $\tilde{\Phi}_4 = \Phi_4^{\mathrm{norm}}$, then we have 2^7 special sections. They are conics in Φ_4 intersecting Γ at one point. The union of a special section S and its dual special section $S^\dagger$ is a plane section of Φ_d. Special sections of degree one are the proper transforms of lines $\langle p_i, p_j \rangle, 0 < i < j \leq 8$. There are 28 of them. There are 70 special sections of degree 2. They are the proper transforms of conics in the plane passing through p_0 and a subset of four points in $\mathcal{P} \setminus \{p_0\}$. Special sections of degree three are dual to special sections of degree one, they are the proper transforms of cubics with a node at p_0 and passing through 6 points in $\mathcal{P} \setminus \{p_0\}$. There is one special section of degree 4, the proper transform of the quartic through $\mathcal{P}$ with a triple point at p_0. Its dual special section is of degree 0, the exceptional curve over p_0. If Φ_4^{norm} is not smooth, some of these conics become reducible or coincide. This happens when the points $p_0, \ldots, p_8$ are not in a general position. For example, the Plücker Complex surface contains only nine special sections.

4.3 Special sections and congruences of lines

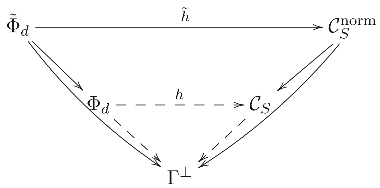
Let $S \subset \Phi_d$ be a special section. The closure of the set of lines in $\mathbb{P}^3$ intersecting S and Γ at one point is a congruence of lines $\mathcal{C}_S$ in the Grassmannian $G_1(\mathbb{P}^3)$ of lines in $\mathbb{P}^3$. Its order (= the number of rays of the congruence passing through a general point in $\mathbb{P}^3$) is equal to one and its class (= the number of rays in a general plane) is equal to $d-2$. It is one of the three possible kinds of congruences of order one in $\mathbb{P}^3$ in Kummer's classification [16] (see a modern exposition of this classification in [1] or [20]). In the Plücker embedding, $\mathcal{C}_S$ is a ruled surface of degree $d-1$ contained in the special line complex $\Omega(\Gamma)$. Its generators are the pencils of lines $\Omega(x, \Pi)$ in a plane $\Pi \in \Gamma^{\perp}$ that pass through the point $x = S \cap \Pi$.

We know that the monoidal surfaces Φ_n which contain Γ with multiplicity $n-1$ depend on $3n$ parameters. They intersect S with multiplicity $n-1$ at $d-3$ points lying on Γ. If we require that, additionally, they pass through $n(d-2)-(d-3)(n-1)+1 = n+d-2$ general points in S, we obtain that Φ_n contains S. Generically, the minimal n for which this is possible is equal to $\lceil \frac{d-2}{2} \rceil = m-1$, where $d = 2m+1$ or $d = 2m$. We expect that there will be a unique such Φ_{m-1} if d is even, and a pencil if d is odd.

There is a unique line on Φ_{m-1} with that is contained in a generator $\Omega(x, \Pi)$ of $\mathcal{C}_S$. This shows that generatrix of the ruled surface Φ_m in $\mathbb{P}^3$ is a directrix of degree m of the ruled surface $\mathcal{C}_S$. Thus, $\mathcal{C}_S$ is a ruled surface of degree $d-1$ that has a directrix of degree $m-1$.

It follows from Lemma 1.3 that $m-1$ is expected to be the minimal degree of a directrix of the ruled surface $\mathcal{C}_S$. For a special position of points $p_1, \ldots, p_{3d-3}$, the smallest degree of a directrix of $\mathcal{C}_S$, and hence, of a section of the conic bundle, could be less than $\lceil \frac{d-1}{2} \rceil$. For example, if $A_d = 0$ in Eq. (3.1), there is a special section of degree 1 given by $x_2 = x_3 = 0$.

A general point in $\mathbb{P}^3$ is contained in a unique ray of $\mathcal{C}_S$ that intersects Φ_d at one point outside $\Gamma \cup S$. This defines a birational map $h : \Phi_d \dashrightarrow \mathcal{C}_S$ over Γ that can be regularized by the following commutative diagram:



Assume that $m = \lceil \frac{d-1}{2} \rceil$ is the minimal degree of a conic section on Φ_d. As we know, in this case, $\mathcal{C}_S^{\mathrm{norm}}$ is isomorphic to $\mathbf{F}_1$ (if d is even) or $\mathbf{F}_0$ (if d is odd). If m is even, the composition of the map $\tilde{h} : \tilde{\Phi}_d \to \mathbf{F}_1$ with the blowing-down map $\mathbf{F}_1 \to \mathbb{P}^2$ is given by the linear system

$$|D| = |mH - (m-1)\Sigma - E|.$$

It is equal to the proper transform on $\tilde{\Phi}_d$ of the linear system $|mH - (m-1)\Gamma - S|$ of monoidal surfaces of degree m containing S. If d is odd, this linear system is one-dimensional, and the map $\tilde{h}$ is given by the linear system

$$|D| = |mH - (m-1)\Sigma - E - (e_0 - e_{3d-3})|.$$

In both cases, the map $\tilde{h}$ blows down the irreducible components of reducible fibers of the conic bundle which do not intersect E.

