ORBITAL COUNTING OF CURVES ON ALGEBRAIC SURFACES AND SPHERE PACKINGS

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To the memory of Andrey Todorov

Abstract. We realize the Apollonian group associated to an integral Apollonian circle packings, and some of its generalizations, as a group of automorphisms of an algebraic surface. Borrowing some results in the theory of orbit counting, we study the asymptotic of the growth of degrees of elements in the orbit of a curve on an algebraic surface with respect to a geometrically finite group of its automorphisms.

1. Introduction

Let Γ be a discrete subgroup of isometries of a hyperbolic space $\mathbb{H}^n$ and let $\mathrm{O}_\Gamma(x_0)$ be its orbit. Consider a family of compact subsets $\mathcal{B}_T$ of $\mathbb{H}^n$ whose volume tends to infinity as $T \to \infty$. The problem of finding the asymptotic of $\#\mathrm{O}_\Gamma(x_0) \cap \mathcal{B}_T$ is a fundamental problem in harmonic analysis, number theory, ergodic theory and geometry. In this paper we discuss an application of this problem to algebraic geometry.

Let X be a smooth projective algebraic surface and let $\mathrm{Num}(X)$ be the group of divisor classes on X modulo numerical equivalence. For any divisor class D we denote by $[D]$ its image in $\mathrm{Num}(X)$. The intersection form on divisor classes defines a non-degenerate symmetric bilinear form on $\mathrm{Num}(X)$ of signature $(1, n)$. Let Γ' be a group of automorphisms of X such that its image Γ in the orthogonal group $\mathrm{O}(\mathrm{Num}(X))$ is an infinite group. This implies that the Kodaira dimension of X is less than or equal to 1.

For any divisor class D we denote by $\mathrm{O}_\Gamma(D)$ the Γ-orbit of $[D]$. Fix an ample divisor class H and an effective divisor class C. For any positive real number T, let

$$N_T(H, C) = \#\{[C'] \in \mathrm{O}_\Gamma(C) : H \cdot C' \leq T\} = \#\{H' \in \mathrm{O}_\Gamma(H) : H' \cdot C \leq T\}.$$

We are interested in the asymptotic of this function when T goes to infinity. As far as I know, this problem was first considered by Arthur Baragar in his two papers [3], [4].[1]

[1] I thank Serge Cantat for these references.

In the present paper we explain how this problem relates to a general orbital counting problem in the theory of discrete subgroups of Lie groups. More precisely, we consider the group $G = \mathrm{SO}(1,n)_0$ realized as the group of orientation preserving isometries of the hyperbolic space $\mathbb{H}^n \subset \mathbb{P}(V)$ associated with the real inner product vector space $V = \mathrm{Num}(X)_{\mathbb{R}}$ of signature $(1,n)$. We will represent points of $\mathbb{H}^n$ (resp. points in its boundary $\partial\mathbb{H}^n$, resp. the points in $\mathbb{P}^n \setminus \overline{\mathbb{H}^n}$) by vectors $v \in V$ with $(v,v) = 1, (v,h_0) > 0$ (resp. positive rays of vectors with $(v,v) = 0, (v,h_0) > 0$, resp. vectors with $(v,v) = -1, (v,h_0) > 0$), where h_0 is a fixed vector with $(h_0,h_0) = 1$. Let $\mathcal{B}_T(e)$ be the set of points x in $\mathbb{H}^n$ whose hyperbolic distance from the point $e \in \mathbb{H}^n$ (resp. from a fixed horosphere with center at e if $e \in \partial\mathbb{H}^n$, resp. from the orthogonal hyperplane H_e with normal vector e if $e \notin \overline{\mathbb{H}^n}$) is less than or equal to T. We use the following result (see [28]) which represents the state of the art in the study of orbital counting.

Theorem 1.1. *Assume that Γ is a non-elementary geometrically finite discrete subgroup of orientation preserving isometries of $\mathbb{H}^n$. Let δ_Γ be the Hausdorff dimension of the limit set $\Lambda(\Gamma)$ of Γ. If $(e,e) \leq 0$, we additionally assume that the orbit $O_\Gamma([e])$ is a discrete set and $\delta_\Gamma > 1$ if $(e,e) < 0$. Then there exists a positive constant $c_{\Gamma,h_0,e}$ depending only on $\Gamma, [h_0]$ and e such that*

$$\lim_{T\to\infty} \frac{\# O_\Gamma(h_0) \cap \mathcal{B}_T}{\exp(T)^{\delta_\Gamma}} = c_{\Gamma,h_0,e}.$$

We apply this theorem to our situation to obtain the following theorem.[2]

Theorem 1.2. *Let X be an abelian surface, or a K3 surface, or an Enriques surface, or a rational surface. Let Γ' be a group of automorphisms of X such that its image Γ in $O(Num(X))$ is a non-elementary geometrically finite discrete group. Fix an ample numerical divisor class $H \in Num(X)$ and an effective numerical divisor class $C \in Num(X)$. Assume that $\delta_\Gamma > 1$ if $C^2 < 0$. Then there exists a positive constant $c_{\Gamma,H,C}$ depending only on $\Gamma, [H]$ and $[C]$ such that*

$$\lim_{T\to\infty} \frac{N_T(H,C)}{T^{\delta_\Gamma}} = c_{\Gamma,H,C}.$$

As we see, the main ingredient of the asymptotic expression is the Hausdorff dimension δ_Γ of the limit set $\Lambda(\Gamma) \subset \partial\mathbb{H}^n$ of Γ. One of the first and the most beautiful example where δ_Γ was computed with many

[2] A recent paper of A. Baragar [5] contains the same result under the assumption that $\delta_\Gamma > \frac{n-1}{2}$.

decimals is the example where Γ is an Apollonian group in $\mathbb{H}^3$ whose limit set is the closure of a countable union of circles that intersect in at most one point, an *Apollonian gasket*. It was proved by D. Boyd [8] that the asymptotic of curvatures of circles in an Apollonian gasket is equal to $c(T)T^{\delta_\Gamma}$ for some function $c(T)$ such that $\lim_{T\to\infty} \frac{c(T)}{T} = 0$. He also showed that $1 < \delta_\Gamma < 2$. It is known now that δ is about 1.305686729... [39]. An Apollonian sphere packing is a special case of a Boyd-Maxwell sphere packing defined by certain Coxeter groups of hyperbolic type introduced by G. Maxwell in [26].[3]. The Hausdorff dimension of the limit set of the Boyd-Maxwell Coxeter groups coincides with the sphere packing critical exponent equal to

$$\inf\{s : \sum_{S\in\mathfrak{P}} r(S)^s < \infty\} = \sup\{s : \sum_{S\in\mathfrak{P}} r(S)^s = \infty\}, \tag{1.1}$$

where $r(S)$ denotes the radius of a sphere S in a sphere packing $\mathfrak{P}$. Although the Hausdorff dimension of the limit set of a geometrically finite group is usually difficult to compute, the bounds for sphere packing critical exponent is easier to find.

We give several concrete examples where the Boyd-Maxwell groups, and, in particular the Apollonian groups, are realized as automorphism groups of algebraic surfaces so that our counting problem can be solved in terms of the sphere packing critical exponent. Since this paper addresses a problem in algebraic geometry, we give a short introduction to the hyperbolic geometry and the theory of sphere packings in a language which we think is more suitable for algebraic geometers. Whenever the details or references are omitted, they can be found in [1], [2], [34], or [40].

It is my pleasure to thank Peter Sarnak for his inspiring talks in Ann Arbor, in April 2014, that gave rise to the present paper. The paper could never be written without the assitance of Jeffrey Lagarias who patiently answered my numerous questions on the theory of Apollonian sphere packings, of Boris Apanasov and Serge Cantat who helped me to correct some of my illiteracy in the theory of Kleinian groups, and of Hee Oh for explaining to me the state of the art in the general orbital counting problem.

I am also thankful to the organizers of the conference at Schiermonnikoog2014 for giving me an opportunity to talk about the beautiful subject of Apollonian circle packings.

[3]We borrow the terminology from [10]

2. Hyperbolic space

Let $\mathbb{R}^{n,1}$ denote the Minkowski space defined by the quadratic form (the *fundamental quadratic form*)

$$q = -t_0^2 + \sum_{i=1}^{n} t_i^2$$

of signature $(n,1)$.[4] We denote by (v,w) the value of the associated symmetric form on two vectors $v,w \in \mathbb{R}^{n,1}$, so that $q(v) = (v,v)$. Let

$$\mathbb{H}^n = \{v \in \mathbb{R}^{n,1} : q(v) < 0\}/\mathbb{R}^* \subset \mathbb{P}^n(\mathbb{R})$$

be the n-dimensional *hyperbolic* (or *Lobachevsky*) space, a Riemannian space of constant negative curvature. Its points are lines $[v] = \mathbb{R} \cdot v$ spanned by a vector $v \in \mathbb{R}^{n,1}$ with $(v,v) < 0$. The set of real points of the *fundamental quadric* $\mathsf{Q} : q = 0$ is called the *absolute*. The union of $\mathbb{H}^n$ and the absolute is the closure $\overline{\mathbb{H}}^n$ of $\mathbb{H}^n$ in $\mathbb{P}^n(\mathbb{R})$. In affine coordinates $y_i = x_i/x_0$, one represents $\mathbb{H}^n$ by the interior of the n-dimensional ball

$$B_n = \{(y_1,\ldots,y_n) : \sum_{i=1}^{n} y_i^2 < 1\}.$$

This is called the *Klein model* of $\mathbb{H}^n$. In this model the Riemannian metric of constant curvature -1 is given by

$$ds^2 = \frac{(1-|y|^2)\sum_{i=1}^n dy_i^2 + (\sum_{i=1}^n y_i dy_i)^2}{(1-|y|^2)^2},$$

where $|y|^2 = \sum_{i=1}^n y_i^2$. One can also represent any point in $\mathbb{H}^n$ by a vector $v = (x_0,\ldots,x_n) \in \mathbb{R}^{n,1}$ with $(v,v) = -1, x_0 > 0$. Thus we obtain a model of $\mathbb{H}^n$ as one sheet of a two-sheeted hyperboloid Y^n in $\mathbb{R}^{n,1}$

$$-t_0^2 + \sum_{i=1}^{n+1} t_i^2 = -1,\ t_0 > 0.$$

This model is called the *vector model* of $\mathbb{H}^n$. The hyperbolic distance $d(v,w)$ in this model between two points in $\mathbb{H}^n$ satisfies

$$\cosh d(v,w) = -(v,w). \tag{2.1}$$

[4]Note that we use the signature $(n,1)$ instead of $(1,n)$ to agree with the standard notations in hyperbolic geometry.

Here and later we use that, for any two vectors v, w in the vector model of $\mathbb{H}^n$, we have $(v, w) < 0$ (see [34], Theorem 3.1.6). The maps

$$B^n \to Y^n,\ (y_1, \dots, y_n) \mapsto \big(\frac{1}{\sqrt{1-|y|^2}}, \frac{y_1}{\sqrt{1-|y|^2}}, \dots, \frac{y_n}{\sqrt{1-|y|^2}}\big)$$

and

$$Y^n \to B_n,\ (t_0, \dots, t_n) \mapsto \big(\frac{t_1}{t_0}, \dots, \frac{t_n}{t_0}\big)$$

are inverse to each other.

The induced metric of $\mathbb{R}^{n,1}$ $ds^2 = -dt_0^2 + \sum_{i=1}^n dt_i^2$, after the scaling transformation $y_i = t_i/q(t)^{1/2}$, defines a Riemannian metric on $\mathbb{H}^n$ of constant curvature -1

$$ds^2 = \frac{1}{(1-|y|^2)^2}\big((1-|y|^2)\sum_{i=1}^n dy_i^2 + (\sum_{i=1}^n y_i dy_i)^2\big), \tag{2.2}$$

where $|y|^2 = y_1^2 + \dots + y_n^2$. Consider the section of B_n by the hyperplane $y_n = 0$. It is isomorphic to the $n-1$-dimensional ball

$$'B_{n-1} = \{(y_1, \dots, y_{n-1}) \in \mathbb{R}^{n-1} : \sum_{i=1}^{n-1} y_i^2 < 1\}. \tag{2.3}$$

The restriction of the metric (2.2) to $'B_{n-1}$ coincides with the metric of $\mathbb{H}^{n-1}$. Thus we may view $'B_{n-1}$ as a ball model of a geodesic hypersurface of $B_n = \mathbb{H}^n$.

Let $p = [1, 0, \dots, 0, -1] \in \mathsf{Q}(\mathbb{R})$ be the southern pole of the absolute. The projection $\mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$ from this point is given by the formula $[t_0, \dots, t_n] \mapsto [t_1, \dots, t_{n-1}, t_0 + t_n]$. Since the hyperplanes $t_0 + t_n = 0, t_0 = 0$ do not intersect $\mathsf{Q}(\mathbb{R})$, $\mathsf{Q}(\mathbb{R}) \setminus \{p\}$ maps bijectively onto $\mathbb{R}^{n-1}$ with coordinates $u_i = \frac{y_i}{y_n+1}$. The rational map defined by the formula

$$\Phi : \mathbb{P}^{n-1} \to \mathsf{Q},\ [x_0, \dots, x_{n-1}] \mapsto [x_0^2 + |x|^2, 2x_0x_1, \dots, 2x_0x_{n-1}, x_0^2 - |x|^2], \tag{2.4}$$

where $|x|^2 = x_1^2 + \dots + x_{n-1}^2$, blows down the hyperplane $x_0 = 0$ to the point p and equals the inverse of the restriction of the projection $\mathsf{Q}(\mathbb{R}) \setminus \{p\} \to \mathbb{P}^{n-1}(\mathbb{R})$. In affine coordinates $u_i = x_i/x_0$ in $\mathbb{P}^{n-1} \setminus \{x_0 = 0\}$ and affine coordinates $(y_1, \dots, y_n)$ in $\mathsf{Q} \setminus \{t_0 = 0\}$ the map is defined by the formula

$$(u_1, \dots, u_{n-1}) \mapsto (y_1, \dots, y_n) = \big(\frac{2u_1}{1+|u|^2}, \dots, \frac{2u_{n-1}}{1+|u|^2}, \frac{1-|u|^2}{1+|u|^2}\big), \tag{2.5}$$

where $|u|^2 = u_1^2 + \dots + u_{n-1}^2$. Let Q^+ be the part of $\mathsf{Q}(\mathbb{R})$ where $y_n > 0$ (the northern hemisphere). The preimage of Q^+ under the map Φ is the subset of $\mathbb{R}^{n-1}$ defined by the inequality $|u|^2 < 1$. It is a unit ball

B_{n-1} of dimension $n-1$. Consider the orthogonal projection α of Q^+ to the ball B_{n-1} from (2.3). The composition

$$\alpha\circ\Phi : B_{n-1} \to {}'B_{n-1},\ (u_1,\dots,u_{n-1}) \mapsto \big(\frac{2u_1}{1+|u|^2},\dots,\frac{2u_{n-1}}{1+|u|^2}\big) \tag{2.6}$$

with the inverse

$${}'B_{n-1} \to B_{n-1},\ (y_1,\dots,y_{n-1}) \mapsto \big(\frac{y_1}{1+\sqrt{1-|y|^2}},\dots,\frac{y_{n-1}}{1-\sqrt{1-|y|^2}}\big)$$

is a bijection which we can use to transport the metric on ${}'B_{n-1}$ to a metric on B_{n-1} with curvature equal to -1. In coordinates $u_1,\dots,u_{n-1}$, it is given by the formula

$$ds^2 = 4(1-|u|^2)^{-2}\sum_{i=1}^{n-1} du_i^2.$$

This is a metric on B_{n-1} of constant curvature -1, called the *Poincaré metric*. The corresponding model of the hyperbolic space $\mathbb{H}^{n-1}$ is called a *conformal* or *Poincaré model* of $\mathbb{H}^{n-1}$.

