PETERSEN GRAPH AND ICOSAHEDRON

IGOR V. DOLGACHEV

ABSTRACT. We discuss some of the various beautiful relations between two omnipresent objects in mathematics: the Petersen graph and the Icosahedron. Both have high symmetry and we show how these symmetry groups arise as symmetry groups of richer geometric objects: algebraic surfaces.

Petersen graph. The *Petersen graph* is a undirected regular 3-valent graph with 10 vertices and 10 edges

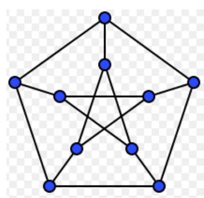

FIGURE 1. Petersen graph

According to a citation from Donald Knuth borrowed from Wikipedia the Petersen graph is "a remarkable configuration that serves as a counterexample to many optimistic predictions about what might be true for graphs in general." The author first encountered this graph on the cover of the Russian translation of Frank Harary's book [3].

FIGURE 2. Julius Petersen

The graph is named after a Danish mathematician *Julius Petersen* (1839-1910), one of the founders of the graph theory. In particular, he is famous with his fundamental work on regular graphs.

As often happens, it had appeared earlier in the work of an English mathematician *Sir Alfred Kempe* (1849-1922) well-known for his work in the invariant theory and the four-color theorem.

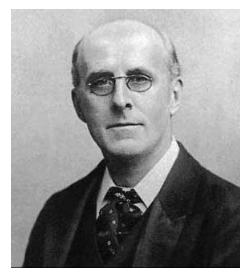

FIGURE 3. Sir Alfred Kempe

Kempe realized the Petersen graph as the graph whose vertices represent lines in a *Desargues configuration* of 10 lines and 10 points in projective plane with two vertices connected by an edge if two lines do not meet at one of the ten points of the configuration.

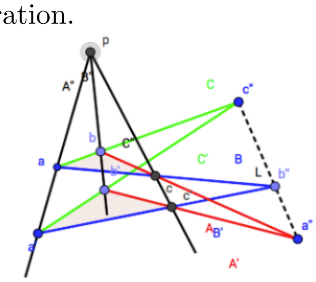

FIGURE 4. Desargues configuration

Recall from projective geometry that the Desatgues configuration is based on the *Desargues's Theorem*: the sides of two perspective triangles in a projective plane intersect at three collinear points. A French mathematician *Girard Desargues* (1591-1661) is considered to be one of the founders of projective geometry.

The *Desargues graph* is a graph with 20 vertices corresponding to lines and points in the Desargues configuration with edges connecting two points if they are realized as a point lying on a line. The graph admits an involutive fixed-point-free automorphism such that the orbit graph is isomorphic to the Petersen graph.

FIGURE 5. Girard Desargues

Quintic del Pezzo surface. Another incarnation of the Petersen graph is as follows. Take four points p_1, p_2, p_3, p_4 in projective plane and join them by pairs to obtain six lines ℓ_{ij}. Now each line ℓ_j intersects three lines $\ell_{ik}, \ell_{j,l}$ and ℓ_{kl}, where $\{i, j, k, l\} = \{1, 2, 3, 4\}$. We are almost there, but need four more lines with the same incidence property.

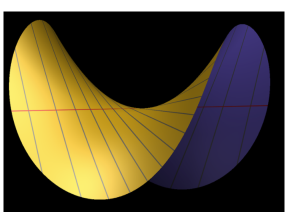

FIGURE 6. Blowing-up

To create them, for each point p_i, consider all directions at this point, i.e. all slopes of lines passing through this point. If we take any line ℓ in the plane that does not contain p_i, the directions correspond to intersection points of lines through p_i with ℓ. So, we obtain that all directions are naturally parameterized by a projective line E_i. The process that replaces a point with the set of directions at this point is called the *blowing-up* of the point. It is one of the surgical tools in algebraic geometry, symplectic geometry, differential topology and the theory of differential equations. The result of the blowing-up is a projective algebraic surface: a two-dimensional projective algebraic variety. Over an open subset (in Zariski topology where closed subsets are the sets of common zeros of polynomials) isomorphic to an affine plane the blown-up surface looks as in the following picture:

After we perform four blowings-up at the points p_1, p_2, p_3, p_4, we will arrive at an algebraic surface $\mathcal{D}_5$ which contains 10 subsets each bijective to the projective line with the intersection graph isomorphic to the Petersen graph. In fact, by choosing an appropriate projective embedding of the surface in 9-dimensional projective space $\mathbb{P}^9$ one can realize them as lines in the projective space. The equations of the surface are very nice: we realize $\mathbb{P}^9$ as the projective space associated with skew-symmetric 5×5-matrices with entries in a chosen ground field (e.g. of real numbers $\mathbb{R}$, or of complex numbers $\mathbb{C}$, or any infinite field you fancy). Then take five general linearly independent equations and add five quadratic equations defined by pfaffians of five principal 4×4-submatrices of our skew-symmetric matrix. The resulting surface is a *quintic del Pezzo surface*, named after an Italian mathematician *Pasquale del Pezzo*, Duke of Caianello (1859-1936). We refer to the rich theory of del Pezzo surfaces to [1].

