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NEWTON POLYHEDRA AND FACTORIAL RINGS

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1. Introduction

Let

$$F=\sum a_{i_1\cdots i_n}T_1^{i_1}\dots T_n^{i_n}\in k[T_1,T_1^{-1},\dots,T_n,T_n^{-1}]=k[T,T^{-1}]$$

be a Laurent polynomial over a field k. The convex hull of the set $\{(i_1,\dots,i_n)\in \mathbf{Z}^n: a_{i_1\cdots i_n}\neq 0\}$ in $\mathbf{R}^n$ is called the Newton polyhedron of F and will be denoted by Δ_F. Following Hovanskiĭ [5] we say that F is nondegenerate if for any face Δ' of Δ_F the hypersurface $\sum_{(i_1,\dots,i_n)\in\Delta'}a_{i_1\cdots i_n}x_1^{i_1}\cdots x_n^{i_n}=0$ is non-singular in $\bar{k}^{*n}$ ($\bar{k}$ is the algebraic closure of the field k). The main result of the present paper is the following:

1.1. Theorem. *Assume that F is nondegenerate and* $\dim\Delta_F\geqslant 4$. *Then the factor ring $A_F=k[T,T^{-1}]/(F)$ is factorial* (= UFD).

For ordinary polynomials $F\in k[T_1,\dots,T_n]=k[T]$ we have

1.2. Corollary. *Assume that F is nondegenerate as a Laurent polynomial,* $\dim\Delta_F\geqslant 4$ *and Δ_F intersects each coordinate hyperplane in $\mathbf{R}^n$. Then the factor ring $k[T]/(F)$ is factorial.*

It is easy to see that the second condition for Δ_F says that F cannot be represented in the form $F=T_iF'$ for any $i=1,\dots,n$. Certainly, this condition is necessary for $k[T]/(F)$ to be a domain.

If F is a homogeneous polynomial, then the hypotheses of the corollary mean that $n\geqslant 5$ and the hypersurface $F=0$ in $\bar{k}^n$ has an isolated singularity at the origin. In that case the result is well known ([1]).

Now let Δ be any compact convex polyhedron in $\mathbf{R}^n$ whose vertices have integral coordinates. The set of all Laurent polynomials $F\in k[T,T^{-1}]$ such that $\Delta_F\subset\Delta$ is a vector space of dimension $\#\Delta\cap Z^n$, we denote it by $\Gamma(\Delta)$. It is shown in [5] that for an algebraically closed field k of characteristic zero almost all Laurent polynomials

$F \in \Gamma(\Delta)$ are nondegenerate (as usually, "almost" means belonging to some open subset of $\Gamma(\Delta)$ in the Zarisky topology).

Conjecture. If $k = \bar{k}$ and $\mathrm{char}(k) = 0$, then for any Δ of dimension ≤ 3 and almost all $F \in \Gamma(\Delta)$ the divisor class group $C(A_F)$ does not depend on F and can be computed via Δ only.

For ordinary polynomials we have a similar conjecture for the divisor class group $C(k[T]/(F))$ assuming additionally that Δ intersects each coordinate hyperplane. The only case where I know that this conjecture has been verified is the homogeneous case: $\Delta = \{(t_1, \dots, t_n) \in \mathbf{R}^n : t_1 + \dots + t_n = d\}$. Here $\Gamma(\Delta)$ is the space of homogeneous polynomials of degree d. For $\dim \Delta = 3$ and almost all such F we have (see [1]) $C(k[T]/(F)) = 0$ for $d = 1$, $d \geq 4$, $= \mathbf{Z}^6$ for $d = 3$ and $= \mathbf{Z}$ for $d = 2$. Also the homogeneous case shows that one can expect that this conjecture is valid for Δ of smaller dimension.

