ON ALGEBRAIC PROPERTIES OF ALGEBRAS OF AUTOMORPHIC FORMS

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§0. Introduction: Let V be a connected complex manifold, $p: L \to V$ a complex line bundle on V, Γ a group of analytical automorphisms of V which acts totally discontinuously with the compact quotient space $X = \Gamma \backslash V$. Suppose that Γ also acts on L compatible with the action on V(i.e., $p(g(v))=g(p(v))$ for any $g \in \Gamma$, $v \in L$). We fix this action and call L an automorphy factor with respect to Γ. For any integer m the m-th tensor power L^m of L is also provided with the structure of an automorphy factor, and the group Γ acts in a natural way on the space of its global sections $H^o(V,L^m)$. Let $A_L(\Gamma)_m = H^o(V,L^m)^\Gamma$ be the subspace of the invariant sections, its elements are called (integral) automorphic forms of weight m with respect to the group Γ and the automorphy factor L. The direct sum $A_L(\Gamma) = \bigoplus_{m \in Z} A_L(\Gamma)_m$ has a natural structure of a graded commutative algebra over the field of complex numbers $\mathbb{C}$, it is called the algebra of automorphic forms with respect to the group Γ and the automorphy factor L.

The most known particular cases of this definition are the algebra of G-invariant polynomials $\mathbb{C}[T_1,\ldots, T_{n+1}]^G$, G a finite subgroup of $GL(n+1,\mathbb{C})$ (corresponds to the case $V = P^n(\mathbb{C})$ the projective space, $\Gamma < PGL(n,\mathbb{C})$ the quotient of G by the subgroup of scalar matrices), and the Poincaré algebra $A(\Gamma)$ of automorphic forms ($V\subset C^n$ a homogeneous bounded domain, $L=\overset{n}{\Lambda}T(V)$ the determinant of the tangent bundle).

There are many results on algebraic properties of the algebras of invariant polynomials (see [11]), and as far as I know their generalizations to the general case were never considered. Here, I announce some results in this direction.

§1. Finiteness. We say that the triple (V,Γ,L) is admissible if the following conditions are satisfied:

(i) Γ contains a subgroup of finite index Γ' which acts freely on V;

(ii) the quotient $\Gamma'\backslash L$ defines the negative or positive (in sense of Kodaira [4]) line bundle on $\Gamma'\backslash V$.

Theorem 1: For any admissible triple (V,Γ,L) the algebra of automorphic forms $A_L(\Gamma)$ is finitely generated over $\mathbb{C}$.

Sketch of the proof: The condition (i) easily implies that one can choose an invariant subgroup of finite index Γ' which acts freely on V. By Kodaira the condition (ii) implies that some power L'^k embeds the quotient $X' = \Gamma'\backslash V$ into the projective space. This immediately shows that the algebra $\bigoplus_{m\in Z} H^o(X',L'^{mk})$ is finitely generated (being equal to the projective coordinate ring of the embedded algebraic variety X'). Using an easy algebraic argument we infer that the algebra $A' = \bigoplus_{m\in Z} H^o(X',L'^m)$ is also finitely generated. Then we identify the algebra $A_L(\Gamma)$ with the algebra of invariants A'^G, $G = \Gamma/\Gamma'$, and use that the finiteness is preserved under passing to the invariant subalgebra for a finite group of automorphisms.

Remarks: 1. Let $n = \dim A_L(\Gamma)$ be the Krull dimension of the algebra $A_L(\Gamma)$. The arguments of the proof of Theorem 1 show that $n = \dim V + 1$ if (V,Γ,L) is admissible.

2. If L' is a positive (resp. negative) line bundle on a compact complex manifold, X', then $H^0(X',L'^m) = 0$ for $m < 0$ (resp. $m>0$). Also, $H^0(X',L'^0) = H^0(X',1_{X'}) = \mathbb{C}$. This shows that $A_L(\Gamma)_m = 0$ for either positive m or negative m and $A_L(\Gamma)_0 = \mathbb{C}$. After obvious regrading of $A_L(\Gamma)$ we may always assume that $A_L(\Gamma)_m = 0$, $m < 0$ and L defines a positive line bundle L'.