FIGURE 7. Pascuale del Pezzo

Let us label 10 lines on a quintic del Pezzo surface by 2-element subsets of $[5] = \{1, 2, 3, 4, 5\}$ as follows. First any line ℓ_{ij} acquires the label $\{ab\}$, where $\{i, j, a, b, 5\} = [5]$ and any 'blown-up line' E_i acquires the label $\{i5\}$. In this labelling two vertices are joined by an edge if and only if their labels are disjoint subsets. Now we see that the symmetric group $\mathfrak{S}_5$ acts naturally by automorphisms of the graph via its natural action on the set [5]. Obviously, the action is transitive on vertices and edges.

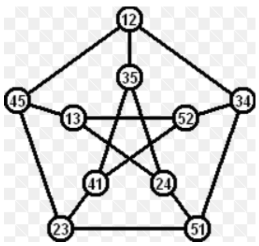

FIGURE 8. Labelled Petersen graph

Using this one can easily show that $\mathfrak{S}_5$ is the full automorphism group of the graph. A natural question now is whether we can realize this symmetry group as a group of automorphisms of an algebraic variety, in our case, the quintic del Pezzo surface. The answer is yes, and it can be done in the following way. First, we choose projective coordinates $(x : y : z)$ in $\mathbb{P}^2$ in such a way that the points p_1, p_2, p_3, p_4 have coordinates $(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1), (1 : 1 : 1)$. Then the symmetrical group $\mathfrak{S}_4$ acts naturally by projective transformations that leave the set $\{p_1, p_2, p_3, p_4\}$ invariant. Namely, we let its subgroup $\mathfrak{S}_3$ act by permutations of coordinates and let one more generating elelent, the transposition (34) act by the transformation $(x : y : z) \mapsto (x - z : y - z : -z)$. Now it suffices to define the action of the transposition (45) which together with $\mathfrak{S}_4$ generates the whole group $\mathfrak{S}_5$.

To do this we use a transformation $\tau : (x : y : z) \mapsto (yz : xz : yz)$. Although the formula does make sense because it is not defined at the points p_1, p_2, p_3 since there is no a point in $\mathbb{P}^2$ with coordinates $(0 : 0 : 0)$. This is an example of a *Cremona transformation*, the foundations of the theory of such transformations was laid by an Italian mathematician *Luigi Cremona* (1830-1903). We are familiar with geometric transformations which are now defined everywhere since our first high-school course in geometry. It is the inversion transformation.

FIGURE 9. Luigi Cremona

If identify the plane with $\mathbb{C}$, then the transformation is given by the formula $z \mapsto r/\bar{z}$.

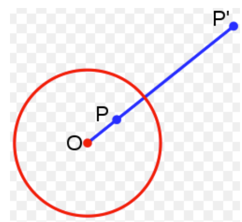

FIGURE 10. Inversion transformation

If we restrict the transformation τ to the open subset of the line at infinity $z = 0$, then it will be given by inversion of the coordinates $(x, y) \mapsto (1/x, 1/y)$. Although our Cremona transformation is not defined at the points p_1, p_2, p_3, it acts naturally on directions at each point p_1, p_2, p_3 and hence can be lifted to an action on the blow-up surface $\mathcal{D}_5$. The whole group $\mathfrak{S}_5$ acts now on $\mathcal{D}_5$, and one can show that there are no other automorphisms of the del Pezzo surface $\mathcal{D}_5$.