Yet, another model of $\mathbb{H}^n$ is realized in the *upper half-space*

$$\mathcal{H}^n = \{(u_1,\dots,u_n)\in\mathbb{R}^n, u_n>0\}.$$

The isomorphism $B_n \to \mathcal{H}^n$ is defined by

$$(u_1,\dots,u_n) = \frac{1}{\rho^2}(2y_1,\dots,2y_{n-1},2(y_n+1)-\rho^2),$$

where $\rho^2 = y_1^2+\cdots+y_{n-1}^2+(y_n+1)^2$. The metric on $\mathcal{H}^n$ is given by

$$ds^2 = \frac{1}{y_n^2}\sum_{i=1}^{n} dy_i^2.$$

A *k-plane* Π in $\overline{\mathbb{H}}^n$ is the non-empty intersection of a k-plane in $\mathbb{P}^n(\mathbb{R})$ with $\overline{\mathbb{H}}^n$. We call it *proper* if its intersection with $\mathbb{H}^n$ is non-empty, and hence, it is a geodesic submanifold of $\mathbb{H}^n$ of dimension k isomorphic to $\mathbb{H}^k$.

A *geodesic line* corresponds to a 2-dimensional subspace U of signature $(1,1)$. It has a basis (f,g) that consists of isotropic vectors with $(f,g)=-1$ (called a *standard basis*). Thus any geodesic line ℓ intersects the absolute at two different points $[f]$ and $[g]$. We can choose a parameterization $\gamma:\mathbb{R}\to l$ of ℓ of the form $\gamma(t)=\frac{1}{\sqrt{2}}(e^t f+e^{-t}g)$ so that $\gamma(-\infty)=[f]$ and $\gamma(+\infty)=[g]$. The hyperbolic distance $d(\gamma(0),\gamma(t))$ from the point $\gamma(0)=\frac{1}{\sqrt{2}}(f+g)$ to the point $\gamma(t)$ satisfies $d(\gamma(0),\gamma(t)) = -(\gamma(0),\gamma(t)) = \cosh t$. Thus $t=d(\gamma(0),\gamma(t))$ is the natural parameter of the geodesic line. The subgroup of $\mathrm{SO}(U)_0 \cong$

$SU(1,1)_0$ is isomorphic to $\mathbb{R}_{>0}$ and consists of transformation g_t given in the standard basis by the matrix $\left(\begin{smallmatrix} e^t & 0 \\ 0 & e^{-t} \end{smallmatrix}\right)$. It can be extended to a group of isometries of $\mathbb{R}^{n,1}$ that acts identically on $U^\perp$. Comparing with the natural parameterization of ℓ, we see that each transformation g_t moves a point $x \in l$ to a point $g_t(x)$ such that $d(x, g_t(x)) = t$. For this reason, it is called the *hyperbolic translate*.

A $n-1$-plane H is called a *hyperplane*. We write it in the form

$$H_{\mathfrak{e}} = \{v \in \mathbb{R}^{n,1} \setminus \{0\} : (v, \mathfrak{e}) = 0\}/\mathbb{R}^*.$$

Since the signature of q is equal to $(n,1)$, $\mathbb{H}^n \cap H_{\mathfrak{e}} = \emptyset$ if and only if $(\mathfrak{e}, \mathfrak{e}) \geq 0$. If $(\mathfrak{e}, \mathfrak{e}) = 0$, the intersection $H_{\mathfrak{e}} \cap \overline{\mathbb{H}}^n$ consists of one point $[\mathfrak{e}]$. We assume that $(\mathfrak{e}, \mathfrak{e}) \geq 0$ and $(\mathfrak{e}, \mathfrak{e}) = 1$ if $(\mathfrak{e}, \mathfrak{e}) > 0$. These properties define $\mathfrak{e}$ uniquely, up to multiplication by a constant (equal to ± 1 if $(\mathfrak{e}, \mathfrak{e}) = 1$). A choice of one of the two rays of the line $\mathbb{R}\mathfrak{e}$ is called an *orientation* of the hyperplane. The complement of $H_{\mathfrak{e}}$ in $\mathbb{H}^n$ consists of two connected components (half-spaces) defined by the sign of $(v, \mathfrak{e})$. We denote the closures of these components by $H_{\mathfrak{e}}^{\pm}$, accordingly. Changing the orientation, interchanges the half-spaces.

Consider a hyperplane $H_{\mathfrak{e}'}$ in $'B_{n-1}$. It is given by an equation

$$a_0 t_0 + \sum_{i=1}^{n-1} a_i t_i = 0$$

for some $(a_0, \ldots, a_{n-1}) \in \mathbb{R}^{n-1,1}$ with norm 1.

Under the map (2.6), the pre-image of $H_{\mathfrak{e}'}$ is equal to the intersection $S(\mathfrak{e}') \cap \bar{B}_{n-1}$, where

$$S(\mathfrak{e}') := -a_0 \sum_{i=1}^{n-1} u_i^2 + 2 \sum_{i=1}^{n-1} a_i u_i - a_0 = 0.$$

If $k = a_0 \neq 0$, we can rewrite this equation in the form

$$k^2 \sum_{i=1}^{n-1} (u_i - a_i/k)^2 = \sum_{i=1}^{n-1} a_i^2 - a_0^2 = 1.$$

This is a sphere of radius $r = \frac{1}{k}$ with the center $c = (a_1/k, \ldots, a_{n-1}/k)$. It is immediately checked that $S(\mathfrak{e}')$ intersects the boundary of B_{n-1} orthogonally (i.e the radius-vectors are perpendicular at each intersection point in the Euclidean inner product). The intersection of $S(\mathfrak{e}')$ with B_{n-1} is a hyperbolic codimension 1 subspace of the Poincaré model of $\mathbb{H}^{n-1}$.

Assume $k = 0$, the equation of $S(\mathfrak{e}')$ becomes $\sum_{i=1}^{n-1} a_i u_i = 0$. The hyperplane $S(\mathfrak{e}')$ should be viewed as a sphere of radius $r = \infty$ with the center at the point at infinity corresponding to p in the compactification

$\mathbb{R}^n \cup \{\infty\} \cong \mathsf{Q}(\mathbb{R})$. For any $k \geq 0$, the intersection $S(\mathfrak{e}') \cap B_{n-1}$ is a geodesic hypersurface in $\mathbb{H}^{n-1}$. It contains the center of the ball if and only if $a_0 = 0$.

Example 2.1. A one-dimensional hyperbolic space can be modeled by an open interval $(-1, 1)$. The Poincaré and the Klein metrics coincide and equal to $dy^2/(1-y^2)^2$. It is isomorphic to the Euclidean one-dimensional space in this case.

Assume $n = 2$. We have two models of a hyperbolic 2-dimensional space $\mathbb{H}^2$ both realized in the unit disk B_2. The the Klein model is defined by the metric

$$ds^2 = \frac{y_1 dy_2^2 + y_2 dy_1^2}{(1 - y_1^2 - y_2^2)^2}.$$

The Poincaré model is defined by the metric

$$ds^2 = \frac{dy_1^2 + dy_2^2}{(1 - y_1^2 - y_2^2)^2}.$$

In the Klein model, a geodesic line in $\mathbb{H}^2$ is a line joining two distinct points on the absolute. In the Poincaré model, a geodesic line is a circle arc intersecting the absolute orthogonally at two distinct points or a line passing through the center of the disk.

The group $\mathrm{Iso}(\mathbb{H}^n)$ of isometries of $\mathbb{H}^n$ is the index 2 subgroup $\mathrm{O}(n,1)'$ of the orthogonal group of $\mathbb{R}^{n,1}$ that preserves each sheet of the hyperboloid representing the vector model of $\mathbb{H}^n$. When n is even, we have $\mathrm{O}(n,1)' = \mathrm{SO}(n,1)$. It is also isomorphic to the group $\mathrm{PO}(n,1)$ of projective transformations of $\mathbb{P}^n$ that leaves invariant the quadric $\mathsf{Q} = \partial\mathbb{H}^n$. It consists of two connected components, the connected component of the identity $\mathrm{PO}(n,1)_0$ is the group of isometries that preserve the orientation. It is isomorphic to the subgroup of $\mathrm{SO}(n,1)_0$ of elements of spinor norm 1.

Via the rational map (2.4), the group $\mathrm{Iso}(\mathbb{H}^n)$ becomes isomorphic to the subgroup of the real Cremona group $\mathrm{Cr}_{\mathbb{R}}(n-1)$ of $\mathbb{P}^{n-1}_{\mathbb{R}}$. It is generated by affine orthogonal transformations of $\mathbb{P}^{n-1}_{\mathbb{R}}$ and the inversion birational quadratic transformation

$$[x_0, \ldots, x_{n-1}] \mapsto [\sum_{i=1}^{n-1} x_i^2, x_0x_1, \ldots, x_0x_{n-1}]. \tag{2.7}$$

This group is known classically as the *Inversive group* in dimension $n-1$. We can also identify Q with a one-point compactification $\hat{E}^{n-1} = \mathbb{R}^{n-1} \cup \{\infty\}$ of the Euclidean space of dimension $n-1$. The point ∞ corresponds to the point on Q from which we project to $\mathbb{P}^{n-1}$. In this

model the Inversive group is known as the Möbius group $\text{Möb}(n-1)$. If we identify, in the usual way, the 2-dimensional sphere with the complex projective line $\mathbb{P}^1(\mathbb{C})$, we obtain that the group $\text{Möb}(2)$ is isomorphic to $\text{PSL}_2(\mathbb{C})$ and consists of Möbius transformations $z \mapsto \frac{az+b}{cz+d}$. The inversion transformation (2.7) is conjugate to the transformation $z \mapsto -1/z$.

It is known that the orthogonal group $\text{O}(n,1)$ is generated by *reflections*

$$s_\alpha : v \mapsto v - 2\frac{(v,\alpha)}{(\alpha,\alpha)}\alpha, \tag{2.8}$$

where $(\alpha,\alpha) \neq 0$. If $(\alpha,\alpha) < 0$, then s_α has only one fixed point in $\overline{\mathbb{H}}^n$, the point $[\alpha] \in \mathbb{H}^n$. If $(\alpha,\alpha) > 0$, then the fixed locus of s_α is a hyperplane in $\overline{\mathbb{H}}^n$. The pre-image of its intersection with the absolute $\partial\mathbb{H}^n$ is the sphere $S(\alpha/(\alpha,\alpha)^{1/2})$. The Möbius transformation corresponding to s_α is the inversion transformation that fixes the sphere.

An isometry γ of a hyperbolic space $\mathbb{H}^n$ is of the following three possible types:

- hyperbolic if γ has no fixed points in $\mathbb{H}^n$ but has two fixed points on the boundary;
- parabolic if γ has no fixed points in $\mathbb{H}^n$ but has one fixed point on the boundary;
- elliptic if γ has a fixed point in $\mathbb{H}^n$.

An element is elliptic if and only if it is of finite order. In the $\hat{E}^{n-1}$ model of the absolute, a parabolic element corresponds to a translation in the affine orthogonal group $\text{AO}(n-1)$.

A discrete subgroup Γ of $\text{Iso}(\mathbb{H}^n)$ is called a *Kleinian group*. It acts discontinuously on $\mathbb{H}^n$ and the extension of this action to the absolute acts discontinuously on the set $\Omega(\Gamma) = \partial\mathbb{H}^n \setminus \Lambda(\Gamma)$, where $\Lambda(\Gamma)$ is the set of limit points of Γ, i.e. points on the boundary that belongs to the closure of the Γ-orbit of a point x in $\mathbb{H}^n$ (does not matter which one). The set $\Omega(\Gamma)$ is called the *discontinuity set* of Γ. The set $\mathbb{H}^n \cup \Omega(\Gamma)$ is the largest open subset of $\overline{\mathbb{H}}^n$ on which Γ acts discontinuously.

A Kleinian group Γ is called *elementary* if it has a finite orbit in $\overline{\mathbb{H}}^n$. It is characterized by the property that it contains a free abelian subgroup of finite index.

The limit set $\Lambda(\Gamma)$ of an elementary Kleinian group is either empty or consists of at most two points, the fixed points of a parabolic or a hyperbolic element in Γ. Otherwise $\Lambda(\Gamma)$ is a perfect set, i.e. has no isolated points. A fixed point of a hyperbolic or a parabolic transformation is a limit point. The limit set of a non-elementary Kleinian

group is equal to the closure of the set of fixed points of parabolic or hyperbolic elements [34], Theorem 12.2.2.

A Kleinian group in dimension 2 is called a *Fuchsian group*. Usually one assumes that Γ is contained in the connected component of $\mathrm{Iso}(2)$, i.e. it preserves the orientation. The discontinuity set $\Omega(\Gamma)$ is either empty, and then Γ is called of the first kind, or consists of the union of open intervals, then γ is of the second kind. The quotient $X = \mathbb{H}^2 \cup \Omega(\Gamma)/\Gamma$ is a Riemann surface. If Γ is of the first kind, one can compactify X by a finite set of orbits of cuspidal limit points. A Fuchsian group with $\mathbb{H}^2/\Gamma$ of finite volume is of the first kind. If Γ is of the second kind, then X is a Riemann surface with possibly a non-empty boundary.

