Enriques Surfaces: Old and New

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> "I ventotto anni trascorsi colà [a Bologna] furono forse i più lieti e fecondi della sua vita. In quella dotta città, dove lo spazio ristretto rende facili e frequenti i contatti tra i professori delle varie facoltà, egli trovò l'ambiente più favorevole per lo scambio delle idee e l'incremento della sua cultura..."
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> G. Castelnuovo, "Commemorazione di F. Enriques", Atti Accad. Naz. Lincei, 1947.

Introduction.

In many aspects this talk repeats my recent talk at a conference in Cortona [Do3]. As is appropriate for the occasion, more emphasis is placed on the history of Enriques surfaces. It is easy to guess that Enriques surfaces were introduced first by Federigo Enriques. We begin with a brief story of how he was led to the discovery of such surfaces.

It is known that a rational algebraic curve X is characterized by the condition that its genus is equal to zero. This was first proven by A. Clebsch [CG]. Following Riemann the genus was defined as the maximal number of linearly independent differential 1-forms of the first kind (i.e. regular everywhere). At that time a standard model of a curve was a plane one, i.e. X was defined by an equation $F(x,y) = 0$, or, in homogeneous coordinates, by a homogeneous equation $F_n(x,y,z) = 0$, where F_n is an irreducible homogeneous polynomial of degree n. A differential form of the first kind can be written in the form $\Phi(x,y)dy/F'_x(x,y)$, where $\Phi(x,y) = 0$ is the affine equation of a curve of degree n-3 which passes through singu points (including those infinitely near) of multiplicity $m > 1$ with order m-1. Each such a cu was called an adjoint curve to X. Since passing through a point with order k imposes k(k-1 conditions, the expected value of the genus is given by the formula:

$$g = (n-1)(n-2)/2-\Sigma m_p(m_p-1)/2,$$

where the sum is taken over all singular points $p \in X$ including those infinitely near, and denotes the corresponding multiplicity. It was proven by Max Noether **[No1]** that this is right number, that is, there are no excessive adjoint curves. Also it was shown that definition coincides with the topological definition of the genus given by B. Riemann. In [Clebsch generalizes this definition to the case of surfaces which were considered as the $F_n(x,y,z,w) = 0$ in the projective 3-dimensional space. The genus ("Flachengeschlecht") defined as the maximal number of linearly independent double integrals of the first kind geometrically, as the maximal number of linearly independent adjoints. Similarily to the cas curves, an adjoint surface is a surface of degree n-4 passing with order m-1 through singular curve of X of multiplicity m, and passing with order r-2 through any singular poin multiplicity r of X. In **[No1]** Noether proved that the genus is a birational invariant. A there is a formula as above (called a postulation formula) which gives the expected numbe such adjoints. For example, if the singular locus of X consists of an irreducible curv degree d and genus ρ with t double points and τ triple points (that can always be achieve projecting a nonsingular model of a surface from $\mathbb{P}^5$), the formula looks like:

$$g = \binom{n-1}{3} - (n-4)\,d+2t+\tau+\rho-1\ .$$

However, it is not true anymore that this number is always equal to the number of line independent adjoints. For example, if X is a ruled surface, the number g is negative. For reason, the first number was called the numerical genus and denoted by by p_n and the nu of adjoints was called the geometric genus and denoted by p_g. The first example of a surfa with $p_n = p_g$, and $p_n \geq 0$ was constructed by Castelnuovo in 1891 **[Ca1]**. In modern te the difference $p_g - p_n$ is equal to q, the irregularity of the surface, and the numerical gen is equal to p_a-1, where $p_a = \chi(X,\mathcal{O}_X)$ is the arithmetic genus of X. For every rational su X, both p_n and p_g are equal to zero, hence a natural question arose as to whether characterizes a rational surface. In Spring of 1894, Castelnuovo started his investigation o question by using his idea of termination of adjoints. To prove the theorem he needed to v that if the linear system of curves |2C| is contained in the linear system |2C'|, where C' adjoint curve, then |C| is contained in |C'|. At the same time, Enriques showed tha bicanonical linear system |2K| = |2C'-2C| is invariant with respect to birational transforma hence its dimension P, called the bigenus, is a birational invariant of a surface [**En1**]. Thus the problem led to the question of whether there exists a non-rational surface with $p_n = p_g = 0$, $P \neq 0$. Castelnuovo asked Enriques for help with this question, and, in July of the same year, Enriques suggested that he consider a surface of degree 6 passing doubly through the edges of the coordinate tetrahedron. The paper of Castelnuovo, containing his famous rationality criterion (X is rational if and only if $p_n = p_g = P = 0$) appeared in 1896 [**Ca2**]. In it, he presented the examples of Enriques. These surfaces can be given by homogeneous equations:

$$F_2(x_2x_3x_4, x_1x_3x_4, x_1x_2x_4, x_1x_2x_3) + x_1x_2x_3x_4G_2(x_1, x_2, x_3, x_4) = 0,$$

where F_2 and G_2 are homogeneous polynomials of degree 2. It is easy to see that these equations represent a 10-dimensional family (up to a linear transformation of variables). For every $m \neq 1$, the m-canonical linear system $|mK|$ of this surface has dimension 0. Castelnuovo gave another example, this time his own, of a non-rational surface with $p_n = p_g = 0$. In his example $P_m = \dim|mK|$ is unbounded. It turned out later, in course of the classification of surfaces, that Enriques's example is essentially the only one for non-rational surfaces with the properties that $p_n = p_g = 0$ and $\dim |mK|$ is bounded (in modern terms, of Kodaira dimension 0). The Castelnuovo example belongs to another class of surfaces, for which $\dim |mK|$ is unbounded but grows at most linearly (of Kodaira dimension 1). These surfaces always contain an elliptic pencil. Their theory, founded by Enriques himself, was completed in the works of K. Kodaira and I. Shafarevich in the sixties of this century. It allows one to classify all such surfaces (cf. [**Do1**], [**CD3**]). Only much later, in 1932, L. Campedelli and L. Godeaux have found independently different examples of surfaces with $p_n = p_g = 0$ for which $\dim |mK|$ grows quadratically (of Kodaira dimension 2). The classification of such surfaces is very far from being completed (cf. [**Do1**], [**BPV**]).

1. Models. Most of the classical results about Enriques surfaces, which from now on mean non-rational surfaces with $p_n = p_g = 0$ and of Kodaira dimension 0, were obtained by Enriques himself [**En1**],[**En2**], and later summarized in his book [**En6**]. In [**En2**] he starts with proving that every Enriques surface contains a pencil of elliptic curves $|F|$ without base points (in modern terminology, an elliptic fibration). In fact, by a very ingenious method, he proves that every curve on an Enriques surface S is linearly equivalent to a positive linear combination of elliptic or smooth rational curves. Every elliptic curve C either moves in a pencil, or taken doubly moves in such a pencil. There are exactly two double elliptic curves 2F and 2F' (which may degenerate) in every elliptic pencil. Then Enriques proves that for every elliptic pencil $|2F_1|$ one can find either i) an elliptic pencil $|2F_2|$ with $F_1 \cdot F_2 = 1$, or ii) a smooth rational curve