Example 4.5 Let Φ_3 be a smooth cubic surface and Γ be one of its lines. The sixteen lines skew to Γ are the special sections of Φ_3.

Choose a special section S. The congruence of lines $\mathcal{C}_S$ is the locus of lines intersecting Γ and S. It is a congruence of lines embedded in the Plücker space as a quadric surface. Each ray of $\mathcal{C}_S$ is the intersection of the planes $\Pi \in \Gamma^\perp$ and $\Pi' \in S^\perp$. The pencils $\Gamma^\perp$ and $S^\perp$ define the two rulings of $\mathcal{C}_S$. The ruling $S^\perp$ corresponds to the monoidal surfaces Φ_1. It is the ruling of directrices of the ruling $\Gamma^\perp$.

Example 4.6 Assume $d = 4$ and Φ_d^{norm} is smooth. A special section S from Corollary 4.3 is a conic that intersects Γ at one point y_0. By Corollary (4.3), there are 128 such conics (see Example 4.4).

The congruence of lines $\mathcal{C}_S$ is a smooth ruled surface of degree 3 in $G_1(\mathbb{P}^3)$. The pencil $\Omega(y_0, \langle S\rangle)$ of rays of $\mathcal{C}_S$ is a generator of $\mathcal{C}_S$. Since each ray of $\mathcal{C}_S$ intersects Γ and a unique point on S, $\Omega(y_0, \langle S\rangle)$ is also a directrix of $\mathcal{C}_S$ of degree one. The surface $\mathcal{C}_S$ isomorphic to $\mathbf{F}_1$ embedded in $\mathbb{P}^5$ by the linear system $|2\mathfrak{f} + \mathfrak{e}|$. It is contained in the special line complex $\Omega(\Gamma)$.

5 Cremona involutions associated with submonoidal surfaces

Recall that a pair $\{a, b\}$ of distinct points on $\mathbb{P}^1$ defines an involutions $\sigma_{a,b}$ uniquely determined by the property that $\sigma_{a,b}(a) = a$ and $\sigma_{a,b}(b) = b$. It sends a point x to a unique point x' such that $x + x'$, $2a$, $2b$ belong to the same linear series g_2^1 of degree 2 of $\mathbb{P}^1$. In classical terminology, the pair $\{x, x'\}$ is harmonically conjugate to the pair $\{a, b\}$. Another involution $\sigma_{a,b;p}$ associated with the pair $\{a, b\}$ requires fixing one point p on $\mathbb{P}^1$. It is uniquely determined by the property that $\sigma_{a,b;p}(a) = b$, $\sigma_{a,b}(b) = a$, $\sigma_{a,b}(p) = p$.

Any involution of $\mathbb{P}^1$ coincides with either $\sigma_{a,b}$ or $\sigma_{a,b;p}$ for some a, b, p. It is easy to check that the involutions $\sigma_{a,b}$ and $\sigma_{a,b;p}$ commute.

A plane submonoidal curve $\mathcal{H}_d$ of degree d with a $(d-2)$-multiple point x_0 defines a Cremona transformations $\mathrm{dJ}_{\mathcal{H}_d,x_0}$ (resp. $\mathrm{dJ}'_{\mathcal{H}_d,x_0}$) of $\mathbb{P}^2$ uniquely determined by the property that its restriction to a general line ℓ passing through x_0 coincides with the involution $\sigma_{a,b}$ (resp. $\sigma'_{a,b;x_0}$), where a, b are the residual intersection points of ℓ with $\mathcal{H}_d$. The involutions are the *de Jonquières birational involutions* of the plane (see [6, 7.3.6]). The fundamental points of $\mathrm{dJ}_{\mathcal{H}_d,x_0}$ are the intersection points x_i of the first polar $P_{x_0}(\mathcal{H}_d)$ with $\mathcal{H}_d$ (and the point x_0 if $d = 2$). The number of them is equal to $2d - 1 = 2g + 3$, where $g = d - 2$ is the geometric genus of $\mathcal{H}_d$. The transformation $dJ'_{\mathcal{H}_d,x_0}$ lifts to a biregular involution of the blow-up $\mathrm{Bl}_{x_0,\ldots,x_{2d-1}}(\mathbb{P}^2)$ of the fundamental points. It sends the exceptional curve over x_0 to the proper transform of the polar curve $P_{x_0}(\mathcal{H}_d)$ and sends other exceptional curves to the proper transforms of the lines $\langle x_0, x_i\rangle$ tangent to $\mathcal{H}_d$ at x_i. If $d \geq 4$, the curve $\mathcal{H}_d$ is a hyperelliptic curve and the points x_i, $i \neq 0$, are the Weierstrass points of $\mathcal{H}_d$. The set of fixed points of the lift of $\mathrm{dJ}'_{\mathcal{H}_d,x_0}$ to X is equal to the proper transform of the curve $\mathcal{H}_d$.