2. Grothendieck's Lefschetz type theorem

Let A be any noetherian normal commutative ring, $X = \mathrm{Spec}(A)$. Recall that A is factorial if and only if $C(X) = 0$ ([3]). Here for any locally noetherian scheme X, $C(X)$ denotes the divisor class group of X, that is the free abelian group generated by the set $X^{(1)}$ of points $x \in X$ such that $\dim \mathcal{O}_{X,x} = 1$ modulo the subgroup of principal divisors $(f) = \sum_{x \in X^{(1)}} \nu_x(f)x$ (f is a rational function on X, $\nu_x(f)$ is the value at f of the discrete valuation defined by the ring $\mathcal{O}_{X,x}$). If all local rings of X are factorial (e.g. X is a regular scheme), then $C(X)$ coincides with the Picard group $\mathrm{Pic}(X)$, the group of isomorphism classes of invertible sheaves on X.

The following result is the main ingredient of the proof of our theorem.

2.1. Theorem (Grothendieck [4]). *Let X be a locally noetherian scheme, $\mathcal{J}$ a quasi-coherent sheaf of ideals on X, $Y = V(\mathcal{J})$ the corresponding closed subscheme of X. Suppose that the following conditions are satisfied*:

(i) *for any point $x \in X - Y$ the local ring $\mathcal{O}_{X,x}$ is factorial of depth ≥ 3 (for example, X is regular of dimension ≥ 3)*;

(ii) $H^i(X, \mathcal{J}^{n+1}/\mathcal{J}^{n+2}) = 0$ *for $i = 1, 2$ and $n \geq 0$.*

Then the restriction homomorphism $r: \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ is bijective.

3. A construction of Hovanskiĭ

To deduce our main result from the Grothendieck theorem we need the following construction of Hovanskiĭ [5] which is based in its turn on the theory of torical varieties (see [2, 7]).

Let Δ be a compact convex polyhedron in $\mathbf{R}^n$ whose vertices $v_1, \ldots, v_m$ have integral coordinates. The function $f_\Delta : \mathbf{R}^n \to \mathbf{R}$,

$$f_\Delta(x) = \min_{y \in \Delta} \langle x, y \rangle = \min_i \langle x, v_i \rangle$$

is called the supporting function of the polyhedron Δ.

3.1. Definition. A finite set $\Sigma = \{\sigma_\alpha\}_\alpha$ of convex rational polyhedron cones in $\mathbf{R}^n$ is said to be a Δ-fan if the following conditions are satisfied:

(i) if σ is a face of some σ_α, then $\sigma = \sigma_\beta$ for some β;
(ii) for any α, β, $\sigma_\alpha \cap \sigma_\beta$ is a face of σ_α and σ_β;
(iii) $\bigcup_\alpha \sigma_\alpha = \mathbf{R}^n$;
(iv) each σ_α has a vertex (i.e. $\pm x \in \sigma_\alpha$ implies $x = 0$);
(v) $f_\Delta | \sigma_\alpha$ is a linear function for each α.

The first three conditions say that Σ is a finite rational polyhedral decomposition of $\mathbf{R}^n$ in terms of [7] or a complete fan in terms of [2]. Given a Δ-fan Σ let X_Σ be the corresponding torical variety. Recall that X_Σ is constructed as a glueing of affine varieties $X_{\sigma_\alpha} = \mathrm{Spec}(A_\alpha)$, where A_α is the subalgebra of the group algebra $k[\mathbf{Z}^n]$ generated by e^r, $\langle r, x \rangle \geq 0$ any $x \in \sigma_\alpha$. The natural injection $A_\alpha \hookrightarrow k[\mathbf{Z}^n]$ defines an open embedding of $T^n = \mathrm{Spec}(k[\mathbf{Z}^n])$ into X_{σ_α} for each α, they are glued to an open embedding of T^n (n-torus) into X_Σ. Condition (iii) of 3.1 implies that X_Σ is a complete algebraic variety.

For each $\sigma_\alpha \in \Sigma$ let M_α be a A_σ-submodule of $k[\mathbf{Z}^n]$ spanned by the e^r's, $\langle r, x \rangle \geq f_\Delta(x)$ for all $x \in \sigma_\alpha$. It can be checked that the M_α's are glued to a coherent sheaf $\mathscr{L}_\Sigma$ on X_Σ, a subsheaf of the constant sheaf $(k[\mathbf{Z}^n])_{X_\Sigma}$. Condition (v) of 3.1 implies that each M_α is a cyclic A_σ-module generated by any e^r, where $f_\Delta(x) = \langle r, x \rangle$, $x \in \sigma_\alpha$. Thus, $\mathscr{L}_\Sigma$ is an invertible sheaf.