3. Geometrically finite Kleinian groups

A non-empty subset A of a metric space is called *convex* if any geodesic connecting two of its points is contained in A. Its closure is also a convex set [34], Theorem 6.2.1. A closed convex subset P of $\mathbb{H}^n$ whose interior is non-empty is equal to the intersection of half-spaces $H^{\pm}_{\mathfrak{e}_i}, i \in I$. It is assumed that none of the half-spaces $H^{\pm}_{\mathfrak{e}_i}$ contains the intersection of other half-spaces. A maximal non-empty closed convex subset of the boundary of P is called a *side*. Each side is equal to the intersection of P with one of the bounding hyperplanes $H_{\mathfrak{e}_i}$. The boundary of P is equal to the union of sides and two different sides intersect only along their boundaries.

A *convex polyhedron* in $\mathbb{H}^n$ is a convex closed subset P with non-empty interior that has finitely many sides. It is bounded by a finite set of hyperplanes H_i.

We choose the vectors $\mathfrak{e}_i$ as in the previous section so that

$$P = \bigcap_{i=1}^{N} H^{-}_{\mathfrak{e}_i}. \tag{3.1}$$

The *dihedral angle* $\phi(H_{\mathfrak{e}_i}, H_{\mathfrak{e}_j})$ between two proper bounding hyperplanes is defined by the formula

$$\cos\phi(H_{\mathfrak{e}_i}, H_{\mathfrak{e}_j}) := -(\mathfrak{e}_i, \mathfrak{e}_j).$$

If $|(\mathfrak{e}_i, \mathfrak{e}_j)| > 1$, the angle is not defined, we say that the hyperplanes are *divergent*. In this case the distance between $H_{\mathfrak{e}}$ and $H_{\mathfrak{e}'}$ can be found from the formula

$$\cosh d(H_{\mathfrak{e}}, H_{\mathfrak{e}'}) = |(\mathfrak{e}, \mathfrak{e}')|. \tag{3.2}$$

For any subset J of $\{1, \ldots, N\}$ of cardinality $n-k$, such that the intersection of $H_{\mathfrak{e}_i}, i \in J$ is a k-plane Π in $\mathbb{H}^n$, the intersection $P \cap \Pi$

is a polyhedron in Π. It is called a *k-face* of P. An $(n-1)$-face of P is a side of P.

The matrix

$$G(P) = (g_{ij}), \quad g_{ij} = (\mathfrak{e}_i, \mathfrak{e}_j),$$

is called the *Gram matrix* of P. There is a natural bijection between the set of its k-dimensional proper faces and positive definite principal submatrices of $G(P)$ of size $n-k$. The improper vertices of P correspond to positive semi-definite principal submatrices of size n.

Recall that a closed subset D of a metric space X is called a *fundamental domain* for a group Γ of isometries of X if

(i) the interior D^o of D is an open non-empty set;
(ii) $\gamma(D^o) \cap D^o = \emptyset$, for any $\gamma \in \Gamma \setminus \{1\}$;
(iii) the set of subsets of the form $\gamma(D)$ is locally finite[5];
(iv) $X = \cup_{\gamma\in\Gamma}\gamma(D)$;

A group Γ admits a fundamental domain if and only if it is a discrete subgroup of the group of isometries of X. For example, one can choose D to be a *Dirichlet domain*

$$D(x_0) = \{x \in X : d(x, x_0) \leq d(\gamma(x), x_0), \text{ for any } \gamma \in \Gamma\}, \tag{3.3}$$

where x_0 is a fixed point in X and $d(x, y)$ denotes the distance between two points. Assume $X = \mathbb{H}^n$. For any $\gamma \in \Gamma \setminus \Gamma_{x_0}$, let H_γ be the hyperplane of points x such that $d(x_0, x) = \rho(x, \gamma(x_0))$. Then $D(x_0) = \cap_{\gamma\in\Gamma\setminus\Gamma_{x_0}} H_\gamma^-$ and, for any $\gamma \notin \Gamma_{x_0}$, $S = \gamma(D(x_0)) \cap D(x_0) \subset H_\gamma$ is a side in $\partial D(x_0)$. Each side of $D(x_0)$ is obtained in this way for a unique γ.

A fundamental domain D for a Kleinian group Γ in $\mathbb{H}^n$ is called *polyhedral* if its boundary $\partial D = D \setminus D^o$ is contained in the union of a locally finite set of hyperplanes H_i and each side S of the boundary is equal to $D \cap \gamma(D)$ for a unique $\sigma_S \in \Gamma$. A Dirichlet domain is a convex polyhedral fundamental domain.

A choice of a polyhedral fundamental domain allows one to find a presentation of Γ in terms of generators and relations. The set of generators is the set of elements $\gamma_S \in \Gamma$, where S is a side of D. A relation $\gamma_{S_t} \circ \cdots \circ \gamma_{S_1} = 1$ between the generators corresponds to a cycle

$$D_0 = D, D_1 = \gamma_{S_1}(D_0), \ldots, D_t = \gamma_{S_t} \circ \cdots \circ \gamma_{S_1}(D_0) = D_0.$$

Among various equivalent definitions of a geometrically finite Kleinian group we choose the following one (see [2], Chapter 4, §1): A Kleinian

[5]This means that every point $x \in \mathbb{H}^n$ is contained in a finite set of subsets $\gamma(D)$

group Γ is called *geometrically finite* if it admits a polyhedral fundamental domain with finitely many sides. [6] It follows from above that such a group is finitely generated and finitely presented. The converse is true only in dimension $n \leq 2$. In dimensions $n \leq 3$, one can show that Γ is geometrically finite if and only if there exists a Dirichlet fundamental domain with finitely many sides (loc. cit., Theorem 4.4). On the other hand, for $n > 3$ there are examples of geometrically finite groups all whose Dirichlet domains have infinitely many sides (loc. cit. Theorem 4.5).

Example 3.1. A convex polyhedron P in $\mathbb{H}^n$ is called a *Coxeter polyhedron* if each pair of its hyperplanes $H_{\mathfrak{e}_i}, H_{\mathfrak{e}_j}$ is either divergent or the dihedral angle between them is equal to π/m_{ij}, where $m_{ij} \in \mathbb{Z}$ or ∞. The *Coxeter diagram* of a Coxeter polyhedron P is a labeled graph whose pairs of vertices corresponding to divergent hyperplanes are joined by a dotted edge, two vertices corresponding to hyperplanes forming the zero angle are joined by a solid edge or by putting ∞ over the edge, and two vertices corresponding to the hyperplanes with the dihedral angle $\pi/m_{ij} \neq 0, \pi/2$ are joined by an edge with the label $m_{ij} - 2$, the label is dropped if $m_{ij} = 3$. The proper (resp. ideal) vertices of the polyhedron P correspond to the subdiagrams which are Coxeter diagrams of finite (resp. affine) root systems of rank n (resp, $n-1$).

A Coxeter polyhedron is a fundamental polyhedron of the reflection Kleinian group Γ_P generated by the reflections $s_{\mathfrak{e}_i}$. For each pair of reflections $s_{\mathfrak{e}_i}, s_{\mathfrak{e}_j}$ with $m_{ij} \neq \infty$, the product $s_{\mathfrak{e}_i} \circ s_{\mathfrak{e}_j}$ is a rotation in angle $2\pi/m_{ij}$ around the codimension 2 face $H_{\mathfrak{e}_i} \cap H_{\mathfrak{e}_j}$. It follows that $(s_i \circ s_j)^{m_{ij}}$ is one of the basic relations. In fact, these relations are basic relations. An abstract Coxeter group is a pair (Γ, S) consisting of a group Γ and a set S of its generators s_i such that $(s_i s_j)^{m_{ij}} = 1$ are the basic relations, where (m_{ij}) is a matrix with diagonal elements equal to 1 and off-diagonal elements equal to integers ≥ 2 or the infinity symbol ∞ if the order of the product is infinite. A realization of a Coxeter group (G, S) in the form of Γ_P, where P is a Coxeter polyhedron in a hyperbolic space $\mathbb{H}^n$ is called a hyperbolic Coxeter group. Compact (resp. of finite volume) simplicial Coxeter polytopes are described by Lanner (resp. quasi-Lanner) diagrams which can be found for example in [40], Chapter 5, Table 4.

[6] Another equivalent definition is given in terms of the convex core of Γ, the minimal convex subset of $\mathbb{H}^n$ that contains all geodesics connecting any two points in $\Lambda(\Gamma)$.

4. Sphere packings

Recall that a real *n-sphere* S in the projective space $\mathbb{P}^n_{\mathbb{R}}$ over reals is the set of real points $Q(\mathbb{R})$ of a quadric Q defined over $\mathbb{R}$ that contains a given nonsingular quadric Q_0 with $Q_0(\mathbb{R}) = \emptyset$ in a fixed hyperplane H_∞. We fix the projective coordinates to assume that $H_\infty : x_0 = 0$ and $Q_0 : x_0 = x_1^2 + \cdots + x_n^2 = 0$. A 1-sphere is called a *circle*, it is a conic passing through the points $[0, 1, \pm i]$ at infinity. We do not exclude the case $n = 1$ where H_∞ is the point $\infty = [0, 1]$ and $Q_0 = \emptyset$. A 0-dimensional sphere is a set of two real points in $\mathbb{P}^1_{\mathbb{R}}$ that may coincide.

Let

$$\mathbb{P}^n_{\mathbb{R}} \dashrightarrow \mathbb{P}^{n+1}_{\mathbb{R}}$$

be the rational map defined by the formula

$$[x_0, \dots, x_n] \mapsto [2x_0^2, -2x_0x_1, \dots, -2x_0x_n, \sum_{i=1}^n x_i^2].$$

Its image is the quadric Q given by the equation

$$q = 2t_0t_{n+1} + t_1^2 + \cdots + t_n^2 = 0. \tag{4.1}$$

This map is given by a choice of a basis in the linear system of quadrics containing the quadric Q_0. A different choice of a basis leads to the map (2.4) whose image is a quadric with equation $-t_0^2 + \sum_{i=1}^{n+1} t_i^2 = 0$. Let $\mathfrak{e} = (a_0, \dots, a_{n+1}) \in \mathbb{P}^{n+1,1}$ with $(\mathfrak{e}, \mathfrak{e}) \geq 0$. We assume that $(\mathfrak{e}, \mathfrak{e}) = 1$ if $(\mathfrak{e}, \mathfrak{e}) > 0$. The pre-image of a hyperplane

$$H_{\mathfrak{e}} : a_0t_{n+1} + a_{n+1}t_0 + \sum_{i=1}^{n+1} a_it_i = 0$$

in $\mathbb{P}^{n+1}_{\mathbb{R}}$ is a quadric in $\mathbb{P}^n_{\mathbb{R}}$ defined by the equation

$$a_0 \sum_{i=1}^n x_i^2 - 2a_{n+1}x_0^2 - 2x_0 \sum_{i=1}^n a_ix_i = 0.$$

If $a_0 \neq 0$, we can rewrite this equation in the form

$$a_0^2 \sum_{i=1}^n (\frac{x_i}{x_0} - \frac{a_i}{a_0})^2 = 2a_0a_{n+1} + \sum_{i=1}^n a_i^2 = 1. \tag{4.2}$$

So, we can identify its real points with an n-dimensional sphere $S(\mathfrak{e})$ in the Euclidean space $\mathbb{P}^n(\mathbb{R}) \setminus \{x_0 = 0\}$ of radius square $r^2 = 1/a_0^2$ and the center $c = [\frac{a_1}{a_0}, \dots, \frac{a_n}{a_0}]$. It is natural to call $|a_0|$ the *curvature* of the sphere. If $a_0 = 0$, we get the union of two hyperplanes H_∞ and $S := \sum_{i=1}^n a_ix_i + a_{n+1}x_0 = 0$. In this case we set $r := \infty$. Note that if we take $\mathfrak{e} = (a_0, \dots, a_{n+1})$ in (4.2) with $(\mathfrak{e}, \mathfrak{e}) = 0, a_0 \neq 0$, we obtain a

quadratic cone with the vertex at c. Its set of real points consists only of the vertex.

One introduces the oriented curvature equal to a_0. We agree that the positive curvature corresponds to the interior of the sphere $S(\mathfrak{e})$, i.e. an open ball $B(\mathfrak{e})$ of radius r. The negative curvature corresponds to the open exterior of the sphere, we also call it the *ball* corresponding to the oriented sphere S and continue to denote it $B(\mathfrak{e})$. It can be considered as a ball in the extended Euclidean space $\hat{E}^n$.

A *sphere packing* is an infinite set $\mathfrak{P} = (S_i)_{i\in I}$ of oriented n-spheres such that any two of them are either disjoint or touch each other. We say that a sphere packing is *strict* if, additionally, no two open balls B_i intersect. An example of a non-strict sphere packing is an infinite set of nested spheres. We assume also that the set $\mathfrak{P}$ is *locally finite* in the sense that, for any $t > 0$, there exists only finitely many spheres of curvature at most t in any fixed bounded region of the space. The condition that two spheres S_i and S_j are disjoint or touch each other is easily expressed in terms of linear algebra. We have the following.

Lemma 4.1. *Let $S(v)$ and $S(w)$ be two n-dimensional spheres corresponding to hyperplanes H_v and H_w. Then their interiors do not intersect if and only if*

$$(v, w) \leq -1.$$

The equality takes place if and only if the spheres are tangent to each other, and hence intersect at one real point.

Proof. Suppose $(v, w) \leq -1 = -(v, v)(w, w)$, then the plane spanned by v, w is not positive definite, hence its orthogonal complement does not contain vectors with negative norm. This implies that the hyperplanes H_v and H_w either diverge or, if $(v, w) = -1$, intersect at one point in the absolute corresponding to the unique isotropic line in the orthogonal complement of the plane. The converse is proven along the same lines. □

Let Γ be a geometrically finite Kleinian group in $\mathbb{H}^{n+1}$ that is identified with a subgroup of the Möbius group $\text{Möb}(n)$. We say that a sphere packing is a *sphere Γ-packing* if it is invariant with respect to the action of Γ on spheres and Γ has only finitely many orbits, say N, on the set. We also say that a sphere Γ-packing is *clustered* if one can choose representatives $S_1, \ldots, S_N$ of the orbits such that $\mathfrak{P}$ is equal to the union of the sets $\{\gamma(S_1), \ldots, \gamma(S_N)\}, \gamma \in \Gamma$. Each such set is called a *cluster* of the sphere packing. We will see later that Apollonian sphere packings and, their generalizations, Boyd-Maxwell sphere packings, are examples of clustered sphere Γ-packings.