R with $F_1 \cdot R = 1$. In case i), the linear system $|2F_1+F_2|$ has two base points and map birationally onto a double plane $z^2 = F_8(x,y)$ with the branch curve of degree 8 compose two lines and a sextic which has a node at the point of intersection of the two lines and has tacnodes whose tangents are the two lines. In case ii) the linear system $|3F_1+R|$ does the s but the branch curve degenerates into the union of two lines and a sextic with 3 tacnodes; of them is situated at the intersection point of the two lines, and another is infinitely near We will refer to these double planes as an Enriques double plane (resp. an Enriques degen double plane). It is difficult to understand on what grounds Enriques asserted that a "gen Enriques surface has a representation as a non-degenerate double plane. The only explan is that he was able to count the number of parameters for isomorphism classes of degen double planes (which is equal to 9) and compare it with the number of parameters of degenerate planes (which is equal to 10). Note that the formula of Noether **[No2]** fo number of moduli of an algebraic surface which existed at the time of writing **[En2]** cou be applied (one of its assumptions was the condition $p_g \geq 3$). A more general formula f number of moduli was obtained by Enriques himself a little later **[En5]**. It agrees counting of constants and gives the answer 10 for the moduli of Enriques surfaces. Enriques shows that, essentially, all Enriques surfaces arise from his earlier constructio sextic surface passing doubly through the edges of the coordinate tetrahedron (an En sextic). First he checks that the number of constants for the sextic construction is also show directly that a general Enriques surface is birationally isomorphic to an Enriques he, starting from the non-degenerate double plane construction, finds an elliptic curve F that $F_1 \cdot F_2 = F_1 \cdot F_3 = 1$ and shows that the linear system $|F_1+F_2+F_3|$ maps S birationall an Enriques sextic. The proof of the existence of such a curve is one of the most o points in Enriques's memoir. This argument has been reconstructed much later in the th Michael Artin at M.I.T **[Ar]** and Boris Averbukh in Moscow **[AS]**, **[Av]**. It ha observed by Castelnuovo that the sextic construction may degenerate but also gi Enriques surface. One of these degenerations corresponds to the case when the edges tetrahedron pass through one point. The sextic acquires a quadruple point and represented by an equation of the form:

$$F_2(l_2l_3l_4, l_1l_3l_4, l_1l_2l_4, l_1l_2l_3) + l_1l_2l_3l_4G_2(l_1, l_2, l_3, l_4) = 0,$$

where the l_i's are linearly dependent homogeneous linear forms in projective coordin x_2, x_3 and x_4. This case occurs when the adjoint linear system $|F_1'+F_2+F_3|$ does no birational map. In the second degeneration two opposite edges become infinitely near case the sextic acquires a triple line and a double line infinitely near to it. Its equation can be given in the form:

$$a(l_1l_2l_3)^2+bl_1l_2^2l_3^2l_4+cl_1^4l_2^2+dl_1^4l_3^2+l_1^2l_2l_3G_2(l_1,l_2,l_3,l_4) = 0,$$

where l_1,l_2,l_3 and l_4 are some linear forms in projective coordinates, a ,b,c,d are constants. This degeneration occurs when one starts with a degenerate Enriques double plane. The corresponding linear system is $|2F_1+F_2+R|$, where F_1 and F_2 are elliptic curves, R is a smooth rational curve, and $F_1 \cdot F_2 = 1$, $F_1 \cdot R = 1$, $F_2 \cdot R = 0$. This result of Enriques was also reconstructed in the theses of Artin and Averbukh. Only recently, it was shown that the last case can always be avoided (see **[CD3]**, Corollary 4.9.2).

In 1901 Gino Fano (born on the same day as Enriques) observed that the congruence of lines in $\mathbb{P}^3$ which are contained in a subpencil of quadrics in a fixed general web of quadrics is an Enriques surface **[Fa1]**. These congruences, called nowadays Reye congruences (cf. **[Co1]**), were introduced much earlier by Darboux **[Da]** and then were studied by Reye **[Re]**. Via its Plücker embedding this congruence is isomorphic to a surface of degree 10 in $\mathbb{P}^5$ lying on a quadric. This construction gives only a 9-parameter family of Enriques surfaces, and later Fano proved that a general Enriques surface can be embedded into $\mathbb{P}^5$ as a surface of degree 10 not necessarily lying on a quadric **[Fa2]**. This surface contains 20 plane cubic [illegible]s $F_{\pm i}$ with $F_i \cdot F_j = 1$ if $i+j \neq 0$ and $F_i \cdot F_{-j} = 0$, $i = 1,\ldots,10$. The linear system $|F_1+F_2+F_3|$ maps the surface birationally onto an Enriques sextic. Conversely, starting from an Enriques sextic surface, Fano finds a quintic elliptic curve C lying on it which together with two elliptic curves F_1 and F_2 coming from the edges of the tetrahedron form the linear system $|F_1+F_2+F_3|$ which maps the surface onto a surface of degree 10 in $\mathbb{P}^5$. Again this is true only generically. Only recently, it was proven that for every Enriques surface S one can find a birational morphism from S onto a surface of degree 10 in $\mathbb{P}^5$ with at most double rational points as singularities **[Co1]**, **[CD3]**. In particular, every Enriques surface which does not contain smooth rational curves is isomorphic to a surface of degree 10 in $\mathbb{P}^5$.

