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Polar Cremona Transformations

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Let $F(x_0, \dots, x_n)$ be a complex homogeneous polynomial of degree d. Consider the linear system $\mathcal{P}_F$ generated by the partials $\frac{\partial F}{\partial x_i}$; we call it the *polar linear system* associated to F. The problem is to describe those F for which the polar linear system is homaloidal, that is, for which the map $(t_0, \dots, t_n) \to \left(\frac{\partial F}{\partial x_0}(t), \dots, \frac{\partial F}{\partial x_n}(t)\right)$ is a birational map. We shall call F with such property a *homaloidal polynomial*. In this paper we review some known results about homaloidal polynomials and also classify them in the cases when F has no multiple factors and either $n = 3$ or $n = 4$ and F is the product of linear polynomials.

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1. Examples

As was probably first noticed by Ein and Shepherd-Barron [ES], many examples of homaloidal polynomials arise from the theory of prehomogeneous vector spaces. Recall that a complex vector space V is called *prehomogeneous* with respect to a linear rational representation of an algebraic group G in V if there exists a nonconstant polynomial F such that the complement of its set of zeros is homogeneous with respect to G. The polynomial F is necessarily homogeneous and an eigenvector for G with some character $\chi : G \to \mathrm{GL}(1)$, and it generates the algebra of invariants for the group $G_0 = \mathrm{Ker}(\chi)$. The reduced part F_{red} of F (i.e., the product of irreducible factors of F) is determined uniquely up to a scalar multiple. A prehomogeneous space is called *regular* if the determinant of the Hessian matrix of F is not identically zero; this definition does not depend on the choice of F. We shall call F a *relative invariant* of V. Note that there is a complete classification of regular irreducible prehomogeneous spaces with respect to a reductive group G (see [KS]).

Theorem 1 [EKP; ES]. *Let V be a regular prehomogeneous vector space. Then its relative invariant is a homaloidal polynomial.*

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Here are some examples.

EXAMPLES 1–4. 1. Any nondegenerate quadratic form Q is obviously a homaloidal polynomial. The corresponding birational map is a projective automorphism. It is also a relative invariant for the group $O(Q) \times \mathrm{GL}(1)$ in its natural linear representation.

2. A reduced cubic polynomial F on V is a relative invariant for a regular prehomogeneous space with respect to a reductive group G if and only if the pair (V, G) is one of the following (up to a linear transformation).

2.1: $G = \mathrm{GL}(1)^3 \subset \mathrm{GL}(3)$, $V = \mathbb{C}^3$, the action is natural, $F = x_0x_1x_2$.

2.2: $G = \mathrm{GL}(3)$, V is the space of quadratic forms on $\mathbb{C}^3$, the action is via the natural action on $\mathbb{C}^3$, F is the discriminant function.

2.3: $G = \mathrm{GL}(3) \times \mathrm{GL}(3)$, $V = \mathrm{Mat}_3$ is the space of complex 3×3 matrices, the action is by $(g, g') \cdot A = gAg'^{-1}$, the polynomial F is the determinant.

2.4: $G = \mathrm{GL}(6)$, $V = \Lambda^2(\mathbb{C}^6)$, the action is via the natural action on $\mathbb{C}^6$; the polynomial F is the pffafian polynomial.

2.5: $G = E_6 \times \mathrm{GL}(1)$, $V = \mathbb{C}^{27} = \mathrm{Mat}_3 \times \mathrm{Mat}_3 \times \mathrm{Mat}_3$ is its irreducible representation of minimal dimension; the polynomial F is the Cartan cubic $F(A, B, C) = |A| + |B| + |C| + \mathrm{Tr}(ABC)$.

Examples 2.2–2.5 correspond to the four Severi varieties: nonsingular nondegenerate subvarieties S of $\mathbb{P}^r$ of dimension $(2r-4)/3$ whose secant variety $\mathrm{Sec}(S)$ is not equal to the whole space. The zero locus of the cubic F in $\mathbb{P}(V)$ defines the secant variety. The singular locus of $\mathrm{Sec}(S)$ is the Severi variety. According to a theorem from [ES], any homaloidal cubic polynomial F such that the singular locus of $F^{-1}(0)$ in $\mathbb{P}(V)$ is nonsingular coincides with one from Examples 2.2–2.5.