Let Γ_i be the stabilizer subgroup of some sphere S_i in a sphere Γ-packing. The sphere corresponds to a geodesic hyperplane H_i in $\mathbb{H}^{n+1}$, hence the ball B_i is the Klein model of $\mathbb{H}^n$ on which Γ_i acts as a geometrically finite Kleinian group. Its limit points are on S_i. We know that they belong to the closure of the set of fixed points of non-elliptic elements in Γ_i, hence they are limit point of the whole Γ.

Let us introduce the following assumptions.

(A1) for any sphere S_i, $\Lambda(\Gamma_i) = S_i$.
(A2) $\bigcup_{i\in I} \bar{B}_i = \hat{E}^n$.

closure

Assumption (A1) implies that

$$\overline{\bigcup_{i\in I} S_i} \subset \Lambda(\Gamma). \tag{4.3}$$

Assumption (A2) means that the packing is *maximal*, i.e. one cannot find a sphere whose interior is disjoint from the interiors of all other spheres in the packing.

If the packing is strict, assumptions (A1) and (A1) imply the equality in (4.3). Indeed, suppose x is a limit point of Γ that does not lie in the closure of the union of the spheres. By (A2), it belongs to the closure of some open ball B_i. Suppose x belongs to the open ball B_i. We mentioned before that the limit set is the closure of the set of fixed points of non-elliptic elements of Γ. Therefore, we can find some open neighborhood of x inside B_i that contains a fixed point of a non-elliptic element γ in Γ. The stabilizer subgroup Γ_i of B_i in Γ is a Kleinian group acting on B_i. It cannot contain a non-elliptic element with a fixed point in the ball. This shows that $\gamma \notin \Gamma_i$ and hence $\gamma(B_i) \subsetneq B_i$. Thus the set $\{\gamma^n(B_i), n \in \mathbb{Z}\}$ consists of nested spheres. This is excluded by the definition of a strict sphere packing.

In the next two sections we give examples of sphere Γ-packings satisfying assumptions (A1) and (A2).

5. Boyd-Maxwell sphere packings

Let P be a Coxeter polytope in $\mathbb{H}^{n+1}$ with the Gram matrix $G(P)$. Let $\mathfrak{e}_1, \dots, \mathfrak{e}_{n+2}$ be the normal vectors of its bounding hyperplanes. We can take the set $(\mathfrak{e}_1, \dots, \mathfrak{e}_{n+2})$ to be a basis of the space $\mathbb{R}^{n+1,1}$ with the fundamental quadric (4.1).

Let ω_j be a vector in $\mathbb{R}^{n,1}$ uniquely determined by the condition

$$(\omega_j, \mathfrak{e}_i) = \delta_{ij}. \tag{5.1}$$

We have

$$\omega_j = \sum_{i=1}^{n} g^{ij}\mathfrak{e}_i, \tag{5.2}$$

where

$$G(P)^{-1} = (g^{ij}) = ((\omega_i, \omega_j)).$$

We call ω_i *real* if $(\omega_i, \omega_i) = g^{ii} > 0$. In this case we can normalize it to set

$$\bar{\omega}_i := \omega_i/\sqrt{g^{ii}}.$$

Let $J = \{j_1 < \cdots < j_r\}$ be the subset of $\{1, \ldots, n+2\}$ such that $\bar{\omega}_j$ is real if and only if $j \in J$. Consider the union

$$\mathfrak{P}(P) = \bigcup_{k=1}^{r} O_{\Gamma_P}(S(\bar{\omega}_{j_k})). \tag{5.3}$$

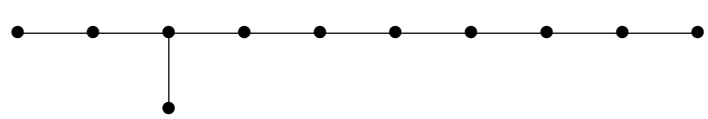
A *Boyd-Maxwell sphere packing* is a sphere packing of the form $\mathfrak{P}(P)$, where P is a simplicial Coxeter polytope. By definition, it is clustered with clusters

$$(\gamma(S(\bar{\omega}_{j_1})), \ldots, \gamma(S(\bar{\omega}_{j_r}))) = (S(\gamma(\bar{\omega}_{j_1})), \ldots, S(\gamma(\bar{\omega}_{j_r})).$$

We call the cluster $(S(\omega_{j_1}), \ldots, S(\omega_{j_r}))$ the *initial cluster*. It is called *non-degenerate* if all ω_i are real. This is equivalent to that all principal maximal minors of the matrix $G(P)$ are negative.

Following G. Maxwell [26], we say that the Coxeter diagram is of *level* l if, after deleting any l of its vertices, we obtain a Coxeter diagram of Euclidean or of parabolic type describing a Coxeter polyhedron in an Euclidean or a spherical geometry. They can be characterized by the property that all $m_{ij} \neq \infty$ (except in the case $n = 1$) and the symmetric matrix (m_{ij}) is non-negative definite, definite in the Euclidean case, and having a one-dimensional radical in the parabolic case. Coxeter diagrams of level $l = 1$ correspond to Lanner and quasi-Lanner Coxeter groups. Coxeter diagrams of level 2 have been classified by G. Maxwell (with three graphs omitted, see [10]). They occur only in dimension $n \leq 10$. For $n \geq 4$ they are obtained from quasi-Lanner diagrams by adding one vertex.

For example, a Coxeter polyhedron in $\mathbb{H}^{10}$ with Coxeter diagram of level 2



defines a sphere packing with only one real ω_i corresponding to the extreme vertex on the right.

The following theorem is proven, under a certain assumption, in loc.cit., Theorem 3.3. The assumption had been later removed in [27], Theorem 6.1.[7]

Theorem 5.1. *Let P be a Coxeter polyhedron in $\mathbb{H}^n$. Then $\mathfrak{P}(P)$ is a sphere Γ_P-packing if and only if the Coxeter diagram of P is of level ≤ 2.*

If P is a quasi-Lanner Coxeter polytope, the Kleinian group Γ_P has a fundamental polyhedron of finite volume. It follows that Γ_P is a Kleinian group of the first kind ([34], Theorem 12.2.13). Therefore, any point on the absolute is its limit point. The stabilizer $\Gamma_{P,i}$ of each $S(\bar{\omega}_i)$ is the reflection group of the Coxeter group defined by the Coxeter diagram of level 1, hence the limit set of $\Gamma_{P,i}$ is equal to $S(\bar{\omega}_i)$. Thus the assumption (A1) is satisfied.

Let $\mathfrak{P}(P)$ be a non-degenerate Boyd-Maxwell sphere packing and let $\mathbf{k} = (k_1, \ldots, k_{n+2})$ be the vector of the curvatures of the spheres $S(\gamma(\omega_1), \ldots, S(\gamma(\omega_{n+2}))$ in its cluster. Let N be the diagonal matrix $\mathrm{diag}(1/\sqrt{g^{11}}, \ldots, 1/\sqrt{g^{n+2}})$. Denote by $\widetilde{G(P)}$ the matrix $NG(P)N$.

Theorem 5.2. *Let $G(P)$ be the Gram matrix of P. Then*

$$ {}^t\mathbf{k} \cdot \widetilde{G(P)} \cdot \mathbf{k} = 0. \tag{5.4} $$

Proof. An entry a_{ij} of the matrix $\widetilde{G(P)}$ is equal to $\frac{1}{g^{ij}}(\mathfrak{e}_i, \mathfrak{e}_j)$. For any $\gamma \in \Gamma_P$, we have $(\gamma(\mathfrak{e}_i), \gamma(\mathfrak{e}_j)) = (\mathfrak{e}_i, \mathfrak{e}_j)$. Thus $\widetilde{G(P)} = \widetilde{G(\gamma(P))}$. So, we may assume that the cluster $S(\gamma(\omega_1)), \ldots, S(\gamma(\omega_{n+2}))$ is equal to the initial cluster $S(\omega_1), \ldots, S(\omega_{n+2})$.

Let J_{n+2} be the matrix of the symmetric bilinear form defined by the fundamental quadratic form q from (4.1). It satisfies $J_{n+2} = J_{n+2}^{-1}$. Let X be the matrix whose jth column is the vector of coordinates of the vector $\bar{\omega}_1, \ldots, \bar{\omega}_{n+2}$. Recall that the first coordinate of each vector $\bar{\omega}_j$ is equal to the curvature of the sphere $S(\bar{\omega}_j)$. By definition of the Gram matrix, we have

$$ {}^tX \cdot J_{n+2} \cdot X = ((\bar{\omega}_i, \bar{\omega}_j) = N^{-1}G(P)^{-1}N^{-1}, $$

hence

$$ {}^tX^{-1} \cdot N^{-1}G(P)^{-1}N^{-1} \cdot X^{-1} = J_{n+2}. $$

Taking the inverse, we obtain

$$ X \cdot \widetilde{G(P)} \cdot {}^tX = J_{n+2}^{-1} = J_{n+2}. $$

[7] I am grateful to Hao Chen for the reference.

The first entry a_{11} of the matrix in the right-hand side is equal to zero. Hence

$$^t\mathbf{k} \cdot \widetilde{G(P)} \cdot \mathbf{k} = 0,$$

□

Assume $n \geq 2$. Suppose $\mathfrak{P}(P)$ is a non-degenerate Boyd-Maxwell sphere packing. Let $\bar{\omega}_1, \ldots, \bar{\omega}_{n+2}$ be defined as (5.2). Then the hyperplanes $H(\bar{\omega}_i)$ bound a convex polyhedron with no proper vertices on the absolute. Let us denote it by $P^\perp$. Since $\mathfrak{P}(P)$ is a sphere packing, no two bounding hyperplanes of $P^\perp$ have a common point in the absolute. By Lemma 4.1,

$$(\bar{\omega}_i, \bar{\omega}_j) \leq -1, \quad i \neq j. \tag{5.5}$$

Therefore, any two bounding hyperplanes of $P^\perp$ either diverge or have the dihedral angle equal to 0. In particular, we obtain that $P^\perp$ is a Coxeter polyhedron and hence its reflection group $\Gamma_P^\perp := \Gamma_{P^\perp}$ is a Kleinian group. The Coxeter diagram of $P^\perp$ has either solid or dotted edges. As a non-weighted graph, it is a complete graph K_{n+2} with $n+2$ vertices. It is clear that it is of level $l \leq 2$ if and only if $n \leq 3$. In this case we can define the *dual Boyd-Maxwell sphere packing*

$$\mathfrak{P}(P)^\perp := \mathfrak{P}(P^\perp).$$

Note that the property $(\mathfrak{e}_i, \bar{\omega}_k) = 0$ for $i \neq k$ is interpreted in terms of spheres as the property that the spheres $S(\mathfrak{e}_i)$ and $S(\bar{\omega}_j)$ are orthogonal to each other (see, for example, [14], Proposition 7.3). This implies that the dual sphere packing can be defined by a cluster that consists of spheres orthogonal to all spheres of a cluster defining $\mathfrak{P}(P)$ except one.

Let A be the matrix of a transformation from $\Gamma_{P^\perp}$ in a basis $(\omega_1, \ldots, \omega_{n+1})$. Then

$$^tA \cdot G(P)^{-1} \cdot A = G(P).$$

Taking the inverse, we obtain

$$A^{-1} \cdot G(P) \cdot {}^tA^{-1} = G(P). \tag{5.6}$$

This shows that under the homomorphism $A \to {}^tA^{-1}$, the group $\Gamma_P^\perp$ becomes isomorphic to a subgroup of the orthogonal group of the quadratic form defined by the matrix $G(P)$.

The group Γ_P is generated by the reflections $s_{\mathfrak{e}_i}$ that act on the dual basis $(\omega_1, \ldots, \omega_{n+2})$ by the formula

$$s_{\mathfrak{e}_i}(\omega_j) = \omega_j - 2(\mathfrak{e}_i, \omega_j)\mathfrak{e}_i = \omega_j - 2\delta_{ij}\mathfrak{e}_i = \omega_j - 2\delta_{ij}\sum_{k=1}^{n+2} g_{ki}\omega_k, \tag{5.7}$$

where δ_{ij} is the Kronecker symbol. This gives explicitly the action of Γ_P on the clusters of the sphere packing and also on the set of their curvature vectors.

The group $\Gamma_{P^\perp}$ is generated by the reflections $s_{\omega_i} = s_{\bar{\omega}_i}$ that act on the basis $(\mathfrak{e}_1, \dots, \mathfrak{e}_{n+2})$ by the formula

$$s_{\omega_i}(\mathfrak{e}_j) = \mathfrak{e}_j - 2(\bar{\omega}_i, \mathfrak{e}_j)\bar{\omega}_i = \mathfrak{e}_j - \frac{2\delta_{ij}}{g^{ii}}\omega_i = \mathfrak{e}_j - \frac{2\delta_{ij}}{g^{ii}}\sum_{k=1}^{n+2} g^{ki} e_k. \tag{5.8}$$

Example 5.3. This is the most notorious and widely discussed beautiful example of a Boyd-Maxwell sphere packing (see, for example, [18], [19], [20],[17],[36]).

We start with a set of pairwise touching non-degenerate n-spheres $S(v_1), \dots, S(v_{n+2})$ with positive radii that touch each other. By Lemma 4.1, the Gram matrix $G = ((v_i, v_j))$ has 1 at the diagonal and -1 off the diagonal. Thus,

$$G = \mathrm{cir}(1, -1, \dots, -1),$$

where $\mathrm{cir}(\alpha_1, \dots, \alpha_{n+2})$ denotes the circulant matrix with the first row $(\alpha_1, \dots, \alpha_{n+2})$. It is easy to compute the inverse of G to obtain

$$G^{-1} = \frac{1}{2n}\mathrm{cir}(n-1, -1, \dots, -1). \tag{5.9}$$

Assume $n \geq 2$. Then

$$\frac{2n}{n-1}G^{-1} = \mathrm{cir}(1, \frac{1}{1-n}, \dots, \frac{1}{1-n}).$$

It is equal to the Gram matrix of a Coxeter polytope P if and only if $n = 2, 3$. The group Γ_P is called the *Apollonian group* and will be denoted by Ap_n. It is a Kleinian group only for $n = 2, 3$. We call P an *Apollonian polyhedron*. Its Coxeter diagram is equal to the complete graph K_4 with solid edges if $n = 2$ and a complete graph K_5 with simple edges if $n = 3$. In each case the Coxeter diagram is of level 2. Therefore, P defines a Boyd-Maxwell sphere packing in dimensions $n = 2$ or 3. In the case $n = 2$, we see that all circles in the cluster defined by the vectors $\bar{\omega}_1, \dots, \bar{\omega}_{n+2}$ mutually touch each other. So, all clusters in this packing define a set of $n + 2$ mutually touching circles. Any two circles in the packing are either disjoint or touch each other.