The transformation $\mathrm{dJ}_{\mathcal{H}_d,x_0}$ lifts to a birational involution of $\mathrm{Bl}_{x_0}(\mathbb{P}^2)$. Its fixed points are the pre-images of the points $x_1, \ldots, x_{2d-2}$

In the special case when $d = 2$, the de Jonquières transformation dJ_{C,x_0} is the orthogonal transformation of the plane that fixes x_0 and fixes the polar line $P_{x_0}(C)$ pointwise. More explicitly, let α be a vector of projective coordinates of x_0 and $C = V(q)$ for some quadratic form q, the transformation dJ_{C,x_0} is the reflection with respect to the line $P_{x_0}(C)$:

$$x = [v] \mapsto [q(\alpha)v - b_q(v,\alpha)\alpha]. \tag{5.1}$$

where b_q is the symmetric bilinear form associated with q.

The de Jonquières transformation dJ'_{C,x_0} is the standard Cremona involution with three fundamental points x_0, x_1, x_2, where x_1, x_2 are the intersection points of the polar line $P_{x_0}(C)$ with C.

Theorem 5.1 *Assume $\tilde{\Phi}_d = \Phi_d^{\mathrm{bp}}$. Fix an isomorphism $\tilde{\Phi}_d \cong \mathrm{Bl}_{\mathcal{P}}(\mathbb{P}^2)$. Assume that $d = 2m$ and $|\Sigma + (m-1)(e_0 - e_{3d-3})|$ contains a smooth curve whose image in $\mathbb{P}^2$ is a curve $\mathcal{H}_{3m-2}$ of degree $3m-2$ containing p_0 with multiplicity $3m-4$ and tangent to each line $\langle p_0, p_i\rangle$, $i = 1, \ldots, 3d-4$, at the point p_i. Then, $\tilde{\Phi}$ admits an involution $\tau_{\mathcal{P}}$ that descends to a projective involution τ of Φ_d that sends a special section S to the dual special section $S^\dagger$. The involution τ does not depend on a choice of the blowing down map $\tilde{\Phi}_d \to \mathbb{P}^2$.*

Proof The points p_i are the intersection points of $\mathcal{H}_{3m-2}$ with its polar curve $P_{p_0}(\mathcal{H}_{3m-1})$. Let $\tau_{\mathcal{P}}$ be the de Jonquières involution $\mathrm{dJ}'_{\mathcal{H}_d,p_0}$ associated with the curve $\mathcal{H}_{3m-1}$ which fixes the curve pointwise. The homaloidal linear system defining $\tau_{\mathcal{P}}$ consists of curves $C^{3m-1}(p_0^{3m-2}, p_1, \ldots, p_{3d-4})$. The exceptional curves of $\tau_{\mathcal{P}}$ are the lines $\langle p_0, q_i\rangle$, and the first polar curve $P_{p_0}(\mathcal{H}_d)$. The latter is the unique monoidal curve $C^{3m-2}(p_0^{3m-3}, p_1, \ldots, p_{3d-4})$ that is tangent to the lines $\langle p_0, p_i\rangle$ at the points p_i. The exceptional lines $\langle p_0, p_i\rangle$ are blown down to the points p_i, and the first polar is blown down to p_0. The birational transformation lifts to a biregular involution $\tau_{\mathcal{P}}$. It sends the exceptional curve E_0 over p_0 to the exceptional curve E' equal to the proper transform of $P_{x_0}(\mathcal{H}_d)$. It swaps the irreducible components of fibers of the conic bundle $\pi : \tilde{\Phi}_d \to \mathbb{P}^1$. It follows that $\tau_{\mathcal{P}}$ acts on the geometric basis $(e_0, e_1, \ldots, e_{3d-3})$ corresponding to the blow-up by

$$\begin{aligned} e_0 &\mapsto (3m-1)e_0 - \sum_{i=1}^{3d-4} e_i - (3m-2)e_{3d-3}, \\ e_i &\mapsto e_0 - e_i - e_{3d-3}, \\ e_{3d-3} &\mapsto (3m-2)e_0 - \sum_{i=1}^{3d-4} e_i - (3m-3)e_{3d-3}. \end{aligned} \tag{5.2}$$

Substituting this in (4.5), we obtain that E is mapped to $E^\dagger$. This implies that the birational transformation of Φ_d induced by $\tau_{\mathcal{P}}$ sends S to $S^\dagger$.

Using (5.2), we check that $\tau_{\mathcal{P}}$ sends $H \sim 2me_0 - \sum_{i=1}^{3d-4} -(2m-2)e_{3d-3}$ to itself. Therefore, $\tau_{\mathcal{P}}$ descends to a projective transformation of Φ_d. Since it leaves invariant 2^{6m-4} pairs $S + S^\dagger$ cut out by monoidal surfaces of degree $m-1$ with Γ as its $(m-3)$-multiple line, we see that the involution does not depend on $\mathcal{P}$. □

Remark 5.2 Assume $m \geq 2$. The plane curve $\mathcal{H}_{3m-2}$ is the image of a smooth hyperelliptic curve of genus $g = 3m-3$ under a map given by the linear system $|g_1^2 + q_1 + \cdots + q_{3m-3}|$, where q_i correspond to the branches of of $\mathcal{H}_d$ at p_0. The moduli space of point sets $\{p_0, p_1, \ldots, p_{6m-4}\}$ realized as the singular point of $\mathcal{H}_d$ and its Weierstrass points is of dimension $2g-1+3m-3 = 9m-10$. It is of codimension $12m-14-(9m-10) = 3m-4$ in the moduli space of $6m-3$ points in the plane. For example, the moduli space of quartic submonodidal surfaces contains a subvariety of codimension two parameterizing surfaces for which there exists a birational involution which swaps special conic sections with their duals.