Moduli interpretation. The group of automorphism of a quintic del Pezzo surface has an obvious manifestation if we identify the complement of 10 lines on $\mathcal{D}_5$ with the orbit space of ordered 5-element subsets of points in the projective line $\mathbb{P}^1$ with respect to the group of projective automorphisms. To do so, given a point p outside the lines ℓ_{ij}, we pass a unique conic through the points p_1, p_1, p_3, p_4, p_5. Using a rational parameterization of the conic by $\mathbb{P}^1$, we find an ordered set of 5 points. The orbit of this set is our point in the orbit space. Adding the 10 lines, we obtain an isomorphism between $\mathcal{D}_5$ and a compactification $\overline{\mathcal{M}}_{0,5}$ of the orbit space that can be viewed as the moduli space of 5-*pointed stable rational curves*. The construction can be extended to any number of points, using a higher-dimensional generalization of the quintic del Pezzo surface $\mathcal{D}_5$. It is constructed as the blow-up of $n+3$ points in $\mathbb{P}^n$ followed by further blowings-up of various linear subspaces spanned by subsets of points of cardinality less than n. This blowing-up realizes the moduli space $\overline{\mathcal{M}}_{0,n}$ of n-pointed stable rational curves. It is a subject of intensive study in modern research in algebraic geometry, and via the *Gromov-Witten theory* in physics. Note that another compactification of the orbit space of n-ordered points in $\mathbb{P}^1$ uses the geometric invariant theory. The work of Kempe was an important contribution to the invariant theory of ordered point sets in $\mathbb{P}^1$ (see [4]).

Sylvester pentahedron. There is another realization of the Petersen graph where the vertices correspond to lines in $\mathbb{P}^3$ lying on a quartic surface. To do this we first consider $\mathbb{P}^3$ as a subspace of $\mathbb{P}^4$ with projective coordinates $(x_1 : x_2 : x_3 : x_4)$ given by a linear equation $L = x_1 + \cdots + x_5 = 0$. Then we define the lines ℓ_{ij} by additional equations $x_i = x_j = 0$ and define the points P_{ij} by additional equations $x_k = x_l = x_m = 0$, where i, j, k, l, m are distinct. One immediately checks that each line ℓ_{ij} contains three points P_{ab}, so this realizes the Petersen graph. The group $\mathfrak{S}_5$ acts on this set of lines by permuting the coordinates, and in this way gives a geometric realization of the symmetry of the graph. We can do more by realizing the lines as lines lying on a certain surface given by equations $H = L = 0$, where H is a homogeneous polynomial of degree 4. To do so, we consider a cubic surface S given by equation $F = L = 0$, where F is a homogeneous polynomial of degree 3.

According to a theorem of a British mathematician *James Sylvester* (1814-1897), one of the principal contributor to the development of the theory of invariants in the 19the century, a general cubic surface can be given in a unique way by equations as above, where F has the form $F = \sum a_i x_i^3$.

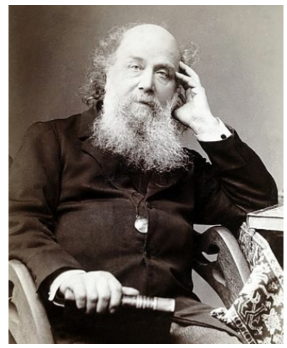

FIGURE 11. James Sylvester

If replace x_5 with $-(x_1 + \ldots + x_4)$ and compute the Hessian $\mathrm{He}(F')$ of the corresponding polynomial F' of degree 4 in $x_1, \ldots, x_4$, i.e. the determinant of the matrix of second partial derivatives of F', we obtain an equation of a quartic surface given by equations $H = L = 0$, where $H = x_1 \cdots x_5(\sum_{i=1}^5 \frac{1}{a_i x_i})$.

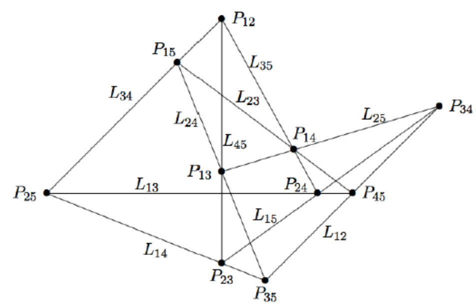

FIGURE 12. Sylvester pentahedron

We call this surface the Hessian of F and denote it by $H(S)$. Now we see that all lines ℓ_{ij} lie on $H(S)$ and the points P_{ij} are its singular points. The union of five hyperplanes $x_i = L = 0$ is called the *Sylvester pentahedron* of the cubic surface S.

The reader will immediately see that the projection of the configuration to a plane from a point outside of the pentahedron becomes a Desargues configuration with perspective triangles whose vertices are the projections of the points P_{13}, P_{14}, P_{15} and P_{23}, P_{24}, P_{25}.

FIGURE 13. Hessian quartic

The Hessian surface $H(S)$ is a special case of a quartic surface whose equation is given by the determinant of a symmetric matrix whose entries are linear forms in projective coordinates. They are named *Cayley quartic symmetroids* after a British mathematician *Arthur Cayley* (1821-1895).