3. The previous remark obviously implies that $A_L(\Gamma)$ has no zero divisors if (V,Γ,L) is admissible (the argument is analogous to the case of the polynomial ring.).

§2. Normality. A commutative ring A without zero divisors is said to be normal if any x from its field of fractions satisfying an algebraic equation $x^n + a_1x^{n-1} +\dots+ a_n = 0$, $a_i \in A$, belongs to A.

Theorem 2: Assume that (V,Γ,L) is admissible. Then $A_L(\Gamma)$ is normal.

The argument of the proof of Theorem 1 shows that $A_L(\Gamma) = A'^G$, where $A' = \bigoplus_{m\in Z} H^0(X',L'^m)$ with L' a positive or negative line bundle on a non-singular algebraic variety X', G a finite group. Using Remark 2 above we may assume that L' is positive. Then, by Grauert's criterion [3], the zero section of L'^{-1} can be blown forms ($V\subset C^n$ a homogeneous bounded domain, $L=\overset{n}{\Lambda}T(V)$ the determinant of the tangent bundle).

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§3. Cohen-Macaulay. Let $A = \bigoplus_{m \in Z} A_m$ be a finitely generated graded algebra over a field k, $n = \dim A$, its Krull dimension. By the Noether normalization theorem we can choose n homogeneous elements $x_1, \ldots, x_n \in A$ such that A is a finite module over the polynomial subalgebra $k[x]$ generated by x_i. We say that A is a Cohen-Macaulay algebra if A is a free module over $k[x]$ (it can be shown that this definition does not depend on the choice of generators x_i.).

Assume now that (V,Γ,L) is an admissible triple; let Γ' be an invariant subgroup of finite index of Γ which acts freely on V, $L' = \Gamma' \backslash L$, $X' = \Gamma' \backslash V$. The quotient group $G = \Gamma/\Gamma'$ acts by functoriality on the cohomology space $H^i(X', L'^m)$, and it is easy to see that the subspace of invariants $A_L^i(\Gamma)$ is independent (up to an isomorphism) of the choice of Γ', it is called the i-th space of automorphy cohomology with respect to Γ and L. Clearly, $A_L^o(\Gamma) = A_L(\Gamma)$.

Theorem 3: Let (V,Γ,L) be an admissible triple. Then the algebra $A_L(\Gamma)$ is Cohen-Macaulay if and only if $A_L^i(\Gamma) = 0$ for $0 < i < \dim V$.

Examples: 1. Let $V = P^n(\mathbb{C})$, Γ a finite group. If we take for Γ' the trivial subgroup then $A_L^i(\Gamma) = \bigoplus_{m \in Z} H^i(P^n(\mathbb{C}), L^m)$. It is well known that any complex line bundle on $P^n(\mathbb{C})$ is isomorphic to some power of the tautological line bundle H on $P^n(\mathbb{C})$. By Serre's theorem, $H^i(P^n(\mathbb{C}),H^k) = 0$, $i \geq 0$, $k \in Z$. Hence, $A_L^i(\Gamma) = 0$, $i > 0$. Thus, we obtain that the algebra of invariant polynomials is always Cohen-Macaulay. This is a well-known result proved, for example, in [5].

2. Let V be a homogeneous bounded domain in $\mathbb{C}^n$, $A(\Gamma) = A_{K_V}(\Gamma)$, $K_V = \overset{n}{\Lambda}\, T(V)$, the corresponding Poincare algebra with respect to some group Γ (see §0). The Kodaira vanishing theorem and Serre's duality (see [4]) easily imply that $A_{K_V}^i(\Gamma) = H^i(X,1_X) \oplus H^{n-i}(X,1_X)$, where $X = \Gamma\backslash V$. Hence, $A(\Gamma)$ is Cohen-Macaulay if and only if $H^i(X,1_X) = 0$, $0 < i < n$. By Matsushima-Shimura [6] it is satisfied in the case $V = H^n$, the product of the upper half-planes.