Let $\mathcal{S}_\Gamma(\Phi_d)$ be the closure of the poles of Γ with respect to smooth residual conics C_Π in the planes $\Pi \in \Gamma^\perp$. We call it the *satellite polar curve* of Φ_d with respect to Γ.

We define a birational transformation Θ_Γ of $\mathbb{P}^3$ uniquely determined by the property that its restriction to any plane $\Pi \in \Gamma^\perp$ coincides with the de Jonquières transformation defined by reflection with respect to the line $\Gamma \subset \Pi$. It is the transformation $\mathrm{dJ}_{C,s}$, where C is the residual conic in the plane and $s = \Pi \cap \mathcal{S}_\Gamma(\Phi_d)$. Its restriction to any line ℓ in $\Pi \in \Gamma^\perp$ that passes through s coincides with the involution $\sigma_{a,b;s} = \sigma_{a,b;g}$, where $g = \ell \cap \Gamma$m and a, b are the residual intersection points of ℓ with Φ_d. The transformation Θ_Γ fixes $\mathcal{S}_\Gamma(\Phi_d)$ and Γ pointwise.

Proposition 5.3 *Fix a blow-up model $\tilde{\Phi}_d \cong \mathrm{Bl}_{\mathcal{P}}(\mathbb{P}^2)$ of Φ_d. Assume that Σ is irreducible. The restriction θ_Γ of the birational involution Θ_Γ to Φ_d is defined by the de Jonquières involution associated with the curve Σ and its $(d-3)$-multiple point p_0.*

Proof Let $\phi : \mathbb{P}^2 \dashrightarrow \Phi_d$ be the rational parameterization of Φ_d defined by the blow-up construction. The image of the line $\ell_p = \langle p, p_0\rangle$ in Φ_d is the residual conic C_Π in the plane $\Pi = \langle \phi(p), \Gamma\rangle$. The image of Σ is the line Γ. The restriction of Θ_Γ to the line ℓ_p is an involution with two fixed points from $\Sigma \cap \ell_p$. This shows that Θ_Γ defines a birational involution of the conic C_Π with fixed point from $\Gamma \cap C_\Pi$. This coincides with the restriction of the reflection involution to the conic. □

Let us find an explicit parametric equation of $\mathcal{S}_\Gamma(\Phi_d)$.

We use equation (3.1) of Φ_d. Its intersection with a general plane $\Pi_{[t_0,t_1]} = V(x_1t_0 - x_0t_1) \in \Gamma^\perp$ is equal to the pole of the line $\Pi_{[t_0,t_1]} = V(y_0)$ with respect to the conic $C_{[t_0,t_1]}$ given by (3.2). It is equal to the intersection point of the polar lines of the conic with respect to the points [0, 0, 1] and [0, 1, 0].

$$\begin{aligned} B(t_0,t_1)y_0 + D(t_0,t_1)y_1 + E(t_0,t_1)y_2 = 0,\\ C(t_0,t_1)y_0 + E(t_0,t_1)y_1 + F(t_0,t_1)y_2 = 0. \end{aligned} \tag{5.3}$$

The intersection point has coordinates

$$[y_0, y_1, y_2] = [\Delta_1(t_0,t_1), \Delta_2(t_0,t_1), \Delta_3(t_0,t_1)],$$

where Δ_i are the signed maximal minors of the matrix of the coefficients of this system of linear equations in y_0, y_1, y_2.

The image of the corresponding point $([t_0,t_1],[y_0,y_1,y_2]) \in \mathbb{P}$ to $\mathbb{P}^3$ is equal to

$$[y_0t_0, y_0t_1, y_1, y_2] = [t_0\Delta_1, t_1\Delta_1, \Delta_2, \Delta_3]. \tag{5.4}$$

After cancelling by a common factor we get a rational parameterization of the satellite polar curve $\mathcal{S}_\Gamma(\Phi_d)$.

It follows from (3.3) that the curve $\mathcal{S}_\Gamma(\Phi_d)$ passes through the singular points of the residual conics. It also passes through the points where they intersect Γ with multiplicity 2.

We expect that, in general, $\mathcal{S}_\Gamma(\Phi_d)$ is a curve of degree $2d-3$ that contains $3d-4$ singular points of residual conics and $2d-4$ points in the small discriminant $\mathcal{D}_2$ in Γ.