3.2. Lemma (Hovanskiĭ). *For any Δ-fan Σ*

(a) $H^0(X_\Sigma, \mathscr{L}_\Sigma) = k[\Delta \cap \mathbf{Z}^n]$, *the subspace of* $k[\mathbf{Z}^n]$ *spanned by* e^r, $r \in \Delta$;
(b) $H^i(X_\Sigma, \mathscr{L}_\Sigma^{\otimes k}) = 0$, $0 < i < \dim \Delta$, $k \in \mathbf{Z}$.

Proof. (a) Clearly,

$$H^0(X_\Sigma, \mathscr{L}_\Sigma) = \bigcap_\alpha M_\alpha = \bigoplus_{\substack{r : \langle r, x \rangle \geq f_\Delta(x) \\ \text{for all } x \in \mathbf{R}^n}} ke^r.$$

Obviously, for any $r \in \Delta$, $\langle r, x \rangle \geq f_\Delta(x)$, $x \in \mathbf{R}^n$, then by the supporting hyperplane lemma of H. Weyl there exists a point x and a number b such that $\langle r, x \rangle < b$ and $\langle y, x \rangle \geq b$ for all $y \in \Delta$. This shows that for such x, $\langle r, x \rangle < f_\Delta(x)$. The assertion is proven.

(b) In notation of [7, Ch. I] the sheaf $\mathscr{L}_\Sigma^{\otimes k}$ coincides with the sheaf $\mathscr{F}_{f_k}$ associated with the function $f_k = kf_\Delta$, the supporting function of the polyhedron $k\Delta = \{kx, x \in \Delta\}$. If $k \geq 0$, then f_k is convex upward and hence by loc. cit. $H^i(X_\Sigma, \mathscr{F}_{f_k}) = 0$ for $i > 0$. If $k < 0$, then we have

$$H^i(X_\Sigma, \mathscr{F}_{f_k}) = \bigoplus_{s \in \mathbf{Z}^n} H^i_{Y_s}(\mathbf{R}^n; k),$$

where $Y_s = \{x \in \mathbf{R}^n : \langle x, s\rangle \geq f_k(x)\}$. Using the standard sequence of local cohomology

$$\cdots \to H^i_{Y_s}(\mathbf{R}^n) \to H^i(\mathbf{R}^n) \to H^i(\mathbf{R}^n - Y_s) \to H^{i+1}_{Y_s}(\mathbf{R}^n) \to \cdots$$

we easily get that $H^i_{Y_s}(\mathbf{R}^n) = 0$ ($H^0(\mathbf{R}^n) \to H^0(\mathbf{R}^n - Y_s)$ is always surjective) and

$$H^i_{Y_s}(\mathbf{R}^n) = H^{i-1}(\mathbf{R}^n - Y_s), \quad i \geq 2.$$

Now

$$Y_s = \{x \in \mathbf{R}^n : \langle x, s\rangle \geq k \min_i \langle x, v_i\rangle\} = \{x \in \mathbf{R}^n : \langle x, s - kv_i\rangle \geq 0, i = 1, \ldots, m\}$$

is a convex cone such that the maximal vector subspace contained in it coincides with the space

$$E_s = \{x \in \mathbf{R}^n : \langle x, s - kv_i\rangle = 0, i = 1, \ldots, m\}.$$

We easily compute its dimension

$$\dim E_s = \begin{cases} n\text{-}\dim \Delta & \text{if } s \text{ belongs to the affine hull of } k\Delta, \\ n\text{-}\dim \Delta - 1 & \text{otherwise.} \end{cases}$$