The double plane construction of Enriques can be modified by considering a morphism of degree 2 onto some other rational surface. This can be obtained by applying some birational transformations to the plane. Thus one may consider any Enriques surface as a double cover of a quadric (the degenerate case corresponds to a singular quadric) (cf. **[Ho]**), or as a double cover of a 4-nodal Del Pezzo surface of degree 4 (in the degenerate case the Del Pezzo surface acquires 2 nodes and one double rational point of type A_3) (cf. **[BP]**, **[Do2]**). There is a generalization of this construction to the case of arbitrary characteristic **[CD3]**. A systematic study of linear systems on Enriques surfaces was undertaken by F. Cossec in **[Co1]**, **[Co2]**.

In particular, all classic models of Enriques and Fano were reconsidered from the uniforn approach by using the arithmetic of the quadratic form defined on the Picard group of surface. It was shown that any Enriques surface admits a non-degenerate double plan construction. This was certainly unknown to Enriques.

2. Enriques surfaces and K3-surfaces. A non-singular quartic surface in $\mathbb{P}^3$ has all th genera p_n, p_g, $P_m = 1$ equal to 1. The class of surfaces with such invariants (later christened b André Weil as surfaces of type K3) has a long history. The ubiquitous Kummer surface belong to this class. Other examples of such surfaces can be obtained by taking a complet intersection of three quadrics in $\mathbb{P}^5$. In his joint work with F. Severi on hyperelliptic surface (surfaces which are covered by a complex torus) Enriques discovered an example of such surface which admits a fixed-point-free involution with quotient isomorphic to an Enrique surface. In **[En4]** he proved that a general Enriques surface S can be obtained as a quotient a K3-surface by a fixed-point-free involution. To show this he considers the double cover $\mathbb{P}^3$ branched along the coordinate tetrahedron and shows that the induced cover of the sext surface is a K3-surface. Nowadays this result is almost trivial (true if the ground field is characteristic different from 2) and, by using standard arguments, follows from the fact that t canonical class K on an Enriques surface is a non-trivial 2-torsion element in the Picard grou In this way the study of Enriques surfaces over a field of characteristic different from 2 reduced to the study of K3-surfaces with fixed-point-free involutions. This relationship pla very important role in the recent work on Enriques surfaces (see **[BPV]**). Many examples Enriques surfaces obtained as quotients of K3-surfaces were given in works of L. Godea (cf. **[Go]**). Only recently it was shown by F. Cossec **[Co2]** and A. Verra **[Ve]** that eve Enriques surface is birationally isomorphic to the quotient of the intersection of three quadr in $\mathbb{P}^5$ by a fixed-point-free involution.