Example 5.4 The polar curve $\mathcal{P}_\Gamma(\Phi_d)$ could be reducible. This happens when the binary forms in the parametric equation have common factors. For example, consider a cubic surface

$$2x_0^2x_2 + 2x_1^2x_3 + x_0x_2^2 + x_1x_3^2 = 0.$$

The surface has two Eckardt points [0, 0, 1, 0] and [0, 0, 0, 1] on $\Gamma = V(x_0, x_1)$. The pencil of polar quadrics is given by

$$t_0(x_0^2 + x_0x_2) + t_1(x_1^2 + x_1x_3) = 0.$$

Its base locus is the union of four lines $V(x_0(x_0+x_2), x_1(x_1+x_3))$. The residual part consists of three lines $V(x_0, x_3)$, $V(x_1, x_2)$ and $V(x_0 + x_2, x_1 + x_3)$. The latter line is given by parametric equation $[t_0, t_1, t_0, t_1]$. The first two of the three lines are irreducible components of the intersection of the surface with the two tritangent planes. The third line coincides with $\mathcal{S}_\Gamma(\Phi_3)$. It does not lie on the surface and joins the singular points of three remaining reducible residual conics.

Proposition 5.5 *The involution Θ_Γ is given by the formula*

$$\begin{aligned} x_0 &\mapsto -x_0\Delta_1(x_0, x_1), \\ x_1 &\mapsto -x_1\Delta_1(x_0, x_1), \\ x_2 &\mapsto x_2\Delta_1(x_0, x_1) - 2\Delta_2(x_0, x_1), \\ x_3 &\mapsto x_3\Delta_1(x_0, x_1) - 2\Delta_3(x_0, x_1). \end{aligned}$$

Proof The restriction of Θ_Γ to a general plane $\Pi_{[t_0,t_1]}$ is equal to the the reflection involution

$$y = [y^*] = [y_0, y_1, y_2] \mapsto [q_t(p_0^*)y^* - 2b_t(y, p_0^*)p_0^*],$$

where $p_0 = [p_0^*]$ is the pole of the line $V(y_0)$ with respect to the conic $C_{[t_0,t_1]} = V(q_t)$, and $b_t(v, w) = \frac{1}{2}(q_t(v + w) - q(v) - q(w))$ is the polar symmetric bilinear form associated with q. Substituting the coordinates $p_0^* = (\Delta_1, \Delta_2, \Delta_3)$ in the equation of the conic (3.2), we find that the restriction of Θ_Γ to the plane is given by

$$[y_0, y_1, y_2] \mapsto (-y_0\Delta_1, y_1\Delta_1 - 2y_0\Delta_2, y_2\Delta_1 - 2y_0\Delta_3].$$

Using the translation of coordinates in $\mathbb{P}$ and in $\mathbb{P}^3$, we get the asserted formula. □

By inspection of the formula, we find that the algebraic degree of Θ_Γ for a general Φ_d is equal to $2d - 3$. Its indeterminacy locus is equal to the union of Γ and $3d - 4$ singular points of the residual conics. Its exceptional divisor is equal to the union of the planes containing $2d - 4$ residual conics that are tangent to Γ. They are blown down to the line Γ. The fixed locus of Θ_Γ is equal to the satellite polar curve.

Proposition 5.6 *The involution Θ'_Γ is given by the formula*

$$\begin{aligned} x_0 &\mapsto x_0(F_d(x_0, x_1, x_2, x_3)\Delta_1(x_0, x_1) - P(x_0, x_1)), \\ x_1 &\mapsto x_1(F_d(x_0, x_1, x_2, x_3)\Delta_1(x_0, x_1) - P(x_0, x_1)), \\ x_2 &\mapsto F_d(x_0, x_1, x_2, x_3)\Delta_2(x_0, x_1) - P(x_0, x_1)x_2, \\ x_3 &\mapsto F_d(x_0, x_1, x_2, x_3)\Delta_3(x_0, x_1) - P(x_0, x_1)x_3. \end{aligned}$$

where $\Phi_d = V(F_d)$.

Proof We use the notation from the proof of the previous proposition. The restriction of Θ_Γ to a general plane $\Pi_{[t_0,t_1]}$ is equal to the de Jonquières involution that fixes the conic $C_{[t_0,t_1]}$. For a general point y in the plane, the involution sends the point y to the unique point y' on the line $\ell_y = \langle y, p_0\rangle$ such that the pair of points $\{y, y'\}$ is harmonically conjugate to the pair $\{a, b\} = \ell_y \cap C_{[t_0,t_1]}$. Recall that, if we fix projective coordinates (u, v) on the line ℓ_y

such that $\{a, b\} = V(\alpha u^2 + 2\beta uv + \gamma v^2)$ and $\{y, y'\} = V(\alpha' u^2 + 2\beta' uv + \gamma' v^2)$, then the condition of harmonic conjugacy is $\alpha\gamma' - 2\beta\beta' + \gamma\alpha' = 0$ [6, 2.1.2]. This allows us to find the explicit formula. The line $\ell_y : [uy^* + vp_0^*]$ intersects the conic $C_{[t_0,t_1]} = V(q_t)$ at the point

$$\{a, b\} = V(u^2 q_t(y^*) + 2uvb_t(y^*, p_0^*) + v^2 q_t(p_0^*)) \tag{5.5}$$

Since y has coordinates $[u, v] = [1, 0]$,

$$\{y, y'\} = V(v(2\beta' u + \gamma' v)).$$

Hence, y' is determined by the condition $q_t(y^*)\gamma' - 2b_t(y^*, p_0^*)\beta' = 0$. The coordinates $[u, v]$ of y' are equal to $[\gamma', -2\beta'] = [b_t(y^*, p_0^*), -q_t(y^*)]$. This gives