Retracting $\mathbf{R}^n$ onto the unit sphere S^{n-1} and applying the Alexander duality we obtain

$$\begin{aligned} H^{i-1}(\mathbf{R}^n - Y_s) &= H^{i-1}(S^{n-1} - Y_s \cap S^{n-1}) = H_{n-1-i}(Y_s \cap S^{n-1}) \\ &= H_{n-1-i}(E_s \cap S^{n-1}) = H_{n-1-i}(S^{\dim E_s - 1}) = 0, \quad \text{if } i < \dim \Delta. \end{aligned}$$

Consequently we get for $i \geq 2$

$$H^i_{Y_s}(\mathbf{R}^n) = H^{i-1}(\mathbf{R}^n - Y_s) = 0, \quad i < \dim \Delta; s \in \mathbf{Z}^n,$$

and hence $H^i(X_\Sigma, \mathscr{F}_{f_k}) = 0$ for $0 < i < \dim \Delta$.

Remark. The assertions of the lemma were stated in [5], proofs were omitted there.

Let us identify the algebra $k[\mathbf{Z}^n]$ with the algebra of Laurent polynomials $k[T, T^{-1}]$ by assigning e^r to $T^r = T_1^{r_1} \cdots T_n^{r_n}$, $r = (r_1, \ldots, r_n)$. Then we may identify the spaces $k[\Delta \cap \mathbf{Z}^n]$ and $\Gamma(\Delta)$ (resp. Laurent polynomials $F \in \Gamma(\Delta)$ and global sections s_F of $\mathscr{L}_\Sigma$). Let Y_F^Σ be the closed subscheme of X_Σ defined by the section s_F and $U_F^\Sigma = Y_F^\Sigma \cap T^n$ be its intersection with the open subset T^n of X_Σ. It follows immediately from our constructions that

(3.2) $\quad U_F^\Sigma \simeq \mathrm{Spec}(A_F)$

and hence does not depend on the choice of a Δ-fan Σ.

3.3. Lemma (Hovanskiĭ [5]). *Suppose that $F \in \Gamma(\Delta)$ is a nondegenerate Laurent polynomial. Then there exists a Δ-fan Σ such that X_Σ and Y_F^Σ are non-singular.*

4. Proof of the main result

Let F be a nondegenerate polynomial with $\dim \Delta_F \geq 4$. Choose some Δ_F-fan Σ satisfying the assertion of Lemma 3.3, let $X = X_\Sigma$, $Y = Y_F^\Sigma$, $U_F = U_F^\Sigma$, $\mathscr{L} = \mathscr{L}_\Sigma$. Since by Lemma 3.1 $H^i(X, \mathscr{L}^{\otimes k}) = 0$ for $k < 0$ and $i < \dim \Delta_F$ we easily get that the conditions of theorem 2.1 are satisfied for the pair (X, Y) (obviously, $Y = V(\mathscr{L}^{\otimes -1})$). Thus the restriction homomorphism $r: \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ is bijective. Now, X contains as an open dense subset the n-torus T^n, since $\mathrm{Pic}(T^n) = 0$ we get that each divisor on X is linearly equivalent to a divisor supported in $X - T^n$. This implies that each divisor on Y is linearly equivalent to a divisor supported in $Y - Y \cap T^n = Y - U_F$. Next, we have a canonical surjection $C(Y) \to C(U_F)$ whose kernel is generated by the classes of divisors supported in $Y - U_F$ [3, Corollary 7.2]. Because Y is non-singular $C(Y) = \mathrm{Pic}(Y)$ and we get that the above surjection is trivial, thus $C(U_F) = C(A_F) = 0$.

To get Corollary 1.2 we notice that under its hypotheses the singular locus of the variety $F = 0$ in $\bar{k}^n$ has codimension ≥ 2. By the Krull–Serre normality criterion [3, Theorem 4.1] this implies that the ring $k[T]/(F)$ is normal. It remains to consider the canonical restriction homomorphism $C(K[T]/(F)) \to C(A_F)$ and apply [3, Corollary 7.3] to get that $C(k[T])/(F)$ is zero as soon as $C(A_F)$ is zero.

Remark. The similar technique is applied to the case of complete intersection rings $k[T, T^{-1}]/(F_1, \ldots, F_r)$ (see the needed definitions and facts in [5]).

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