3. Automorphisms. Enriques was the first who observed that a general Enriques surfac admits infinitely many birational automorphisms. His argument goes as follows. Take linear system $|F_1+F_2+F_3|$ which maps the surface onto a non-degenerate Enriques sextic. T the pencils $|2F_2|$ and $|2F_3|$ cut out on every curve F from $|2F_1|$ two linear pencils of degre Their difference defines on each F a divisor class ε of degree 0. For every point $x \in F$ the s $x+\varepsilon$ is linearly equivalent to a unique point y on F, and the correspondence $x \to y$ defin birational automorphism of S. He shows that ε is of infinite order, hence the obtai automorphism of S is of infinite order. Of course, this argument requires a justificat certainly lacking in his paper. It is not clear why all the translation automorphisms of each member of a pencil are induced by an automorphism of the whole surface. The necessary technical tool for the needed justification is the notion of the Jacobian variety and its principal homogeneous spaces for an elliptic curve over a functional field. This was used in the later work of I. Shafarevich [AS]. In fact, Enriques applied this argument to a larger class of algebraic surfaces, namely, surfaces admitting an elliptic fibration. His study of such surfaces is, in my opinion, is one of the best proofs of Enriques' genius. Careful reading of his work on these surfaces reveals that, a long time before the fundamental works on elliptic surfaces by K. Kodaira and I. Shafarevich, he was aware of such concepts as principal homogeneous spaces, logarithmic transformation and Jacobian surfaces.

It is not clear whether Enriques and his contemporaries understood that the group of birational isomorphisms of a minimal non-ruled surface is equal to the group of biregular automorphisms. After all, the theory of minimal models was clarified only much later in the works of Oskar Zariski.

At the very end of [En 2] Enriques asks whether there exists an Enriques surface with only finitely many automorphisms. This question was answered in 1910 by Fano. In **[Fa 2]** he constructs a special web of quadrics in $\mathbb{P}^3$ whose Reye congruence is an Enriques surface ith finitely many automorphisms. I have discovered this result of Fano only very recently, while visiting the University of Torino and looking through Fano's archive. Just prior to this I have published a paper [Do2] with an example of an Enriques surface with finitely many automorphisms, wrongly believing that this was the first example of such a kind. The example of Fano is different, his argument for the proof of finiteness needs justification, and his claim about the structure of the corresponding finite group is wrong. Very recently, V. Nikulin **[Ni2]** classified all Enriques surfaces with finitely many automorphisms from the point of view of the structure of the Picard group of its K3-cover. An explicit geometric classification (by means of equations) was given later by S. Kondō **[Ko]**. There are 7 classes of such surfaces.

The first explicit computation of the (infinite) group of automorphisms of a general Enriques surface was given by W. Barth and C. Peters in [BP]. Independently, this result, in a much more general context, was obtained by Nikulin **[Ni1]**. By acting on the Picard group the automorphism group is represented in the orthogonal group of a certain even unimodular lattice of signature (1,9). Its image is equal to its level 2 congruence subgroup factored by the subgroup generated by the transformation $x \to -x$. A purely geometric calculation of the automorphism group of a generic Reye congruence was given in **[CD2]** (though, again the result can be deduced from Nikulin's results based on the transcendental techniques o period spaces of K3-surfaces).