We have $G(P)^{-1} = \frac{n-1}{2n}G$, thus $P^\perp$ has the Gram matrix equal to G. It follows that $P^\perp$ is always a Coxeter polyhedron with Coxeter diagram equal to a complete graph K_{n+2} with solid edges. It defines a sphere packing only if $n \leq 3$. Clearly, it is non-degenerate, and hence by Theorem 5.1, it is maximal. If $n = 2, 3$, it is the dual to the Apollonian packing. Its clusters consist of $n + 2$ mutually tangent

n-spheres. If $n = 2$, the dual of an Apollonian circle packing is an Apollonian circle packing. Its clusters are orthogonal to clusters of the original packing as shown on Figure 1 below. The group $\mathrm{Ap}_n^\perp$ is called the *dual Apollonian group*.

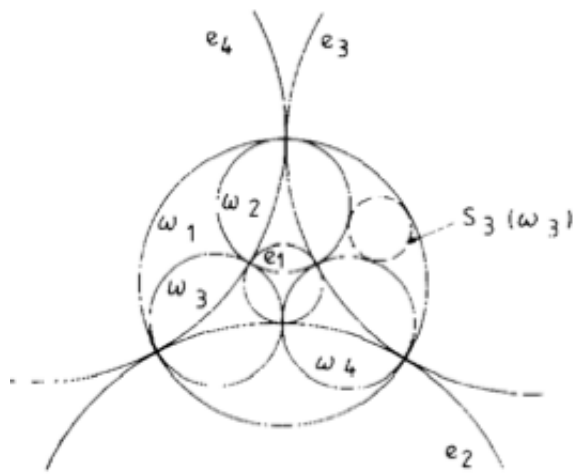


FIGURE 1. Dual Apollonian circle clusters (from [26])

Let $H_{\bar\omega_1}, \dots, H_{\bar\omega_{n+2}}$ be the bounding hyperplanes of an Apollonian Coxeter polyhedron and $s_{\mathfrak{e}_i}$ be the corresponding reflections generating the Apollonian group Ap_n. Let $S(\mathfrak{e}_1), \dots, S(\mathfrak{e}_{n+2})$ be the cluster of the dual Apollonian sphere packing. The formula (5.7) specializes to give $s_{\mathfrak{e}_i}(\bar\omega_j) = \bar\omega_j$, and

$$s_{\mathfrak{e}_i}(\bar\omega_i) = -\bar\omega_i + \frac{2}{n-1}\sum_{j\neq i} \bar w_j.$$

For example, $s_{\mathfrak{e}_1}$ is represented in the basis $(\bar\omega_1, \dots, \bar\omega_{n+2})$ by the matrix

$$A_1 = \begin{pmatrix} -1 & 0 & 0 & \dots & 0 \\ \frac{2}{n-1} & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{2}{n-1} & 0 & 0 & \dots & 1 \end{pmatrix}.$$

So, the Apollonian group is generated by $n+2$ matrices $A_1, \dots, A_{n+2}$ of this sort.

Formula (5.8) shows that the dual Apollonian group $\mathrm{Ap}_n^\perp$ acts on the Apollonian packing by acting on spheres $S(\mathfrak{e}_i)$ via the reflections $s_{\bar\omega_i}$.

We have $s_{\bar{\omega}_i}(\mathfrak{e}_j) = \mathfrak{e}_j$ if $j \neq i$ and

$$s_{\bar{\omega}_i}(\mathfrak{e}_i) = -\mathfrak{e}_i + 2 \sum_{k=1, k\neq i}^{n+2} \mathfrak{e}_j$$

For example, $s_{\bar{\omega}_1}$ is represented by the matrix

$$B_1 = \begin{pmatrix} -1 & 2 & 2 & \dots & 2 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

The equation from Theorem 5.2 becomes

$$n(k_1^2 + \cdots + k_{n+2}^2) - (k_1 + \cdots + k_{n+2})^2 = 0. \tag{5.10}$$

It is known as *Descartes's equation* or *Soddy's equation*[8]

Using (5.6), we see that $\mathrm{Ap}_n^{\perp}$ acts on the solutions of the equation (5.4) by the contragradient representations. It is generated by the matrices ${}^tA^{-1} = {}^tA_1, \dots, {}^tA_{n+2}^{-1} = {}^tA_{n+2}$. However, if $n \neq 2$, it does not leave the Apollonian packing invariant.

Assume $n > 1$. Consider a solution $(k_1, \dots, k_{n+2})$ of the Descartes' equation which we rewrite in the form

$$\sum_{i=1}^{n+2} k_i^2 - \frac{2}{n-1} \sum_{1\leq i\leq j\leq n+2} k_i k_j = 0. \tag{5.11}$$

Thus k_{n+2} satisfies the quadratic equation

$$t^2 - \frac{2t}{n-1} \sum_{i=1}^{n+1} k_i - \frac{2}{n-1} \sum_{1\leq i<k\leq n+1}^{n+2} k_i = 0.$$

It expresses the well-know fact (the *Apollonian Theorem*) that, given $n+1$ spheres touching each other, there are two more spheres that touch them. Thus, starting from an Apollonian cluster $(S_1, \dots, S_{n+2})$, we get a new Apollonian cluster $(S_1, \dots, S_{n+1}, S'_{n+2})$ such that the curvatures of S_{n+2} and S'_{n+2} are solution of the above quadratic equation. The curvatures of S'_{n+2} is equal to

$$k'_{n+2} = -k_{n+2} + \frac{2}{n-1} \sum_{i=1}^{n+1} k_i.$$

[8]From a "poem proof" of the theorem in the case $n = 2$ in "Kiss Precise" by Frederick Soddy published in Nature, 1930:

Four circles to the kissing come. / The smaller are the benter. / The bend is just the inverse of / The distance from the center. / Though their intrigue left Euclid dumb / There's now no need for rule of thumb. / Since zero bend's a dead straight line / And concave bends have minus sign, / The sum of the squares of all four bends / Is half the square of their sum.

So the new cluster $(S_1, \ldots, S_{n+1}, S'_{n+2})$ is the cluster obtained from the original one by applying the transformation $s_{\mathfrak{e}_{n+2}}$.

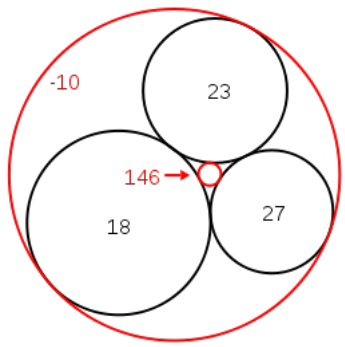


FIGURE 2. Apollonian circle cluster with curvature vector $(-10, 18, 23, 27)$

If we start with a cluster as in the picture, then all circles will be inclosed in a unique circle of largest radius, so our circle packing will look as in following Figure 3.

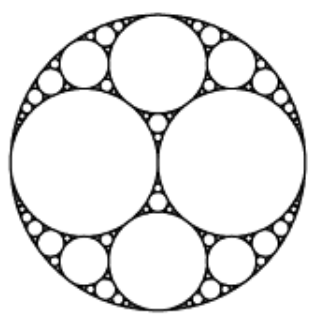


FIGURE 3. Apollonian circle packing

Another possibility is when we start with a cluster of three circles touching a line. Then, the circle packing will look like in the following Figure 4.

The following example is taken from [7].

Example 5.4. We take P with the Gram matrix

$$G(P) = \mathrm{cir}(1, -1, 0, -1).$$

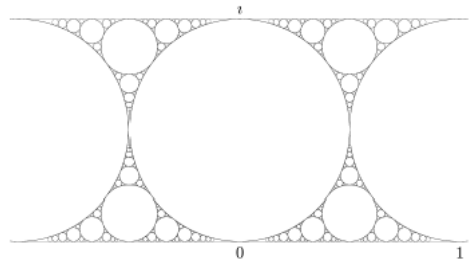
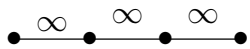


FIGURE 4. Apollonian circle packing in a band

It is a Coxeter polyhedron with the Coxeter diagram



The dual polyhedron $P^{\perp}$ has the Gram matrix $G(P)^{-1}$

$$G(P)^{-1} = \frac{1}{2}\mathrm{cir}(1, -1, -2, -1).$$

Example 5.5. Let us consider the exceptional case of Apollonian packings when $n = 1$ (this case was briefly mentioned in [17]). Here 0-dimensional spheres are sets of two or one points in $\mathbb{P}^1(\mathbb{R})$ that we can identify with the boundary of the unit disk model of the hyperbolic plane $\mathbb{H}^2$. We start with three 0-dimensional spheres touching each other. This means that each pair has a common point. Let p_{12}, p_{13}, p_{23} be the common points. We use the Poincaré model of $\mathbb{H}^2$. Then the corresponding hyperplanes $H(\omega_1), H(\omega_1), H(\omega_1)$ are three geodesics joining pairs of these points. They form a *ideal triangle* with vertices on the boundary.

FIGURE 5. Ideal triangle

The Apollonian group is not defined in this case. However, the dual Apollonian group $\mathrm{Ap}_1^{\perp}$ is defined. It is a Coxeter group with Coxeter

diagram equal to the complete graph K_3 with solid edges. It is a quasi-Lanner Coxeter group generated by the reflections in sides of the ideal triangle.

The fundamental polyhedron of $\mathrm{Ap}_1^\perp$ is equal to the closure of the interior of the triangle. Its Gram matrix is equal to $\mathrm{cir}(1,-1,-1)$. Since it is of finite volume, the group is a Fuchsian group of the first kind with 3 cusps. The limit set of $\mathrm{Ap}_1^\perp$ is the whole circle. However, the packing is not strict since the interiors of our 0-dimensional spheres are one of the two intervals in $\mathbb{P}^1(\mathbb{R})$ with end points equal to the sphere. Obviously we have a nested family of intervals. The set of 0-dimensional spheres in the sphere $\mathrm{Ap}_1^\perp$-packing is a proper subset of $\Lambda(\mathrm{Ap}_1^\perp)$. It consists of cusp points and is countable.

The Descartes equation becomes

$$k_1k_2 + k_1k_3 + k_2k_3 = 0. \tag{5.12}$$

6. Integral packings

Let $K \subset \mathbb{R}$ be a totally real field of algebraic numbers of degree $d = [K : \mathbb{Q}]$ and $\mathfrak{o}_K$ be its ring of integers. A free $\mathfrak{o}_K$-submodule L of rank $n+2$ of $\mathbb{R}^{n+1,1}$ is called a $\mathfrak{o}_K$-lattice. Let $(\mathfrak{e}_1, \dots, \mathfrak{e}_{n+2})$ be a basis of L. Assume that the entries of its Gram matrix $G = ((\mathfrak{e}_i, \mathfrak{e}_j)) = (g_{ij})$ belong to $\mathfrak{o}_K$ Let $\sigma_i : K \hookrightarrow \mathbb{R}, i = 1, \dots, d-1$, be the set of non-identical embeddings of K into $\mathbb{R}$. We assume additionally that the matrices $G^{\sigma_i} = (\sigma_i(g_{ij}))$ are positive definite. Let

$$f = \sum_{1 \le i,j \le n+2} g_{ij} t_i t_j$$

be the quadratic form defined by the matrix G and let $\mathrm{O}(f, \mathfrak{o}_K)$ be the subgroup of $\mathrm{GL}_{n+2}(\mathfrak{o}_K)$ of transformations that leave f invariant. The group $\mathrm{O}(f, \mathfrak{o}_K)$ is a discrete subgroup of $\mathrm{O}(\mathbb{R}^{n+1,1})$ with a fundamental polyhedron of finite volume, compact if f does not represent zero. By passing to the projective orthogonal group, we will view such groups as Kleinian subgroups of $\mathrm{Iso}(\mathbb{H}^{n+1})$. A subgroup of $\mathrm{Iso}(\mathbb{H}^{n+1})$ which is commensurable with a subgroup of the form $\mathrm{PO}(f, \mathfrak{o}_K)$ is called *arithmetic*. Two groups $\mathrm{PO}(f', \mathfrak{o}'_K)$ and $\mathrm{PO}(f, \mathfrak{o}_K)$ are commensurable if and only if $K = K'$ and f is equivalent to $\lambda f'$ for some positive $\lambda \in K$. In particular, any subgroup of finite index in $\mathrm{PO}(f, \mathfrak{o}_K)$ is an arithmetic group.

We will be interested in the special case of an *integral lattice* where $K = \mathbb{Q}$. A Kleinian group Γ in $\mathbb{H}^{n+1}$ is called *integral* if it is commensurable with a subgroup of $\mathrm{PO}(f, \mathbb{Z})$ for some integral quadratic

form f of signature $(n+1,1)$. We will be dealing mostly with non-arithmetic geometrically finite Kleinian groups. However, it follows from [6], Proposition 1, that such a group is always Zariski dense in $\mathrm{PSO}(n+1,1)$ or $\mathrm{PO}(n+1,1)$ if it acts irreducibly in $\mathbb{H}^{n+1}$. In terminology due to P. Sarnak, Γ is a *thin group*.

We will use the usual terminology of the theory of integral quadratic forms, or more abstractly, *integral quadratic lattices*, free abelian groups L of finite rank equipped with a symmetric bilinear form with values in $\mathbb{Z}$. By tensoring with $\mathbb{R}$, we embed L in $L_{\mathbb{R}} := L \otimes \mathbb{R}$. We assume that the quadratic form of L extends to a quadratic form on $L_{\mathbb{R}}$ of signature $(n+1,1)$, this will identify L with an integral lattice in the previous sense. The set $L^\vee$ of all vectors $v \in L_{\mathbb{R}}$ such that $(x,v) \in \mathbb{Z}$ for all $x \in L$ is called the *dual lattice*. It is a free abelian group spanned by the dual basis of a basis of L in $L_{\mathbb{R}}$. The Gram matrix of this basis is the inverse of the Gram matrix of the basis of L. The quotient group $L^\vee/L$ is called the *discriminant group* of L. Its order is equal to the minus of the determinant of the Gram matrix of any basis of L. Note that $L^\vee \subset L_{\mathbb{R}}$ is not, in general, an integral lattice. However, it becomes integral if we multiply its quadratic form by the exponent of the discriminant group of L. We denote by $\mathrm{O}(L)$ the orthogonal group of L, the subgroup of $\mathrm{GL}(L)$ that preserves the bilinear form. A reflection isometry of $\mathbb{R}^N$ defined by a vector $v \in L$ belongs to $\mathrm{O}(L)$ if and only if $\frac{2(v,x)}{(v,v)} \in \mathbb{Z}$ for all $x \in L$.