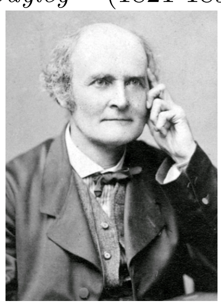

FIGURE 14. Arthur Cayley

Enriques surface. Although the Hessian surface $H(S)$ is singular at the points P_{ij}, we can transform it to a nonsingular surface $\tilde{S}$ by blowing up the singular points, i.e. by adding all tangent directions at these points of germs of nonsingular curves lying on the surface. The resulting surface is an example of a *K3 surface* (see for their theory [5]).

FIGURE 15. Federigo Enriques

The Cremona involution defined by $(x_1 : \ldots : x_5) \mapsto (\frac{1}{a_i x_1} : \ldots : \frac{1}{a_5 x_5})$ lifts to an automorphism of $\tilde{S}$ of order 2. It exchanges the directional curves E_{ij} blown-up from points P_{ij} with the line L_{ij}. The intersection graph of the 20 curves E_{ij}, L_{ij} obtained in this way is the Desargues graph. The quotient of the K3-surface $\tilde{S}$ by the Cremona involution that acts free of fixed points is an example of an *Enriques surface* named after an Italian mathematician *Federigo Enriques* (1871-1946).

The blown-up surface admits an embedding in $\mathbb{P}^5$ as a surface of degree 10. In this embedding the orbits $\{E_{ij}, L_{ij}\}$ are represented by 10 lines whose intersection graph is the Petersen graph.

Clebsch diagonal cubic surface. Let us consider a special cubic surface where $F = \sum x_i^3$. It is called *Clebsch diagonal cubic surface* and it is named after a German mathematician *Alfred Clebsch* (1833-1872).

FIGURE 16. Alfred Clebsch

It has an obvious symmetry group isomorphic to $\mathfrak{S}_5$ and it acts on its Hessian surface by projective automorphisms. It is the only cubic surface with $\mathfrak{S}_5$ symmetry. This realizes the symmetry of the Petersen graph by projective automorphisms of a cubic surface.

The Enriques surface obtained from the Hessian of the Clebsch diagonal surface is one of very rare Enriques surfaces whose group of automorphisms is finite [8]. In our case the group is isomorphic to $\mathfrak{S}_5$. So, we obtain yet another incarnation of the Petersen graph and its group of symmetry as the intersection graph of 10 lines lying on an Enriques surface with the full automorphism group $\mathfrak{S}_5$.

It was discovered in 1849 by Arthur Cayley and an Irish mathematician *George Salmon* (1819-1904). that a nonsingular cubic surface contains exactly 27 lines. Each face $x_i = 0$ of the Sylvester pentahedron intersected by the other four faces along three *diagonals* of, a complete quadrilateral with vertices P_{ij} lying in the plane.

These diagonals define 15 lines on the surface and this explains its name. The remaining 12 lines form a *double-six*: the union of two sets of six skew lines, each intersecting 5 lines from the other set.

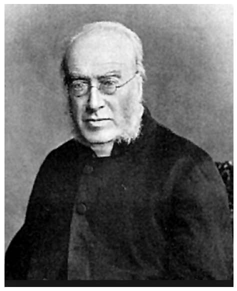

FIGURE 17. George Salmon

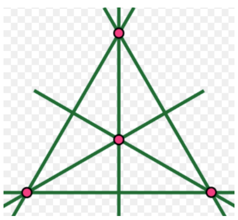

FIGURE 18. Complete quadrilateral

It is known that a nonsingular cubic surface can be obtained as a projectively embedded blowing-up of 6 points in a projective plane no three of which are collinear and not all of them lie on a conic. One realizes the 27 lines as 15 lines joining a pair of points, 6 lines coming from directions at the six points, and 6 conics passing through all points except one.

The Clebsch diagonal cubic surfaces is the only cubic surface where all 27 lines are defined over reals and then can be seen on a picture of the surface. The six points must be chosen in a special way and how to choose them is related to the icosahedron.

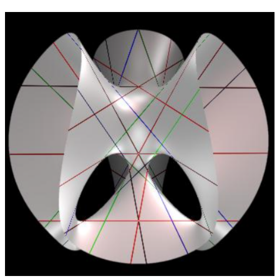

FIGURE 19. Clebsch surface

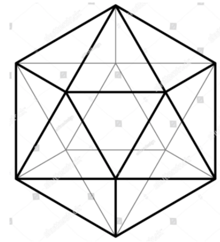

FIGURE 20. Icosahedron

Icosahedron. Finally we arrive at the icosahedron. As one of Platonic solids it is omnipresent in mathematics since antiquity. It is a regular convex polyhedron with 12 vertices, 30 edges and 20 faces.