3. Let us identify $\mathbb{P}^{n^2-1}$ with the space $\mathbb{P}(\mathrm{Mat}_n)$. The map $A \to A^{-1}$ is obviously birational and it is given by the polar linear system of the polynomial $A \to \det(A)$. The polynomial is a relative invariant from Example 2.3 (extended to any dimension).

4. The polynomial $F = x_0(x_0x_2 + x_1^2)$ is homaloidal. It is a relative invariant for a prehomogeneous space with respect to a nonreductive group.

2. Multiplicative Legendre Transform

This section is borrowed almost entirely from [EKP]. Let $F \in \mathrm{Pol}_d(V)$ be a homogeneous polynomial of degree d on a complex vector space V of dimension $n+1$. We denote by F' or by dF the derivative map $V \to V^*$, $v \to (dF)_v$. If no confusion arises then we also use this notation for the associated rational map $\mathbb{P}(V) \to \mathbb{P}(V^*)$. If we choose a basis in V and the corresponding dual basis in V^*, we will be able to identify both spaces with $\mathbb{C}^n$ and also the map F' with the polar map defined in the introduction. Suppose F is homaloidal, that is, F' defines a birational map $\mathbb{P}(V) \to \mathbb{P}(V^*)$. Then, obviously, $d \ln F = F'/F$ defines a birational map $V \to V^*$.

LEMMA 1. *Let f be a homogeneous function of degree k on V (defined on an open subset) such that* $\det(\mathrm{Hess}(\ln f))$ *is not identically zero. Then there exists a homogeneous function f_* on V^* of degree k such that, on some open subset of V,*

$$f_*(d\ln f) = 1/f. \tag{2.1}$$

Proof. Recall first the definition of the *Legendre transform*. Let Q be a function on V defined in an open neighborhood of a point v_0 such that $\det \mathrm{Hess}(Q)(v_0) \neq 0$. Let $dQ(v_0) = p_0 \in V^*$. Then the Legendre transform $L(Q)$ of Q is the function $L(Q)$ on V^* defined in an neighborhood of p_0 such that

$$L(Q)(p) = p(v_p) - Q(v_p), \tag{2.2}$$

where v_p is the unique critical point of the function $v \to p(v) - Q(v)$ in a neighborhood of v_0.

Since the critical point v_p satisfies $p = dQ(v_p)$, we obtain from (2.2) an equality of functions on a neighborhood of v_p in V:

$$L(Q)(dQ(v)) = dQ(v)(v) - Q(v).$$

Now let us apply this to $Q = \ln f$. We have

$$L(\ln f)(d\ln f(v)) = d\ln f(v)\cdot v - \ln f(v).$$

Recall that a homogeneous function H of degree k satisfies the Euler formula:

$$kH(v) = dH(v).$$

Applying this to $H = \ln f$, we have

$$e^{L(\ln f)-k}(d\ln f) = 1/f.$$

It remains to define f_* by

$$\ln f_* = L(\ln f) - k. \tag{2.3}$$

It is immediately checked that f_* is homogeneous of degree k. □

The function f_* is called the *multiplicative Legendre transform* of f.

THEOREM 2 [EKP]. *Let $F \in \mathrm{Pol}_d(V)$ be such that* $\det \mathrm{Hess}(\ln F)$ *is not identically zero. Then F is homaloidal if and only if its multiplicative Legendre transform F_* is a rational function. Moreover, in this case*

$$d\ln F_* = (d\ln F)^{-1}. \tag{2.4}$$

Proof. Suppose F is homaloidal. Then $d\ln F$ is a rational map of topological degree 1 in its set of definition. It follows from the definition of the Legendre transform that $L(\ln F)$ is one-valued on its set of definition. Differentiating (2.1), we obtain $(d\ln F_*)\circ(d\log F) = \mathrm{id}$; this checks (2.4). Since $d\ln F_* = dF_*/F$ is a homogeneous rational function, the function F_* must be rational. Conversely, if F_* is rational then differentiating (2.1) yields (2.4) locally. Since $d\ln F_*$ is rational, we have (2.4) globally and hence $d \ln F$ is invertible. This implies that dF defines a birational map, and hence F is homaloidal. □