$$y' = b_t(y^*, p_0^*)y^* - q_t(y^*)p_0^*.$$

Let A be the matrix from (3.3). We have $b_t(y^*, p_0^*) = y^* \cdot A \cdot p_0^* = y_0|A| = y_0 P(t_0, t_1)$. This yields

$$y' = [y_0^2 P(t_0, t_1) - q_t(y^*)\Delta_1, y_0 y_1 P(t_0, t_1) - q_t(y^*)\Delta_2, y_0 y_2 P(t_0, t_1) - q_t(y^*)\Delta_3],$$

where $P(t_0, t_1)$ is the discriminant of the conic bundle and $\Delta_i = \Delta_i(t_0, t_1)$. Going back to the coordinates in $\mathbb{P}^3$, we get the asserted formula. □

By inspection of the formulas, we find that the algebraic degree of Θ'_Γ for general Φ_d is equal to the union of Γ, the satellite polar curve $\mathcal{S}_\Gamma(\Phi_d)$, and the singular residual conics. Its exceptional divisor consists of the union of planes spanned by singular conics and the planes that contain an irreducible component of a singular conic and tangent to $\mathcal{S}_\Gamma(\Phi_d)$ at the singular point of the conic. The degree of the jacobian is equal to $4(3d - 4)$, and it is equal to $2(3d - 4) + (3d - 4) + (3d - 4)$. The closure of the locus of fixed points on the domain of the definition of Θ'_Γ is equal to Φ_d.

Remark 5.7 Instead of taking the satellite polar curve $\mathcal{S}_\Gamma(\Phi_d)$ intersecting a general plane $\Pi \in \Gamma^\perp$ at one point, one can take a special section S of the conic bundle. The Cremona involution $\sigma_{\Phi_d,S}$ is uniquely defined by the property that its restriction to any line contained in $\Pi \in \Gamma^\perp$ and passing through the point $S \cap \mathcal{S}_\Gamma(\Phi_d)$ coincides with the involution $\sigma_{a,b}$ that fixes the intersection points of the line with Φ_d different from s and a point on Γ. Similarly, we define the involution $\sigma'_{\Phi_d,S}$ that swaps the points a and b.

For example, a choice of a line on a smooth cubic surface Φ defines sixteen Cremona involutions $\sigma_{\Phi,S}$, and sixteen Cremona involutions $\sigma'_{\Phi,S}$.

Remark 5.8 We do not know whether the natural homomorphism

$$\mathrm{Bir}(\mathbb{P}^3, \Phi_d) \to \mathrm{Bir}(\Phi_d)$$

is surjective, nor do we know its kernel. The surjectivity is known in the case $d \leq 4$ because the surfaces are Cremona equivalent to a plane [18]. Even if the map is surjective, we do not know whether any birational involution of Φ_d lifts to a birational involution of $\mathbb{P}^3$.

6 The satellite polar variety

The construction of the Cremona involutions and the satellite polar curve $\mathcal{S}_\Gamma(\Phi_d)$ can be extended to a submonoidal hypersurface Φ_d^n in $\mathbb{P}^{n+1}$ which contains a linear subspace Γ of codimension $m > 1$ with multiplicity $d-2$. It is given by equation

$$A_d(x_0,\ldots,x_{m-1}) + 2\sum_{i=m}^{n+1} l_i x_i + \sum_{i,j=m}^{n+1} q_{ij}x_ix_j = 0,$$

where l_i (resp. $q_{ij} = q_{ji}$) are homogeneous polynomials of degree $d-1$ (resp. $d-2$) in $t_0,\ldots,t_{m-1}$. The residual quadrics are cut out by the linear system $\Gamma^\perp$ of dimension m of $n-m+1$-dimensional linear subspaces containing Γ.

As in the case of surfaces, we can define the *polar variety* $\mathcal{P}_\Gamma(\Phi_d^n)$ to be the residual part of $(d-2)\Gamma$ in $\cap_{x\in\Gamma} P_x(\Phi_d^n)$. Its irreducible component $\mathcal{S}_\Gamma(\Phi_d)$ equal to the closure of singular points of quadrics $Q_t \subset \Pi_t$, $t \in \Gamma^\perp$, of corank 1 is called the *satellite polar variety*.

Let us find its parametric equation. First we write Π_t as the span of Γ and a point $[t_0,\ldots,t_{m-1},0,\ldots,0]$. This gives the coordinates $[y,x_m,\ldots,x_{n+1}]$ in Π_t such that any point in Π_t has coordinates $[t_0y,\ldots,t_{m-1}y,x_m,\ldots,x_{n+1}]$ in $\mathbb{P}^{n+1}$. Substituting this in equation of Φ_d^n, we obtain the equation of the residual quadric Q_t

$$A_d(t_0,\ldots,t_{m-1})y^2 + 2y\sum_{i=m}^{n+1} l_i(t_0,\ldots,t_{m-1})x_i + \sum_{i,j=m}^{n+1} q_{ij}(t_0,\ldots,t_{m-1})x_ix_j = 0. \quad (6.1)$$