4. Rational and elliptic curves on Enriques surfaces. Another of "belle quest concernenti la superficie F_6" asked in **[En2]** was the following: What is the distributio linear systems of a given order, especially isolated elliptic curves, on an Enriques se Today we understand this question as a more general question about the Picard group c Enriques surface. It was studied in detail in many recent w (**[Co2],[Co3],[CD1],[CD3]**). Let N_S be the Picard lattice of an Enriques surface S, the group of divisors modulo numerical equivalence equipped with the intersection form. It be shown (rather easily, if the ground field is the field of complex numbers, and much easily in the general case) that this lattice is an even unimodular lattice of signature (1,9), as such is isomorphic to the lattice E_{10}, the direct sum of the standard hyperbolic plane U the root lattice E_8 of a simple Lie algebra of type E_8 (taken with the opposite sign). Though lattices of all Enriques surfaces S are isomorphic as abstract lattices, their semigroup positive or ample divisor classes depend essentially on S. By Riemann-Roch, every div class D with $D^2 \geq 0$ is effective, however, if $D^2 < 0$, D is never effective unless the sur contains smooth rational curves R with $R^2 = -2$ (in which case the surface is said to be n). For instance, every surface represented by a degenerate Enriques sextic or a degene double plane is nodal. The converse was proven only recently by Cossec (**[Co3],[CD** Applying reflection transformations $x \rightarrow x+(x\bullet e)e$ of N_S, where e is the class of a sm rational curve (a nodal curve), allows one to transform every divisor D to a divisor D' whic numerically effective (i.e. $D'\bullet C \geq 0$ for every curve C'). If D is such a divisor, $D^2 = 0$, an is not divisible by an integer, then D is an isolated curve of arithmetic genus 1. For suc curve |2D| is an elliptic pencil (if the characteristic is different from 2). If D is numeric effective (nef) and $D^2 > 0$ then the property of the map f_D given by the linear system depends very much on the number $\Phi(D)$ which is equal to the minimum of the intersect numbers $D\bullet F$, where F is any elliptic curve. Thus $\Phi(D) = 1$ if and only if f_D has base poi $\Phi(D) = 2$ if and only if f_D is a birational map onto a non-normal surface, or $D^2 = 4$, 6 o and f_D is a 4 to 1 map onto $\mathbb{P}^2$ if $D^2 = 4$, and f_D is a 2 to 1 map onto a rational surface if D 6 or 8. Finally, if $\Phi(D) \geq 3$, the map f_D is a birational map onto a surface with only dou rational points as singularities. In particular, if F is unnodal, f_D is an isomorphism onto image. The study of the arithmetic of the lattice E_{10} shows that $\Phi(D) \geq 3$ could happen onl $D^2 \geq 10$. The Fano model of a surface of degree 10 in $\mathbb{P}^5$ corresponds to such a divisor w

$D^2 = 10$. The sextic model corresponds to a divisor D with $D^2 = 6$ and $\Phi(D) = 2$, and the double plane model corresponds to D with $D^2 = 4$ and $\Phi(D) = 1$.

Returning to the original question of Enriques, we see that it asks (the part concerning isolated elliptic curves) about the description of all elliptic curves F whose classes are primitive vectors in N_S and $F \cdot D = n$ for a fixed class of a nef divisor D with $\Phi(D) = 2$, $D^2 = 6$, and a fixed positive integer n (degree of F). Let $E(D)_n$ be the set of such F's. Each such a set is finite. If S is general, then the set of n for which $E(D)_n \neq \emptyset$ is infinite. We do not know how to compute the function $f(n) = \#(E(D))_n$. We know that $f(2) = 6$ (the elements of $E(D)_2$ are mapped to the six edges of the tetrahedron), $f(3) = 98$, and $f(4) = 756$. The complete answer to this question can be obtained by further study of the arithmetic of the lattice E_{10}.

Another interesting question is the distribution of nodal curves on a nodal Enriques surface. This question was never considered in the classic literature and has been studied only recently. We refer to **[Do3]** for a survey of some results and problems concerning this question. Assuming the automorphism group of a surface is known, one can compute the number of its orbits in the set of linear systems of given genus. For example, the number of the orbits in the set of isolated elliptic curves on a general Enriques surface is equal to 1054, the number of linear systems |D| with $D^2 = 6$ and $\Phi(D) = 2$ on a general Enriques surface (i.e. the number of different representations as an Enriques sextic) is equal to 10,792,910 (see ̄]). Similar computations can be made for a general nodal surface (**[CD2]**). We could be proud to show these kinds of results to Enriques.

5. Moduli. Nothing was known in the old days about the moduli space of Enriques surfaces except that the number of moduli is equal to 10. We refer to a survey of the modern development in **[Do3]**.

6. Enriques surfaces over fields of positive characteristic. All classic work on surfaces silently assumed that the ground field was the field of complex numbers. However, transcendental methods were never popular among Italian algebraic geometers. Thus many of the proofs could be extended almost word by word to any characteristic. In the seventies the Enriques classification of surfaces was extended to the case of arbitrary characteristic in the works of E. Bombieri and D. Mumford **[M]**, **[BM1]**, **[BM2]**. The case of Enriques surfaces was given special attention. It turned out in their work that only the case of characteristic 2 presents special difficulties. Since then much work was done on the study of this case. We refer for complete references to **[CD3,CD4]**.

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