Remark 6.1. Let Γ be an integral Kleinian group of isometries of $\mathbb{H}^n$ of finite covolume, for example, the orthogonal group of some integral quadratic lattice L. It is obviously geometrically finite, however, for $n > 3$ it may contain finitely generated subgroups which are not geometrically finite. In fact, for any lattice L of rank ≥ 5 that contains a primitive sublattice defined by the quadratic form $q = -x_0^2+x_1^2+x_2^2+x_3^2$, the orthogonal group $\mathrm{O}(L)$ contains finitely generated but not finitely presented subgroups (see [21]).[9]

A sphere packing $\mathfrak{P} = (S_i)_{i\in I}$ is called *integral* if the following conditions are satisfied.

(i) The corresponding norm one vectors v_i in $\mathbb{R}^{n+1,1}$ span an integral lattice L;
(ii) There exists a positive integer λ such that $\lambda(v_i, v_j) \in \mathbb{Z}$ for all $i, j \in I$. The smallest such c is called the *exponent*.
(iii) The curvatures of spheres S_i are integers.

[9] I thank Boris Apanasov who informed me about this fact.

After multiplying the quadratic form of L, the restriction of the fundamental quadratic form to L, by the exponent we obtain an integral quadratic lattice $\mathrm{int}(L)$. If Γ is a Kleinian group that leaves an integral sphere packing invariant, then it obviously leaves $\mathrm{int}(L)$ invariant and becomes a subgroup of $\mathrm{O}(\mathrm{int}(L))$.

It follows from formula (5.7) that a Boyd-Maxwell sphere packing defined by a Coxeter polytope P with Gram matrix $G(P) = (g_{ij})$ is integral if and only if the curvatures of the spheres from the initial cluster are integers and $g_{ij} \in \mathbb{Q}$. The quadratic lattice is defined by the matrix $eG(P)$, where e is the smallest positive integer such that $eg_{ij} \in \mathbb{Z}$.

Example 6.2. Let P be an Apollonian polyhedron in $\mathbb{H}^{n+1}, n > 1$. It is defined by the matrix $\mathrm{cir}(1, \frac{1}{1-n}, \dots, \frac{1}{1-n})$ whose inverse is equal to $\frac{n-1}{2n}\mathrm{cir}(1, -1, \dots, -1)$. Recall that it is a Coxeter polyhedron only for $n = 2, 3$. The Apollonian packing is an integral Coxeter packing with exponent 1. The Apollonian group Ap_n becomes a subgroup of the orthogonal group $\mathrm{O}(\mathcal{A}p(n))$, where $\mathcal{A}p(n)$ is an integral quadratic lattice of rank $n+2$ with the quadratic form

$$q_n = \sum_{i=1}^{n+2} t_i^2 - 2 \sum_{1 \le i < j \le n+2} t_i t_j. \tag{6.1}$$

Recall that an integral quadratic lattice is called *even* if all its values are even integers, it is called *odd* otherwise. Obviously, the lattice $\mathcal{A}p(n)$ is odd. For any lattice L, we denote by L^{ev} its largest even sublattice. Obviously, $\mathrm{O}(L) \subset \mathrm{O}(L^{\mathrm{ev}})$. Let us find an orthogonal decomposition of the lattice $\mathcal{A}p(n)^{\mathrm{ev}}$.

The quadratic form (6.1) is given in a basis $\bar{\omega}_1, \dots, \bar{\omega}_{n+2}$ of $\mathcal{A}p(n)$. Consider a new basis $f_1, \dots, f_{n+2}$ formed by the vectors $\bar{\omega}_1, \bar{\omega}_1 + \bar{\omega}_2, -\bar{\omega}_1 - \bar{\omega}_3, \bar{\omega}_1 + \bar{\omega}_2 + \bar{\omega}_3 - \bar{\omega}_4, \bar{\omega}_4 - \bar{\omega}_5, \dots, \bar{\omega}_{n+1} - \bar{\omega}_{n+2}$. The Gram matrix of this basis is the following matrix.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 4 & -2 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & -2 & 4 & -2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & -2 & 4 \end{pmatrix}.$$

This shows that, for any $n > 1$,

$$\mathcal{A}p(n) \cong \langle 1 \rangle \oplus \mathsf{U}(2) \oplus \mathsf{A}_{n-1}(2). \tag{6.2}$$

Here we use the standard notation from the theory of integral quadratic lattices:U is the integral hyperbolic plane, A_n is the even positive definite root lattice defined by the Cartan matrix of type A_n, $\langle k \rangle$ is the

lattice of rank 1 spanned by a vector v with $(v,v)=k$, and $M(t)$ denotes the quadratic lattice M with its integral quadratic form multiplied by a rational number t.

We have

$$\mathcal{A}p(n)^{\mathrm{ev}} = \mathsf{A}_1(2)\oplus \mathsf{U}(2)\oplus \mathsf{A}_{n-1}(2). \tag{6.3}$$

The discriminant group of $\mathcal{A}p(n)$ (resp. $\mathcal{A}p(n)^{\mathrm{ev}}$) is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n\oplus\mathbb{Z}/2n\mathbb{Z}$ (resp. $(\mathbb{Z}/2\mathbb{Z})^n\oplus\mathbb{Z}/4\mathbb{Z}\oplus\mathbb{Z}/2n\mathbb{Z}$).

Next we consider the dual Apollonian sphere packing. Its lattice is the dual lattice of $\mathcal{A}p(n)^\vee$ defined by the quadratic form from the left-hand side of (5.10) multiplied by $\frac{n-1}{2n}$.

$$q_n^\perp := (n-1)\sum_{i=1}^{n+2} t_i^2 - 2\sum_{1\le i<j\le n+2} t_it_j.$$

Its exponent is equal to $2n$ and coincides with the exponent of the discriminant group of $\mathcal{A}p(n)$. We have

$$\begin{aligned}\mathcal{A}p(n)^\perp := \mathrm{int}(\mathcal{A}p(n)^\vee) = \mathcal{A}p(n)^\vee(2n) &\cong (\langle 1\rangle\oplus\mathsf{U}(1/2)\oplus\mathsf{A}_{n-1}^\vee(1/2))(2n)\\ &= \langle 2n\rangle\oplus\mathsf{U}(n)\oplus\mathsf{A}_{n-1}^\vee(n).\end{aligned}$$

The lattice $\mathsf{A}_{n-1}^\vee$ is generated by A_{n-1} and a vector

$$\beta = \frac{1}{n}(\omega_1+2\omega_2+\cdots+(n-1)\omega_{n-1}),$$

where $\omega_1,\dots,\omega_{n-1}$ is the standard basis defined by simple roots. In the basis $(\omega_2,\dots,\omega_{n-1},\beta)$, the matrix of $\mathsf{A}_{n-1}^\vee(n)$ is equal to

$$\begin{pmatrix} 2n & -n & 0 & 0 & \dots & 0 & 0 & 0 & 0\\ -n & 2n & -n & 0 & \dots & 0 & 0 & 0 & 0\\ 0 & -n & 2 & -n & \dots & 0 & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \dots & -n & 2n & -n & 0\\ 0 & 0 & 0 & 0 & \dots & 0 & -n & 2n & n\\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & n & n-1 \end{pmatrix} \tag{6.4}$$

Thus we obtain that $\mathrm{Ap}_n^\perp$ is even if n is odd, and, for n even

$$(\mathcal{A}p(n)^\perp)^{\mathrm{ev}} \cong \langle 2n\rangle\oplus\mathsf{U}(n)\oplus(A_{n-1}^\vee(n))^{\mathrm{ev}}, \tag{6.5}$$

where $(A_{n-1}^\vee(n))^{\mathrm{ev}}$ is defined by the matrix (6.4) in which the submatrix $\left(\begin{smallmatrix}2n & n\\ n & n-1\end{smallmatrix}\right)$ is replaced with $\left(\begin{smallmatrix}2n & 2n\\ 2n & 4n-4\end{smallmatrix}\right)$.

Note that the Apollonian packing in dimension $n=1$ exists but it is not a Boyd-Maxwell sphere packing. We have

$$\mathcal{A}p_1\cong\langle 1\rangle\oplus U(2),\quad \mathcal{A}p_1^{\mathrm{ev}}\cong(\mathsf{A}_1\oplus U)(2).$$

The lattice $\mathcal{A}p_1^{\mathrm{ev}}(1/2)=\mathsf{A}_1\oplus\mathsf{U}$ is from Nikulin's list of integral even hyperbolic quadratic lattices of rank 3 and signature $(2,1)$ such that the subgroup of the orthogonal group generated by reflections $s_v, (v,v)=2$

is of finite index (a 2-reflexive lattice) [31]. Each such lattice is a sublattice of $\mathsf{A}_1 \oplus \mathsf{U}$. According to Nikulin's classification of even 2-reflexive hyperbolic lattices of rank ≥ 5, none of the lattices $\mathcal{A}p(n)^{\mathrm{ev}}$ or $\mathcal{A}p(n)^{\mathrm{ev}}$ is 2-reflexive [29].

If $n = 2$, the lattices $\mathcal{A}p_2$ and $\mathcal{A}p_2^{\perp}$ are isomorphic. According to Vinberg's list of 2-reflexive even hyperbolic lattice of rank 4 [41], the lattice $\mathcal{A}p_2^{\mathrm{ev}}(\frac{1}{2}) \cong A_1 \oplus A_1 \oplus U$ is 2-reflexive.

Note that all examples from [7] are examples of integral Boyd-Maxwell sphere packings.

7. Orbital counting

Let A be a subset of a metric space E. For any $s > 0$, let $c_s(A)$ be the infimum of $\sum r_i^s$ such that there is a countable cover of A with balls of radius $r_i > 0$. The *Hausdorff dimension* $\dim_H(A)$ of A is defined to be the infimum of s such that $c_s(A) = 0$.

For example, if the Lebesque measure of A is equal to 0, then it follows from the definition that $\dim_H(A) \leq n$. The Hausdorff dimension coincides with the Lebesque measure if the latter is positive and finite. Also it is known that the topological dimension of A is less than or equal to its Hausdorff dimension.

The Hausdorff dimension is closely related to the *fractal dimension* of a *fractal set* A, i.e. a set that can be subdivided in some finite number $N(\lambda)$ of subsets, all congruent (by translation or rotation) to one another and each equal to a scaled copy of A by a linear factor λ. It is defined to be equal to $\frac{\log N(s)}{\log(1/\lambda)}$. For example, the Cantor set consists of two parts A_1 and A_2 (contained in the interval $[0, 1/3]$ and $[1/3, 1]$), each rescaled version of the set with the scaling factor $1/3$. Thus its fractal dimension is equal to $\log 2/\log 3 < 1$. It is clear that the Hausdorff dimension of a bounded fractal set of diameter $D = 2R$ is less than or equal to the fractal dimension. In fact, such a set can be covered by $N(\lambda)$ balls of radius λR, or by $N(\lambda)^2$ balls of radius $\lambda^2 R$. Since $\lambda < 1$, we get $\mu_s(A) = \lim_{k\to\infty} \frac{N(\lambda)^k}{(\lambda^k R)^s}$ which is zero if $N(\lambda)\lambda^s < 1$ or $s > \log N/\log(1/\lambda)$. In fact, the Hausdorff dimension of the Cantor set coincides with its fractal dimension $\log 2/\log 3$.

By a theorem of D. Sullivan [37], for a geometrically finite non-elementary discrete group Γ, the Hausdorff dimension δ_Γ of $\Lambda(\Gamma)$ is positive and coincides with the *critical exponent* of Γ equal to

$$\inf\{s > 0 : \sum_{g\in\Gamma} e^{-sd(x_0,g(x_0))} < \infty\}, \tag{7.1}$$

where x_0 is any point on $\mathbb{H}^n$. Using this equality Sullivan shows that

$$\delta_\Gamma = \overline{\lim_{T\to\infty}} \frac{\log N_T}{R}, \tag{7.2}$$

where N_T is the number of orbit points y with hyperbolic distance from x_0 less than or equal to R. He further shows in [38], Corollary 10, that, under the additional assumption that Γ has no parabolic fixed points, there exist some constants c, C such that, as $T \to \infty$,

$$ce^{T\delta_\Gamma} \leq N_T \leq Ce^{T\delta_\Gamma}. \tag{7.3}$$

In particular, asymptotically, as $T \to \infty$,

$$N_T \sim c(T)e^{T\delta_\Gamma},$$

where $\lim\limits_{T\to\infty} \frac{c(T)}{T} = 0$.

If $\delta_\Gamma > \frac{1}{2}(n-1)$, then P. Lax and R. Phillips show that, for any geometrically finite non-elementary discrete group Γ, the function $c(T)$ is a constant depending only on Γ. When Γ is of finite covolume, then δ_Γ is known to be equal to $n-1$, and the result goes back to A. Selberg.

The assumption on δ_Γ has been lifted by T. Roblin [35]. Applying (2.1), we will be able to obtain the assertion of Theorem 1.2 from the introduction in the case when $C^2 < 0$. To do the remaining cases where (e, e) is non-negative we need to replace the family of hyperbolic balls with another family of sets of growing volume.

Let e be a nonzero vector in $\mathbb{R}^{n,1}$. We assume that $(e, e) = \pm 1$ if $(e, e) \neq 0$ and we fix an isotropic e' with $(e, e') = -1$ if $(e, e) = 0$. The group $G = \mathrm{SO}(n, 1)_0$ acts in the projective space $\mathbb{P}(\mathbb{R}^{n,1})$ with three orbits $G \cdot [e]$. Let $G_{[e]}$ be the stabilizer subgroup of $[e]$ in G. If $(e, e) < 0$, then $G_{[e]}$ is a maximal compact subgroup in $K \subset G$ isomorphic to $\mathrm{SO}(n)$ and the orbit $G \cdot [e]$ is the homogeneous space of left cosets G/K isomorphic to $\mathbb{H}^n$.