§4. <u>Gorenstein</u>. Let A be a Cohen-Macaulay graded algebra over a field k, A[x] be the polynomial subalgebra of A such that A is a free finite module over A[x]. A is said to be <u>Gorenstein</u> algebra if $x_1,\ldots, x_n$ generate an irreducible ideal of A. There is a lot of other equivalent characterizations of Gorenstein rings for which we refer to [1], [11].

We say that an automorphy factor L is <u>nonramified in</u> codimension $\leq k$ if the set F of all points $w \in L^* = L-\{\text{zero section}\}$ with nontrivial isotropy subgroup is of codimension $\geq k$ in L^*. Notice that F is clearly a line bundle over its projection onto V. If $v \in V$ is a point with nontrivial isotropy subgroup Γ_v, then L defines the isotropy representation $\rho_v\colon \Gamma_v \to GL(L_v) = \mathbb{C}^*$, L_v the fibre of L over the point v. L is nonramified in codimension $\leq k$ if and only if there exists an open subset $U \subset V$ with

$\operatorname{codim}(V-U,V) \geq k+1$ such that for any $v \in U$ the representation ρ_v is faithful (i.e. $\operatorname{Ker}(\rho_v)=\{1\}$).

Theorem 4: Let (V,Γ,L) be an admissible triple with L nonramified in codimension ≤ 1. Assume that the algebra $A_L(\Gamma)$ is Cohen-Macaulay. Then $A_L(\Gamma)$ is Gorenstein if and only if for some integer k the automorphy factor L^k is isomorphic to $K_V = \overset{n}{\wedge} T(V)$ (as Γ-bundles).

Examples: 1. In the case of the algebras of invariant polynomials $\mathbb{C}[T_1,\ldots, T_{n+1}]^G$ the non-ramifiedness at codimension ≤ 1 means that the group G has no pseudoreflections (i.e. for any $g \in G$ $\operatorname{rk}(g-1)>1$). Then the condition of the theorem is equivalent to the condition $G \subset SL(n+1,\mathbb{C})$. In this form this theorem is due to Watanabe [13].

2. If $\overset{n}{\wedge} T(V)$ is a trivial line bundle with the trivial Γ-action (for example, $V=\mathbb{C}^n$, Γ is a group of translations), then $A_L(\Gamma)$ is Gorenstein as soon as it is Cohen-Macaulay.

3. The Poincare algebras $A(\Gamma)$ are always Gorenstein if they are Cohen-Macaulay.

§5. Complete Intersection. Let A be a graded algebra finitely generated over a field k, $x_1,\ldots, x_N$, its homogeneous generators assumed to be of positive degree $q_1,\ldots, q_N$. Then A is a holomorphic image of the polynomial algebra $k[T_1,\ldots, T_N]$ graded by the condition $\deg(T_i) = q_i$. We say that A is a complete intersection if $x_1,\ldots, x_N$ can be chosen in such a way that the kernel of the map $k[T_1,\ldots, T_N] \to A$, $T_i \to x_i$, is generated by a regular sequence of homogeneous elements $F_1,\dots, F_r$ of $k[T_1,\dots, T_N]$ (regular means that F_i is not a zero divisor in the quotient $k[T_1,\dots, T_N]/(F_1,\dots, F_{i-1})$). We say then that A has weights $q_1,\dots, q_N$ and multidegree $(d_1,\dots, d_r)$, where $d_i = \deg(F_i)$. Any complete intersection algebra is Gorenstein.

I do not know any sufficient criteria for an algebra $A_i(\Gamma)$ to be a complete intersection, and it hard to believe that such exist. However, there is a hope to classify them in some cases. This has been done in the cases of the Poincare algebras $A(\Gamma)$, dim V = 1 ([2], different proofs were given later in [10], [12]), and $\Gamma = \{I\}$, dim V = 2 (M. Reid).