FIGURE 21. Felix Klein

In the introduction of his *Lectures on the Icosahedron* [7], a German mathematician *Felix Klein* (1849-1925) writes: "A special difficulty, which presented itself in the execution of my plan, lay in the great variety of mathematical methods entering in the theory of the Icosahedron." Since Klein's time the variety of different methods and connections to different fields of mathematics has greatly increased.

We can circumscribe a sphere in $\mathbb{R}^3$ around an icosahedron in such a way that the planes passing through opposite edges cut the sphere in large circles. This gives a regular tiling of the sphere in 20 spherical triangles.

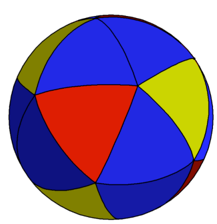

FIGURE 22. Spherical triangles

Now we pass to the real projective plane by identifying the antipodal points on the sphere. The images of the vertices become a 6-point subset of $\mathbb{P}^2(\mathbb{R})$. If we blow-them up, we get an algebraic surface which can be embedded in $\mathbb{P}^3$ with the image equal to the Clebsch diagonal cubic surface. The symmetry group of an icosahedron is isomorphic to the alternating group $\mathsf{A}_5 \subset \mathfrak{S}_5$.

The images of large circles corresponding to the edges are 15 lines in the plane. They connect 15 pairs of the six points. They have a peculiar property that the lines joining three disjoint pairs intersect at one point. It is an *Eckardt point* on a cubic surface, named after a German mathematician F.E. Eckardt (unfortunately, no bio has been found). The number ten of Eckardt point on a cubic surface is almost a record, the only surface that beats it is the Fermat cubic surface whose equation can be given by the sum of four powers of coordinates in $\mathbb{P}^3$. It has 18 Eckardt points.

The images of 10 antipodal pairs on the sphere corresponding to the centers of the 20 faces of an icosahedron are mapped to an $\mathfrak{A}_5$-orbit of 10 points in the plane. Each point lies on three of the 15 lines obtained from the edges. In the dual projective we obtain 10 lines and 15 intersection points which realize the Petersen graph. In fact, one does not need to go to the dual plane. We can realize this configuration in the plane itself. To do this, we use the fact that the action of the icosahedron group $\mathfrak{A}_5$ in the projective plane is defined by its real 3-dimensional irreducible linear representation, and this implies that there is an invariant conic in the plane. It defines the self-duality of the plane. The configuration of 10 lines can be obtained as the projection to the plane of the ten lines lying the del Pezzo surface in $\mathbb{P}^5$ under an $\mathfrak{A}_5$-equivariant projection $\mathbb{P}^5 \to \mathbb{P}^2$ [2].

References

[1] I. Dolgachev, *Classical algebraic geometry. A modern view*, Cambridge Univ. Press, 2012.

[2] I. Dolgachev, B. Farb, E. Looijenga, *Geometry of the Wiman pencil, I: algebro-geometric aspects.*, to appear in European J. Math..

[3] F. Harary, *Graph theory.* Addison-Wesley Publishing Co., Reading, Mass.-Menlo Park, Calif.-London. 1969.

[4] B. Howard, J. Millson, A. Snowden, R. Vakil,*The ideal of relations for the ring of invariants of n points on the line.* J. Eur. Math. Soc. (JEMS) **14** (2012), no. 1, 1-60.

[5] D. Huybrechts, *Lectures on K3 surfaces.* Cambridge Studies in Advanced Mathematics, 158. Cambridge University Press, Cambridge, 2016.

[6] A. Kempe, *On Regular Difference Terms.* Proc. Lond. Math. Soc. **25** (1893/94), 343-359.

[7] F. Klein, *Vorlesungen Über das Ikosaeder und die Auflsung der Gleichungen vom fünften Grade.* Reprint of the 1884 original. Edited, with an introduction and commentary by Peter Slodowy. Birkhäuser Verlag, Basel; B. G. Teubner, Stuttgart, 1993

[8] S. Kondō, *Enriques surfaces with finite automorphism group*, Japan J. Math. **12** (1986), 192–282.

Department of Mathematics, University of Michigan, 525 E. University Av., Ann Arbor, Mi, 49109, USA

E-mail address: idolga@umich.edu