If $(e, e) = 0$, then $G_{[e]}$ is a parabolic subgroup of G. If we identify $\mathsf{Q}\backslash\{[e]\}$ with the Euclidean space E^{n-1} (via the projecting from $[e]$), we obtain a surjective homomorphism from $G_{[e]}$ to the group P of affine orthogonal transformations (motions) of the Euclidean space E^{n-1}. This homomorphism splits by a subgroup G_{ce} of $G_{[e]}$ that stabilizes any nonzero vector ce on the line $[e]$. The kernel of the homomorphism is the group A_l of hyperbolic translations along the hyperbolic line l corresponding to the subspace spanned by e and e'. This shows that $G_{[e]} = O(n-1) \cdot A_l \cdot N_{[e]}$, where $O(n-1)$ is the group of rotations around the line l and $N_{[e]} \cong \mathbb{R}^{n-1}$ is the normal subgroup of translations of P, called a *horospherical subgroup* of G. The decomposition of $G_{[e]}$ from above is induced by the *Iwasawa decomposition* $G = G_x \cdot A_l \cdot N_{[e]}$ of

G, where x is any point on l. The group $G_{[e]}$ acts transitively on the quadric Q and on $\mathbb{H}^p$.

If $(e,e) > 0$, then $G_{[e]} \cong \mathrm{SO}(n-1,1)_0$, the homogeneous space $G_{[e]}\backslash G$ is known as a *de Sitter space*. By taking the orthogonal complement of the line spanned by e, we identify the points of the de Sitter space with oriented hyperplanes $H_{\mathfrak{e}}$ in $\mathbb{H}^n$. For example, if $n = 2$, the de Sitter space is the set of oriented geodesic lines in the hyperbolic plane.

Let us consider the *pencil of geodesic lines* $\mathcal{P}(e)$ defined by indefinite planes $U \subset \mathbb{R}^{n,1}$ containing e. If $(e,e) < 0$, then $\mathcal{P}(e)$ consists of geodesic lines containing the point $x_0 = [e] \in \mathbb{H}^n$. It is called an *elliptic pencil*. If $(e,e) > 0$, it consists of parallel geodesic lines perpendicular to the hyperplane H_e. It is called a *hyperbolic pencil*. Finally, if $(e,e) = 0$, it consists of geodesic lines whose closure in $\overline{\mathbb{H}^n}$ contains the point $[e]$. The pencil is called *parabolic pencil* with center at $[e]$.

The parametric equation of the geodesic line from the pencil $\mathcal{P}(e)$ is equal to

$$\gamma(t) = \begin{cases} v\sinh t + e\cosh t, (v,e) = 0, (v,v) = 1 & \text{if } (e,e) = -1, \\ v\exp(t) + e\sinh t, (v,e) = 1, (v,v) = -1 & \text{if } (e,e) = 0, \\ v\cosh t - e\sinh t, (v,e) = 0, (v,v) = -1 & \text{otherwise.} \end{cases}$$

(see [1], Chapter 4, 2.3). In the elliptic case, we have $(\gamma(t), e) = -\cosh t$, so that $d(\gamma(t), [e]) = t$. So, moving along the geodesic line for the distance $t \geq 0$, we get the set of points in $\mathbb{H}^n$ equidistant from $[e]$. This is a *geodesic sphere*

$$H_e^t := \{[v] \in \mathbb{H}^n : -(v,e) = \cosh t\} = \{[v] \in \mathbb{H}^n : d([v],[e]) = t\}$$

with the center at $x_0 = [e]$. The group $G_{[e]} \cong \mathrm{SO}(n)$ acts transitively on each H_e^t and, projecting from $[e]$ along the lines from the pencil $\mathcal{P}(e)$ identifies each geodesic sphere with the absolute as homogenous spaces with respect to $G_{[e]}$. So, the orbits of $G_{[e]}$ in $\mathbb{H}^n$ are Riemannian homogeneous spaces of constant positive curvature.

If $(e,e) = 0$, we have $(\gamma(t), e) = \exp(t)$. The orbits of $N_{[e]}$ are *horospheres*

$$H_e^t := \{[v] \in \mathbb{H}^n : -(v,e) = \exp(t)\}.$$

In the vector model of $\mathbb{H}^n$, a horosphere is equal to the intersection of $\mathbb{H}^n$ with a sphere in the Euclidean space tangent to Q at the point $[e]$. In the Klein model, they are ellipsoids tangent to the boundary of the ball. Any horosphere inherits a Riemannian metric of zero curvarure and hence isomorphic to an Euclidean space.

If $(e,e) > 0$, we have $(v,e) = (\gamma(0), e) = 0$, i.e. $[\gamma(0)] \in H_e$. Since $(\gamma(t), e) = -\sinh t$, after time t we move from a point on the hyperplane

H_e to a point $[\gamma(t)]$ on the hypersurface

$$H_e^t = \{[w] \in \mathbb{H}^n : -(w,e) = \sinh t\}.$$

The hypersurfaces of this form are called *equidistant hypersurfaces*. They are orbits of the group $G_{[e]}$ and inherit a Riemannian metric of constant negative curvature. In the Klein model, they are ellipsoids whose closures in $\overline{\mathbb{H}^n}$ intersect the closure $\bar{H}_e$ of the hyperplane H_e along its boundary. For example, if $n = 2$, they are ellipses tangent to the absolute at the two points in which the closure of the geodesic line H_e intersects the absolute.

In each of the three cases, let us consider the following sets $B_T(e)$.

If $(e,e) = -1$,

$$B_T(e) = \bigcup_{t=0}^{T} H_e^t = \{[v] \in \mathbb{H}^n : 1 \leq -(v,e) \leq \cosh T\}.$$

This is just a hyperbolic closed ball with center at $[e]$ and radius T.

If $(e,e) = 0$,

$$B_T(e) = \bigcup_{t=0}^{T} H_e^t = \{[v] \in \mathbb{H}^n : 1 \leq |(v,e)| \leq \exp(T)\}.$$

If $(e,e) = 1$,

$$B_T(e) = \bigcup_{t=0}^{T} H_e^t = \{[v] \in \mathbb{H}^n : |(v,e)| \leq \sinh(T)\}.$$

Fix a point $x_0 = [v_0] \in \mathbb{H}^n$ and let $K = G_{x_0}$ so that $G/K = \mathbb{H}^n$. Let $H = \{1\}$ (resp. $N_{[e]}$, resp. $G_{[e]}$) if $(e,e) = -1$ (resp. $(e,e) = 0$, resp. $(e,e) = 1$).

Consider the following subsets of G/H

$$\mathcal{S}_T(e, x_0) := G_{x_0} \cdot A_T \cdot H/H,$$

where $A_T = \{a_t \in A : 0 \leq t \leq \log T\}$.

The following result from [33], Theorem 1.2 and [28], Corollary 7.14 (see a nice survey of some of these results in [32]) is crucial for our applications.

Theorem 7.1. *Let Γ be a torsion free geometrically finite non-elementary discrete subgroup of $SO(n,1)_0$. Let $[e] \in \mathbb{P}(\mathbb{R}^{n,1})$. If $(e,e) \geq 0$, assume that the orbit $O_\Gamma([e])$ is a discrete set and also that $\delta_\Gamma > 1$ if $(e,e) > 0$. Then*

$$\lim_{T\to\infty} \frac{\Gamma \cdot H/H \cap \mathcal{S}_T(e, x_0)}{T^{\delta_\Gamma}} = c_{\Gamma, x_0, [e]},$$

where $c_{\Gamma, x_0, [e]}$ is a positive constant that depends only on $\Gamma, x_0, [e]$.

Note that the results from [28] also give error terms.

Corollary 7.2. *Keep the same assumptions. Fix a point $x_0 \in \mathbb{H}^n$. Then*

$$\lim_{t\to\infty} \frac{\#\{[v] \in O_\Gamma(x_0) : |(e,v)| \le T\}}{T^{\delta_\Gamma}} = c_{\Gamma,x_0,[e]},$$

where $c_{\Gamma,x_0,[e]}$ is a positive constant that depends only on $\Gamma, x_0, [e]$.

Remark 7.3. The assumption $\delta_\Gamma > 1$ is not needed if the group $\Gamma \cap N_{[e]}$ acts on ∂H_e without parabolic fixed points. Also, the assumption that Γ is torsion free made in loc.cit. is needed only for an explicit formula for the constant.

Remark 7.4. In the case when $(e,e) < 0$ (resp. $(e,e) = 0$ and $n = 3$), Theorem 7.1 follows from [33], Theorem 1.2 (resp. [22], Theorem 2.10) that gives an asymptotic of the number of orbit points in a ball $\{[v] \in \mathbb{H}^n : ||v|| \le T\}$, where $||x||$ is the Euclidean norm. One only needs to use that if $v = (x_0, \dots, x_n)$ with $\sum_{i=0}^n x_i^2 \le T^2$ and $-x_0^2 + \sum_{i=1}^n x_i^2 = -1$ (resp. $= 0$), then $(v, (1, \dots, 0))^2 = x_0^2 \le (T^2+1)/2$ (resp. $\le T^2$).

In the case $(e,e) > 0$, and Γ and $\Gamma \cap G_{[e]}$ are of finite covolume (the second condition is always satisfied if $n \ge 3$ [11]), Theorem (7.1) follows from [15], [16]. In this case $\delta_\Gamma = n - 1$.

8. Algebraic geometrical realization

Let X be a smooth projective algebraic surface over an algebraically closed field $\Bbbk$. The group of its automorphisms defines a natural left action $\gamma \mapsto (g^{-1})^*(\gamma)$ on its lattice $\mathrm{Num}(X)$ of algebraic cycles modulo numerical equivalence. As usual, we denote its rank by $\rho(X)$, this is the *Picard number* of X. By the Hodge Index Theorem, the intersection form on algebraic cycles defines a symmetric bilinear form on $\mathrm{Num}(X)$ of signature $(1,n)$. We denote by V_X the corresponding real vector space $\mathrm{Num}(X)_\mathbb{R}$ equipped with the induced symmetric bilinear form of signature $(1,n)$. We assume that the image $\mathrm{Aut}(X)^*$ of the homomorphism $\mathrm{Aut}(X) \to \mathrm{O}(\mathrm{Num}(X))$ is infinite. It is known that this could happen only if the Kodaira dimension of X is less than or equal to 1. Also, if it is one, then $\mathrm{Aut}(X)^*$ contains a finitely generated abelian subgroup of finite index. All of this is rather well-known, see, for example, [12]. Let Γ be an infinite subgroup of $\mathrm{Aut}(X)^*$. Since the automorphism group preserves the connected component of $\{v \in \mathrm{Num}(X)_\mathbb{R} : (v,v) > 0\}$ containing an ample class on X, we obtain that Γ is a discrete subgroup in $\mathrm{O}(V_X) \cong \mathrm{O}(n,1)$. Also, since Γ preserves an integral lattice, its orbits are discrete sets. Therefore, we can apply Theorem 7.1 to obtain the following.

Theorem 8.1. *Assume that Γ is a geometrically finite non-elementary subgroup of $Aut(X)^*$. Let H, C be effective numerical divisor classes in $Num(X)'$ such that H is ample. Then*

$$\lim_{t\to\infty} \frac{\#\{C' \in O_\Gamma(C) : (H, C')/(H, C) \le T\}}{T^{\delta_\Gamma}} = c_{\Gamma,C,H}, \tag{8.1}$$

where $c_{\Gamma,H,C}$ is a positive constant depending only on $\Gamma, [H], [C]$ but does not depend on C^2 and H^2.

Now let us give some examples where we can apply the previous result. First, we note that we may always consider Γ up to a finite group, i.e. we either consider its subgroup of finite index, or extend it by a finite group, or do both. It affects only the constant in the asymptotic formula.

Let X be a complex algebraic K3 surface. It is known that the subgroup of $\mathrm{O}(\mathrm{Num}(X))$ generated by transformations $g^*, g \in \mathrm{Aut}(X)$ and reflections $s_r, (r, r) = -2$ is a subgroup of finite index (this is also true if the characteristic $p \neq 2$ [24]). Thus, if $\mathrm{Num}(X)$ does not contain classes r with $(r, r) = -2$, then $\mathrm{Aut}(X)$ is a Kleinian group of finite covolume.[10] The Hausdorff dimension δ_Γ in this case is equal to $n - 1$. So, it is more interesting to realize proper thin Kleinian groups of automorphisms.

Assume that we find a surface X and vectors $\alpha_1, \dots, \alpha_r \in \mathrm{Num}(X)$ such that $\frac{2}{(\alpha_i,\alpha_i)}\alpha_i \in \mathrm{Num}(X)^\vee$, so that the reflections s_{α_i} define orthogonal transformations of $\mathrm{Num}(X)$. Moreover, assume that there exist automorphisms $g_i \in \mathrm{Aut}(X)$ such that $g_i^* = s_{\alpha_i}$. Let P be the convex polyhedron with normal vectors $\mathfrak{e}_i = \frac{1}{\sqrt{-(\alpha_i,\alpha_i)}}\alpha_i$. Assume that it is a Coxeter polyhedron of level ≤ 2 with the reflection group Γ_P generated by $s_{\alpha_1}, \dots, s_{\alpha_r}$. Then the limit set of Γ_P is a Boyd-Maxwell sphere packing and its Hausdorff dimension δ_{Γ_P} is equal to the critical exponent (1.1). Now we can apply Theorem 8.1.

Example 8.2. Let $M = \mathcal{A}p(2)^{\mathrm{ev}} \cong \mathsf{A}_1(2) \oplus \mathsf{U}(2) \oplus A_1(2)$ be the even sublattice of the Apollonian lattice and Ap_2 be the Apollonian group. Let $\omega_1 = \mathfrak{e}_1, \dots, \omega_4 = \mathfrak{e}_4$ be a basis in $\mathcal{A}p(2)$ with Gram matrix equal to $G = \mathrm{cir}(1, -1, -1, -1) = (g_{ij})$. The lattice $M(-\frac{1}{2}) \cong \mathsf{A}_1(-1) \oplus \mathsf{U}(2) \oplus A_1(-2)$ can be primitively embedded in the lattice $\mathrm{Num}(X) \cong \mathsf{U} \oplus E_8(-1)$, where X is an Enriques surface. We identify the image with $M(-\frac{1}{2})$. The group Ap_2 contains a subgroup Ap_2' of index 2 that consists of isometries of $M(-\frac{1}{2})$ that extend to the whole lattice

[10] By Riemann-Roch theorem, such a divisor class is either effective or anti-effective, and one of its irreducible component is a smooth rational curve.

acting identically on the orthogonal complement. Thus the subgroup $\Gamma = \mathrm{Ap}_2' \cap \mathrm{O}(\mathrm{Num}(X))'$ is realized by automorphisms of X. One can show that the index of Γ in Ap_2' is equal to 2. Now we can apply Theorem 8.1 with $\delta_\Gamma = \delta_{\mathrm{Ap}_2}$.