The equation of Γ in Π_t is $y=0$. Let $[y_0,\alpha_m,\ldots,\alpha_{n+1}]$ be the coordinates of the pole P_t of Q_t with respect to the hyperplane $\Gamma \subset \Pi_t$. Then, the polar hyperplane of P_t with respect to the quadric Q_t is the hyperplane $\Pi_t = V(y)$. We obtain a system of linear equations with unknowns $y_0,\alpha_m,\ldots,\alpha_{n+1}$

$$A = \begin{pmatrix} l_m(t_0,\ldots,t_{m-1}) & q_{mm}(t_0,\ldots,t_{m-1}) & \ldots & q_{mn+1}(t_0,\ldots,t_{m-1}) \\ \vdots & \vdots & \vdots & \vdots \\ l_{n+1}(t_0,\ldots,t_{m-1}) & q_{n+1m}(t_0,\ldots,t_{m-1}) & \ldots & q_{n+1n+1}(t_0,\ldots,t_{m-1}) \end{pmatrix} \quad (6.2)$$

defined by the partial derivatives of Q_t with respect to the variables $y,x_m,\ldots,x_{n+1}$.

This gives us a rational parameterization of $\mathcal{S}_\Gamma(\Phi_d)$

$$\mathfrak{r}: \mathbb{P}^{m-1} \dashrightarrow \mathcal{S}_\Gamma(\Phi_d), \quad (6.3)$$

given by

$$[x_0,\cdots,x_{n+1}] = [t_0\Delta_1,\cdots,t_{m-1}\Delta_1,-\Delta_2,\cdots,(-1)^{n+2-m}\Delta_{n+3-m}], \quad (6.4)$$

where Δ_i is the maximal minor of the matrix A obtained by deleting the ith column. It is of degree equal to $(n+2-m)(d-2)+1$.

Since the codimension of the space of matrices of size $(n-m+2)\times(n-m+3)$ and corank ≥ 1 is equal to 2, the subvariety of $\Gamma^\perp$ given by vanishing of the minors $\Delta_m.\ldots,\Delta_{n+1}$ is expected to be of codimension 2 in $\mathbb{P}^{m-1}$. So, if $m \geq 3$, the map $\mathfrak{r}$ is not a regular map.

As in the case of submonoidal surfaces, we can define two commuting birational transformations of $\mathbb{P}^{n+1}$ that leave Φ_d^n invariant and restrict to the identity on $\mathcal{S}_\Gamma(\Phi_d^n)$.

Let Θ_Γ (resp. Θ'_Γ) be the birational involution of $\mathbb{P}^{n+1}$ uniquely defined by the property that its restriction to any line in Π_t that passes through the point $s = \Pi_t \cap \mathcal{S}_\Gamma(\Phi_d^n)$ is the involution that interchanges (resp. fixes) the intersection points of the line with the residual quadric Q_t.j

The restriction of Θ_Γ (resp. Θ'_Γ) to Φ_d is a non-trivial birational involution θ_Γ of Φ_d^n (resp. the identity).

Example 6.1 In the case $m = n + 1$, i.e. Γ is a point, the satellite polar variety $\mathcal{S}_\Gamma(\Phi_d^n)$ coincides with the monoidal polar hypersurface $P_\Gamma(\Phi_d^n)$ of degree $d-1$. The residual conics are just pairs of points, the fibers of the projection Φ_d^n from the point Γ. The polar hypersurface passes through all ramification points of the projection. A general line in $\mathbb{P}^{n+1}$ passing through Γ intersects $\mathcal{S}_\Gamma(\Phi_d^n)$ at one point s different from the point Γ.

The involution Θ_Γ is defined by the projection that its projection to any line ℓ passing through Γ coincides with the involution $\sigma_{a,b,s} = \sigma_{a,b;\Gamma}$, where s is the residual point of the intersection of ℓ with $P_\Gamma(\Phi_d^n)$, and a, b are the residual intersection points of the line with Φ_d^n. Its restriction to Φ_d^n is defined by the projection involution.

The involution Θ'_Γ is a de Jonquières transformation associated to a submonoidal hypersurface. Its restriction to ℓ coincides with the involution $\sigma'_{a,b}$. Both involutions were studied by M. Gizatullin in [8]. He gives some explicit formulas for these involutions as well as some relations between them. J. Blanc studied dynamical properties of the compositions of involutions Θ_Γ on a cubic hypersurface [2].

Example 6.2 Assume that Φ_d^n is a general n-dimensional cubic hypersurface Φ_3^n in $\mathbb{P}^{n+1}$ with a fixed linear subspace $\Gamma \subset \mathbb{P}^{n+1}$ of codimension m. It is known that the dimension of the variety of linear subspaces $\Gamma \subset \Phi_d^n$ of codimension m in $\mathbb{P}^{n+1}$ is greater or equal than $(n+2-m)m - \binom{n-m+d+1}{d}$ [9, Lecture 12]. So, we can always find a subspace Γ of codimension m in Φ_3^n if m satisfies $(n+2-m)m - \binom{n-m+4+d}{3} \geq 0$. Of course, we can always take $m = n+1$ or n, since a cubic hypersurface of any dimension always contains a line. The satellite variety $\mathcal{S}_\Gamma(\Phi_3)$ is a rational variety of dimension $m-1$ given by a rational parameterization of degree $n - m + 2$.