<u>Theorem 5</u>: Let (V,Γ,L) be an admissible triple. Suppose that L is nonramified in codimension ≤ 1, and $A_L(\Gamma)$ is a complete intersection with weights $q_1,\dots, q_N$ and multidegree $(d_1,\dots, d_r)$. Let k be the integer determined by the equality $L^k = K_V$ (see Theorem 4). Then

$$d_1 +\dots+ d_r - q_1 -\dots- q_N = k.$$

<u>Remarks</u>: 1. It is easy to show that $A_L(\Gamma)$ cannot be a complete intersection if dim V $\geq$ 2, L is nonramified in codimension $\leq$ 2, and $\Gamma \neq \{1\}$.

2. It would be very interesting to find a generalization of the Chevalley theorem which gives a necessary and sufficient condition for $A_L(\Gamma)$ to be the polynomial ring (Recall that the Chevalley theorem asserts that the algebra of invariants $k[T_1,\dots, T_n]^G$ is a graded polynomial algebra if and only if G is generated by pseudo-reflections.).

§6. Halphen Algebras. Assume that the automorphy factor $K_V = \bigwedge^n T(V)$ is positive or negative. Let F be the group of all automorphy factors L with respect to Γ such that $L^k = K_V^m$ for some pair of integers k,m. The F-graded vector space

$$\tilde{A}(\Gamma) = \bigoplus_{L \varepsilon F} H^o(V,L)^\Gamma$$

has a natural structure of the F-graded algebra over $\mathbb{C}$. For the reasons below, I call it the Halphen algebra of the group Γ. Clearly, $\tilde{A}(\Gamma)$ contains the Poincare algebra $A(\Gamma) = \bigoplus_{L\varepsilon\{K_V\}} H^o(V,L)^\Gamma$ as its graded subalgebra.

Theorem 6: Assume that the triple (V,Γ,L) is admissible and $H^1(X,1_X) = 0$, where $X = \Gamma\backslash V$. Then the Halphen algebra $\tilde{A}(\Gamma)$ is a finitely generated normal $\mathbb{C}$-algebra.

Example: Suppose that dim V = 1. Then under the conditions of the theorem we have $X = P^1(\mathbb{C})$, the projective line. Let $\pi: V \to X$ be the projection onto the quotient space. Then there is a finite set of points $x_1,\ldots, x_r$ in X such that π is a local isomorphism for any point v ε V with $\pi(v) \notin \{x_1,\ldots, x_r\}$, and π is ramified at any v with $\pi(v) = x_i$ with index of ramification $e_i > 1$. Also, we have that the number $t = 1/e_1 +\ldots+ 1/e_r$ is either larger than r - 2 (in this case, the universal covering of V is the Riemann sphere) or less than r - 2 (in this case, the universal covering of V is the unit disc.). Using the ideas of Poincare [9] it can be proven that $\tilde{A}(\Gamma)$ is a complete intersection algebra. It is generated by some special r functions (called by Poincare the Halphen functions) $f_1,\ldots, f_r$ with the defining relations

$$f_1^{e_1} + f_i^{e_i} + a_i f_r^{e_r} = 0, \quad i = 2, \ldots, r - 1$$

where a_i are some nonzero constants (see [2], [7], [8]). In some sense $\widetilde{A}(\Gamma)$ is the maximal nonramified abelian extension of the Poincare algebra $A(\Gamma)$.

§7. Final Remarks. As it is seen all our results require the assumption on (V,Γ,L) to be admissible. It would be very interesting to obtain some results without this assumption. Also, it would be nice to consider the case of non-compact quotients $\Gamma\backslash V$ but say of finite volume. There are many interesting explicit examples of the analogous algebras $A_L(\Gamma)$ in this case, some of them are complete intersections.

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