COROLLARY 1. *Let $F(x_0, \dots, x_n)$ be a homaloidal polynomial of degree $k > 2$, and assume that F_* is a reduced polynomial. Then*

$$k | 2(n+1).$$

Proof. By Theorem 2,

$$dF_* \circ dF = F^{k-1}(x) F_*(x)(x_0, \dots, x_n).$$

This implies that the image of the hypersurface $F = 0$ under the birational map $dF : \mathbb{P}^n \to \mathbb{P}^n$ is contained in the set of base points of the polar linear system of F_*. Since F_* is reduced, the latter is a closed subset of codimension > 1. Thus $F = 0$ is contained in the set of critical points of dF (considered as a map of vector spaces) and hence F divides the Hessian determinant. The assertion follows from this. □

A natural question (posed in [EKP]) is: For which homogenous polynomials F is the multiplicative Legendre transform F_* a polynomial function? A polynomial with this property will be called a *homaloidal EKP-polynomial*. It is easy to see that F_* has the same degree as F and that $(F_*)_* = F$. It is conjectured that any homaloidal EKP-polynomial is a relative invariant of a regular prehomogeneous space (the converse is proved in [EKP]). In this case $F_* = F$, up to a scaling.

A remarkable result of [EKP] is the following theorem.

THEOREM 3. *A homaloidal EKP-polynomial of degree* 3 *coincides with one from Examples 2.*

EXAMPLE 5. Consider the polynomial F from Example 4. We have

$$d \ln F = \left(\frac{2x_0x_2 + x_1^2}{x_0(x_0x_2 + x_1^2)}, \frac{2x_1}{x_0x_2 + x_1^2}, \frac{x_0}{x_0x_2 + x_1^2} \right).$$

Inverting this map, we obtain

$$(d \ln F)^{-1} = \left(\frac{8x_2}{4x_0x_2 + x_1^2}, \frac{4x_1}{4x_0x_2 + x_1^2}, \frac{4x_0x_2 - x_1^2}{(4x_0x_2 + x_1^2)x_2} \right)$$
$$= d \ln \frac{(4x_0x_2 + x_1^2)^2}{x_2}.$$

Thus, the multiplicative Legendre transform of F equals

$$F_* = \frac{(4x_0x_2 + x_1^2)^2}{x_2};$$

it is a homogeneous rational but not polynomial function.

3. Plane Polar Cremona Transformations

Here we shall classify all homaloidal polynomials in three variables with no multiple factors.

Since the set of common zeros of the polars $\partial_i F$ is equal to the set of nonsmooth points of the subscheme $V(F)$, this is equivalent to requiring that the polars $\partial_i F$ have no common factors, that is, the linear system $\mathcal{P}_F$ has no fixed part.

Let $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a rational map defined by homogeneous polynomials (P_0, P_1, P_2) of degree d without common factors. Let $\mathcal{J}(f) \subset k[x_0, x_1, x_2]$ be the ideal generated by the polynomials P_0, P_1, P_2. The corresponding closed subscheme $B_f = V(\mathcal{J}(f))$ of $\mathbb{P}^2$ is the base locus subscheme of the linear system spanned by P_0, P_1, P_2. The quotient sheaf $\mathcal{O}_{\mathbb{P}^2}/\mathcal{J}(f)$ is artinian, and we denote by $\tilde{\mu}_x(f)$ the length of its stalk at a point $x \in V(\mathcal{J}(f))$.

LEMMA 2.

$$\sum_{x \in \mathbb{P}^2} \tilde{\mu}_x(f) = d^2 - d_t,$$

where d_t is the degree of the map f.

Proof. See [Fu, 4.4]. □

Recall that, for any singular point x of $V(F)$, we have the conductor invariant δ_x defined as the length of the quotient module $\bar{\mathcal{O}}_{C,x}/\mathcal{O}_{C,x}$, where $\bar{\mathcal{O}}_{C,x}$ is the normalization of the local ring $\mathcal{O}_{C,x}$. Let r_x denote the number of local branches of C at x. We have the following lemma.