Of course, one can also realize $\mathcal{A}p(2)^{\mathrm{ev}}$ as the Picard lattice of a K3 surface X. Since the lattice does not represent -2, a subgroup of finite index of Ap_2 is realized as a group of automorphisms of a K3 surface. According to Vinberg's classification of 2-reflexive lattices of rank 4 [41], the 2-reflection group of the lattice $\mathcal{A}p(2)$ is of finite index in the orthogonal group of the lattice. Thus Ap_2 is a thin geometrically finite group of automorphisms of X.

The same is true for the group Ap_3; it contains a subgroup of finite index that can be realized as a thin geometrically finite group of automorphisms of either an Enriques or a K3 surface.

Example 8.3. Consider the Coxeter group $\Gamma(a,b,c)$ of a Coxeter polyhedron P in $\mathbb{H}^2$ with Gram matrix $\begin{pmatrix} 1 & -a & -b \\ -a & 1 & -c \\ -b & -c & 1 \end{pmatrix}$, where a,b,c are rational numbers ≥ 1. If $a=b=c=1$, the fundamental domain is an ideal triangle. If $a,b,c>2$, the fundamental triangle is a pair of pants as on the following picture.

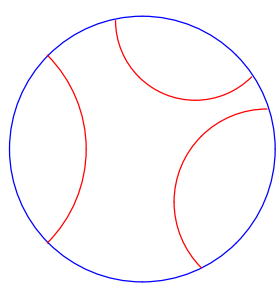


Figure 7: Hyperbolic triangle with no vertices

Let L be the quadratic form defined by the integral matrix $A = -eG(P)$, where e is the exponent of the Coxeter matrix. Let $L^\vee$ be the dual lattice. Suppose we can embed $M = \mathrm{int}(L^\vee)^{\mathrm{ev}}$ in $\mathrm{Num}(X)$ of some K3 surface and realize the reflection group Γ_P as a group of automorphisms of X. Then we can apply Theorem 8.1 to any nef divisor class contained in the sublattice M.

For example, let us consider the case $(a,b,c) = (a,a,1)$. The fundamental triangle has one ideal point and looks like in the following figure 8.3.

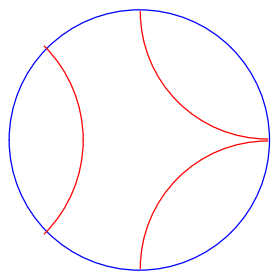


Figure 8: Hyperbolic triangle with one ideal point

Let $\mathcal{H}^2 \to \mathbb{H}^2$ be the map from the upper-half plane to the unit disk given by the map $z \mapsto \frac{z-i}{z+i}$. One can show that the pre-image of the sides of our triangle are the lines $x = 1, x = -1$ and the upper half-circle of radius $r = 1/a$ with center at the origin. Recall that $a \leq -1$ so that the half-circle is between the vertical lines. Let us re-denote our group by Γ_r. It was shown by C. McMullen in [25] that as $r \to 0$, we have

$$\delta_{\Gamma_r} = \frac{r+1}{2} + O(r^2),$$

while for $r \to 1$, we have

$$\delta_{\Gamma_r} \sim 1 - \sqrt{1-r}.$$

We have

$$G(P)^{-1} = \frac{1}{4a^2} \begin{pmatrix} 0 & -2a & -2a \\ -2a & a-1 & -a-1 \\ -2a & -a-1 & a-1 \end{pmatrix}$$

If $a > 1$ is odd, the matrix $2a^2 G(P)^{-1}$ defines an even integral lattice L_r. The group Γ_r acts on this lattice as a reflection group in the sides of the triangle. The lattice $L_r(-2)$ is realized as the Picard lattice of a K3 surface X. Since $L_r(-2)$ does represent -2, the surface X does not contain smooth rational curves. This implies that a subgroup of finite index of Γ_r acts on X by automorphisms. So, we may apply Theorems 8.1 with the Hausdorff dimension computed by McMullen. Note that when $r = a = 1$, the lattice L_r is isomorphic to the Apollonian lattice $\mathcal{A}p_1$ and the group Γ_1 is isomorphic to the Apollonian group Ap_1.

Let us give another example of a realizable group $\Gamma(a, b, c)$. It is taken from [3]. Let X be a K3 surface defined over an algebraically closed field

of characteristic $\neq 2$ embedded in $\mathbb{P}^2 \times \mathbb{P}^2$ as a complete intersection of hypersurfaces of multi-degree $(1,1)$ and $(2,2)$. Let $p_1, p_2 : X \to \mathbb{P}^2$ be the two projections. They are morphisms of degree 2 branched along a plane curve B_i of degree 6. We assume that B_1 is nonsingular and B_2 has a unique double point q_0 so that the fiber $p_2^{-1}(q_0)$ is a smooth rational curve R that is mapped isomorphically under p_1 to a line. We assume that X is general with these properties. More precisely, we assume that $\mathrm{Pic}(X)$ has a basis (h_1, h_2, r), where $h_i = p_i^*(\text{line})$ and r is the class of R. The intersection matrix of this basis is equal to

$$\begin{pmatrix} 2 & 4 & 1 \\ 4 & 2 & 0 \\ 1 & 0 & -2 \end{pmatrix}.$$

Let $s = p_1^*((p_1)_*(r)) - r = h_1 - r$. It is a class of smooth rational curve S on X. The pre-image of the pencil of lines through q_0 is an elliptic pencil $|F|$ on X with $[F] = h_2 - r$. The curve S is a section of the elliptic fibration defined by the linear system $|F|$ and the curve R is its 2-section that intersects S with multiplicity 3. Consider the following three automorphisms of X. The first two Φ_1 and Φ_2 are defined by the birational deck transformations of the covers p_1 and p_2. The third one Φ_3 is defined by the negation automorphism of the elliptic pencil with the group law defined by the choice of S as the zero section.

It is easy to compute the matrix of each Φ_i in the basis $(f, s, r) = ([F], [S], [R])$ with the Gram matrix

$$\begin{pmatrix} 0 & 1 & 2 \\ 1 & -2 & 3 \\ 2 & 3 & -2 \end{pmatrix}.$$

We have $\Phi_1^*(s) = r, \Phi_1^*(r) = s$ and $f' = \Phi_1^*(f) = af + bs + cr$. Since Φ_i^2 is the identity and $(f, f) = 0$, we get $a = -1$ and $b = c$. Since $(f', s) = (f, r) = 2$, we easily get $b = c = 3$. The matrix of Φ_1, and similarly obtained matrices of Φ_2 and Φ_3 are as follows.

$$A_1 = \begin{pmatrix} -1 & 0 & 0 \\ 3 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 4 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 0 & 14 \\ 0 & 1 & 4 \\ 0 & 0 & -1 \end{pmatrix}.$$

The transformations $\Phi_1' = \Phi_1 \circ \Phi_2 \circ \Phi_1, \Phi_2, \Phi_3$ are the reflections with respect to the vector α_i, where

$$\alpha_1 = -4f + 13s + 10r, \ \alpha_2 = 4f - 2s + r, \ \alpha_3 = 7f + 2s - r.$$

The Gram matrix of the vectors $\alpha_1, \alpha_2, \alpha_3$ is equal to

$$G = \begin{pmatrix} -22 & 143 & 220 \\ 143 & -22 & 22 \\ 220 & 22 & -22 \end{pmatrix} = -22 \begin{pmatrix} 1 & -\frac{13}{2} & -10 \\ -\frac{13}{2} & 1 & -1 \\ -10 & -1 & 1 \end{pmatrix}.$$

So, the group generated by Φ_1', Φ_2, Φ_3 coincides with the triangle group $\Gamma(\frac{13}{2}, 10, 1)$. The fundamental triangle P has one ideal vertex. The reflection group Γ_P is a subgroup of infinite index of the group Γ of automorphisms of X generated by Φ_1, Φ_2, Φ_3. Baragar proves that Γ is isomorphic to $\mathrm{Aut}(X)$ (for sufficiently general X). He finds the following bounds for δ_Γ

$$.6515 < \delta_\Gamma < .6538.$$

This implies that

$$\delta_{\Gamma_P} < .6538.$$

Example 8.4. This is again due to Baragar [4]. We consider a non-singular hypersurface X in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of type $(2, 2, 2)$. It is a K3 surface whose Picard lattice contains the Apollonian lattice $\mathcal{A}p(1)$. If X is general, then the Picard lattice coincides with this lattice. We assume that one of the projections $p_{ij} : X \to \mathbb{P}^1 \times \mathbb{P}^1$, say p_{12}, contains the whole $\mathbb{P}^1$ as its fiber over some point $q_0 \in \mathbb{P}^1 \times \mathbb{P}^1$. All the projections are degree 2 maps. Let $F_i, i = 1, 2, 3$, be the general fibers of the projections $p_i : X \to \mathbb{P}^1$. Each F_i is an elliptic curve whose image under the map p_{jk} is a divisor of type $(2, 2)$. Let f_1, f_2, f_3, r be the classes of the curves F_1, F_2, F_3, R. We assume that X is general with these properties so that $\mathrm{Pic}(X)$ is generated by these classes. The Gram matrix of this basis is equal to

$$\begin{pmatrix} 0 & 2 & 2 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 2 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}.$$

It is easy to see that

$$\mathrm{Pic}(X) \cong \mathsf{U} \oplus \left(\begin{smallmatrix} -4 & 2 \\ -2 & -8 \end{smallmatrix}\right).$$

According to Vinberg's classification of 2-reflective hyperbolic lattices of rank 4 [1], the Picard lattice is not 2-reflective. Hence the image of the group $\mathrm{Aut}(X)$ in $\mathrm{O}(\mathrm{Pic}(X))$ is of infinite index.

Let Φ_{ij} be the automorphisms of X defined by the deck transformations of the projections p_{ij}. Let Φ_4' be defined as the transformation Φ_3 in the previous example with respect to the elliptic pencil $|F_3|$ with section R. The transformation Φ_{12}^* leaves the vectors f_1, f_2, r invariant,

and transforms f_3 to $2f_1+2f_2-f_3-r$. Thus Φ_{12}^* is the reflection with respect to the vector $\alpha_1=-2f_1-2f_2+2f_3+r$.

The transformation Φ_{13}^* leaves f_1, f_3 invariant and transforms r in $r'=f_1-r$. It also transforms f_2 to some vector $f_2'=af_1+bf_2+cf_3+dr$. Computing $(f_2', f_1)=(f_2, f_1), (f_2', f_3)=(f_2, f_3), (f_2', r)=(f_2, f_1-r)$, we find that $f_2'=2f_1-f_2+2f_3$. Similarly, we find that $\Phi_{23}^*(f_2)=f_2, \Phi_{23}^*(f_3)=f_3, \Phi_{23}^*(r)=f_2-r$ and $\Phi_{23}^*(f_2)=-f_1+2f_2+2f_3$.

It follows from the definition of a group law on an elliptic curve that

$$\Phi_4'^*(f_3)=f_3,\ \Phi_4'^*(r)=r,\ \Phi_4'^*(f_i)=-f_i+8f_3+4r,\ i=1,2$$

Consider the transformations

$$\Phi_1=\Phi_{12},\ \Phi_2=\Phi_{13}\circ\Phi_{12}\circ\Phi_{13},\ \Phi_3=\Phi_{23}\circ\Phi_{12}\circ\Phi_{23},\ \Phi_4=\Phi_4'\circ\Phi_{12}\circ\Phi_4'.$$

These transformations act on $\mathrm{Pic}(X)$ as the reflections with respect to the vectors

$$\begin{aligned}
\alpha_1 &= -2f_1-2f_2+2f_3+r,\\
\alpha_2 &= \Phi_{13}^*(\alpha_1)=-5f_1+2f_2-2f_3-r,\\
\alpha_3 &= \Phi_{23}^*(\alpha_1)=2f_1-5f_2-2f_3-r,\\
\alpha_4 &= \Phi_4(\alpha_1)=2f_1+2f_2-30f_3-15r.
\end{aligned}$$

The Gram matrix of these four vectors is equal to

$$\begin{pmatrix} -14 & 14 & 14 & 210\\ 14 & -14 & 84 & 182\\ 14 & 84 & -14 & 182\\ 210 & 182 & 182 & -14\end{pmatrix} = -14\begin{pmatrix} 1 & -1 & -1 & -15\\ -1 & 1 & -6 & -13\\ -1 & -6 & 1 & -13\\ -15 & -13 & -13 & 1\end{pmatrix}$$

Let P be the Coxeter polytope defined by this matrix. The Coxeter group Γ_P is generated by the reflections $\Phi_i^*, i=1,2,3,4$.

Baragar proves that the automorphisms Φ_{ij} and Φ_4' generate a subgroup Γ of $\mathrm{Aut}(X)$ of finite index. His computer experiments suggest that

$$1.286<\delta_\Gamma<1.306.$$

Our reflection group Γ_P generated by $\Phi_1,\dots,\Phi_4$ is of infinite index in Γ. So, we obtain

$$\delta_{\Gamma_P}<1.306.$$

Example 8.5. We consider a general Coble rational surface [9]. The orthogonal complement of the canonical class in $\mathrm{Num}(X)$ is isomorphic to $\mathsf{U}\oplus\mathsf{E}_8(-1)$. One can prove that the automorphism group of X is isomorphic to the automorphism group of a general Enriques surface [9] (true in any characteristic). We do the same as in Example 8.2 to realize Γ as an automorphism group of X. Since X is rational, it

gives a realization of Γ as a group of Cremona transformations of $\mathbb{P}^2$. Taking the class of a line, we obtain the asymptotic of the growth of the function $\deg \Phi$, where $\deg \Phi$ is the algebraic degree of a Cremona transformation Φ from Γ.

Remark 8.6. Taking into account Remark 6.1, we see that, for any even hyperbolic lattice of rank > 4 that contains primitively the lattice $M = \mathsf{U} \oplus \mathsf{A}_1(-1) \oplus \mathsf{A}_1(-1)$, the orthogonal group $\mathrm{O}(L)$ contains finitely generated subgroups which are not geometrically finite. Embedding M primitively in the lattice $\mathsf{U} \oplus \mathsf{E}_8(-1)$, we obtain that a general Enriques surface contains finitely generated groups of automorphisms which are not geometrically finite.

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