Following [8], one defines the decomposition (resp. the inertia) group of Φ_d^n as the subgroup of the Cremona group $\mathrm{Cr}(n+1)$ that leaves Φ_d^n invariant (resp. restricts to the identity on Φ_d^n).

It is known that in the case of surfaces, the group $\mathrm{Bir}(\Phi_3)$ is generated by the transformations θ_Γ, where Γ is a point [17]. Is it true for $d > 3$? This would imply that the decomposition group is mapped surjectively onto $\mathrm{Bir}(\Phi_d)$. I do not know whether the same is true in dimension greater than 2, i.e. whether $\mathrm{Bir}(\Phi_d^n)$ is generated by transformations θ_Γ, and whether it is enough to take Γ to be a point. Also, I do not know whether the inertial group of Φ_d^n is generated by the transformations Θ'_Γ.

References

1. Arrondo, E.: Line congruences of low order. Milan J. Math. **70**, 223–243 (2002)
2. Blanc, J.: Dynamical degrees of (pseudo)-automorphisms fixing cubic hypersurfaces. Indiana Univ. Math. J. **62**(4), 1143–1164 (2013)
3. Beltrametti, M., Carletti, E., Gallarati, D., Bragadin, G.: Lectures on curves, surfaces and projective varieties. A classical view of algebraic geometry. Translated from the,: Italian original by Francis Sullivan, p. 2009. EMS Textbooks in Mathematics, European Mathematical Society (EMS), Zürich (2003)
4. Deopurkar, A., Patel, A.: Orbits of linear series on the projective line. arXiv:2211.16603v1 [math.AG] (2022)
5. Déserti, J., Han, F.: On cubic birational maps of $\mathbb{P}^3$. Bull. Soc. Math. France **144**(2), 217–249 (2016)
6. Dolgachev, I.: Classical Algebraic Geometry. A Modern View. Cambridge University Press, Cambridge (2012)
7. Edge, W.: The Theory of Ruled Surfaces. Cambridge University Press, Cambridge (1931)
8. Gizatullin, M.: The decomposition, inertia and ramification groups in birational geometry. Algebraic geometry and its applications (Yaroslavl, 1992), 39–45, Aspects Math., E25, Friedr. Vieweg, Braunschweig, (1994)

9. Harris, J.: Algebraic Geometry. A First Course, Graduate Texts in Mathematics, 133. Springer, New York (1995)
10. Hartshorne, R.: Algebraic Geometry. Graduate Texts in Mathematics, No. 52. Springer, New York, Heidelberg (1977)
11. Helsø, M.: Rational quartic symmetroids. Adv. Geom. **20**(1), 71–89 (2020)
12. Hudson, H.: Cremona Transformations in Plane and Space. Cambridge University Press, Cambridge (1927)
13. Jessop, C.: Quartic Surfaces with Singular Points. Cambridge University Press, Cambridge (1916)
14. Jessop, C.: A treatise of the line complex. Cambridge University Press, 1903 [reprinted by Chelsea Publ. Co, New York] (1969)
15. Kneser, M.: Quadratische Formen. Revised and edited in collaboration with Rudolf Scharlau. Springer, Berlin (2002)
16. Kummer, E.: Über die algebraischen Strahlensysteme inbesondere über die ersten und zweiten Ordnung, Abhandlungen der Königlichen Akademie der Wissenschaften zu Berlin (1866), 1–120. [Collected Papers, ed. A. Weil, vol. 2, 453-572. Springer-Verlag. 1975]
17. Manin, Y.: Hypersurfaces cubiques. II. Automorphismes birationnels en dimension deux. Invent. Math. **6**, 334–352 (1969)
18. Mella, M.: Birational geometry of rational quartic surfaces. J. Math. Pures Appl. (9) **141**, 89–98 (2020)
19. Montesano, D.: Su una classe di trasformazioni involutorie dello spazio. Rendiconti Ist. Lombardo **21**, 688–690 (1888)
20. Ran, Z.: Surfaces of order 1 in Grassmannians. J. Reine Angew. Math. **368**, 119–126 (1986)
21. Salmon, G.: A treatise on the analytic geometry of three dimension, Hodges, Foster and Co. Dublin. 1862. (5th edition, revised by R. Rogers, vol. 1-2, Longmans, Green and Co., 1912–1915; reprinted by Chelsea Publ. Co. vol. 1, 7th edition, 1857, vol. 2, 5th edition, 1965)
22. Sarkisov, V.: Birational automorphisms of three-dimensional algebraic varieties representable as a conic bundle. Uspekhi Mat. Nauk **34**(4), 207–208 (1979)
23. Sturm, R.: Die Gebilde ersten und zweiten Grades der Liniengeometrie in synthetischer Behandlung, Theil 2, Leipzig. Teubner, B.G (1898)
24. Sturm, R.: Die Lehre von den geometrischen Verwandentschaften, vol. IV. Teubner, Leipzig. B.G (1909)
25. Zariski, O.: Algebraic surfaces. Second supplemented edition. With appendices by S. S. Abhyankar, J. Lipman, and D. Mumford. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 61. Springer, New York, Heidelberg (1971)