LEMMA 3. *Let $\tilde{\mu}_x = \tilde{\mu}_x(f)$, where f is the map defined by the polar linear system $\mathcal{P}_F$. For any $x \in C$,*

$$\tilde{\mu}_x \leq 2\delta_x - r_x + 1. \tag{3.1}$$

Proof. Without loss of generality, we may assume that $x = (1, 0, 0)$. Let $\tilde{P}(X, Y)$ denote the dehomogenization of a homogeneous polynomial P with respect to the variable x_0. Applying the Euler formula $dF = x_0 F_0 + x_1 F_1 + x_2 F_2$, we obtain that

$$\mathcal{J}_x = \left(\tilde{F}, \frac{\partial \tilde{F}}{\partial X}, \frac{\partial \tilde{F}}{\partial Y}\right)_x.$$

By Jung–Milnor's formula (see [Mi, Thm. 10.5), the length μ_x of the module $\left(k[X, Y]/\left(\frac{\partial \tilde{F}}{\partial X}, \frac{\partial \tilde{F}}{\partial Y}\right)\right)_x$ is equal to $2\delta_x - r_x + 1$. It only remains to observe that $\tilde{\mu}_x \leq \mu_x$. □

The next lemma is a well-known formula for the arithmetic genus of a plane curve.

LEMMA 4.

$$p_a(C) = \frac{(d-1)(d-2)}{2} = \sum_{i=1}^{h} g_i + \sum_x \delta_x - h + 1, \tag{3.2}$$

where h is the number of irreducible components C_i of C and g_i is the genus of the normalization of C_i.

The next formula is an easy consequence of the incidence relation count for pairs of lines, but just for fun we give a high-brow proof of this.

COROLLARY 2. *Let $\{L_1, \dots, L_s\}$ be a set of lines in $\mathbb{P}^2$. Let a_i denote the number of points that belong to $i \geq 2$ distinct lines. Then*

$$s(s-1) = \sum_{i=2}^{s} a_i i(i-1). \tag{3.3}$$

Proof. We apply the previous formula to the curve $L = L_1 + \cdots + L_s$. Each singular point of L lies on the intersection of $i \geq 2$ lines. It is isomorphic locally to the singular point of the affine curve given by an equation $\prod_{j=1}^{i}(\alpha_j X + \beta_j Y) = 0$. It is easy to compute δ_x, which is equal to $i(i-1)/2$. Since $r_x = i$, by Lemma 4 we have

$$\frac{(s-1)(s-2)}{2} = \sum_{i=2}^{s} \frac{a_i i(i-1)}{2} - s + 1.$$

This is equivalent to the claimed formula. □

THEOREM 4. *Let F be a homaloidal polynomial in three variables without multiple factors. Then, after a linear change of variables, it coincides with one from Examples 1, 2.1, or 4. In other words, $C = V(F)$ is one of the following curves:*

(i) *a nonsingular conic;*
(ii) *the union of three nonconcurrent lines;*
(iii) *the union of a conic and its tangent.*

Proof. Since $\mathcal{P}_F$ is homaloidal, we can apply Lemma 2 and obtain

$$d^2 - 2d = \sum_{x \in C} \tilde{\mu}_x. \tag{3.4}$$

By Lemma 3,

$$d^2 - 2d \leq \sum_{x \in C} (2\delta_x - r_x + 1).$$

By Lemma 4,

$$d^2 - 3d = 2\sum_{i=1}^{h} g_i + 2\sum_{x \in C} \delta_x - 2h. \tag{3.5}$$

Let $C_1, \dots, C_h$ be irreducible components of C and let $d_i = \deg C_i$. Using (3.4) and (3.5), we obtain

$$\sum_{i=1}^{h} (2 - d_i) = -d + 2h \geq 2\sum_{i=1}^{h} g_i + \sum_{x \in C} (r_x - 1) \geq 0. \tag{3.6}$$

The rest of the proof consists of analyzing this inequality. First observe that each point of intersection of two irreducible components gives a positive contribution to the sum $\sum_{i=1}^{k}(r_i - 1)$. This immediately implies that $d_i = 1$ for some i unless C is an irreducible conic. In the latter case it is obviously nonsingular (otherwise, the polar linear system is a pencil); this is case (i) of the theorem. So we may assume that $C_1, \ldots, C_s$ are lines. It follows from (3.6) that

$$0 \geq \sum_{i=s+1}^{h}(2 - d_i) \geq 2\sum_{i=1}^{h} g_i + \sum_{x \in C}(r_x - 1) - s. \tag{3.7}$$

If $s = 1$, then each point of intersection of C_1 with other component of C contributes at least 1 to the sum $\sum_{i=1}^{k}(r_i - 1)$. Hence $C = C_1 + C_2$, where L intersects C_2 at one point and $d_2 = 2$. This is case (iii) of the theorem.

Assume that $s \geq 2$. Let $x_1, \ldots, x_N$ be the intersection points of the lines $C_1, \ldots, C_s$, and let a_j be the number of points among them that belong to $j \geq 2$ lines. Then $\sum_{j=2}^{s} a_j = N$, and

$$\sum_{x \in C}(r_x - 1) - s \geq \sum_{i=1}^{N}(r_i - 1) - s \geq \sum_{j=2}^{s} ja_j - N - s = \sum_{j=2}^{s}(j-1)a_j - s. \tag{3.8}$$

By (3.3),

$$s = \sum_{j=2}^{s} \frac{j}{s-1} a_j(j-1).$$

Assume that not all lines pass through one point, that is, $a_s = 0$. Then $j \leq s - 1$ for all j with $a_j \neq 0$. In this case

$$s \leq \sum_{j=2}^{s} a_j(j-1), \tag{3.9}$$

and the equality holds if and only if $a_j = 0$ for all $j \neq s - 1$. If p_i is a point lying on $s - 1$ lines, then the remaining line must intersect other lines at points different from p_i; this gives that $a_2 \neq 0$. So, if the equality holds, we have $s = 3$ and $a_2 = N = 3$. If $h \neq s$, then C_h is of degree > 1. Its points of intersection with three lines give positive contribution to the sum $\sum_{x \neq x_1, \ldots, x_N}(r_x - 1) - s$. Thus (3.8) is a strict inequality, contradicting (3.7); C is therefore the union of three nonconcurrent lines, which is case (ii) of the theorem.

It remains to consider the case when all lines pass through one point. In this case, $s < h$ (see Lemma 7) and so C_h is of degree > 1. Assume $x_1 \in C_h$. Then $r_1 \geq s + 1$ and

$$\sum_{x \in C}(r_x - 1) - s = (r_1 - 1 - s) + \sum_{x \neq x_1}(r_x - 1) \geq 0. \tag{3.10}$$

It follows from (3.7) that C_h is a nonsingular conic. Since $s \geq 2$, one of the lines is not tangent to C_h at x_1 and hence intersects C_h at some point $x \neq x_1$. Thus

(3.10) is a strict inequality, which contradicts (3.7). If $x_1 \notin C_h$, then C_h intersects each line so that we have $\sum_{x \neq x_1}(r_x - 1) \geq s$ and

$$\sum_{x \in C}(r_x - 1) - s = (r_1 - 1 - s) + \sum_{x \neq x_1}(r_x - 1) \geq s - 1 > 0;$$

again we have a contradiction. □

Let us note the following combinatorial fact, which follows from the proof of Theorem 4 in the case when C is the union of lines.

COROLLARY 3. *Let C consist of s lines $l_1, \dots, l_s$. For each line l_i, let k_i be the number of singular points of C on l_i and let t be the total number of singular points. Assume that $t > 1$. Then*

$$\sum_{i=1}^{s}(k_i - 1) \geq t,$$

with equality if and only if $t = 3$ and $s = 3$.

Proof. Let d be the degree of the map given by the polar linear system of the polynomial defining C. We resolve the indeterminacy points by blowing up the singular points of C. Let E_p be the exceptional curve blow-up from the point p, let h be the class of a general line, and let m_p be the multiplicity of a singular point p. Then

$$d = \Big((s-1)h - \sum_{p \in \mathrm{Sing}(C)}(m_p - 1)E_p\Big)^2 = (s-1)^2 - \sum_{p \in \mathrm{Sing}(C)}(m_p - 1)^2. \quad (3.11)$$

Let $a_i = \#\{p : m_p = i\}$. Applying equality (3.3), we can rewrite (3.11) as follows:

$$\begin{aligned} d &= s(s-1) - (s-1) - \sum_{i=2}^{s} a_i(i-1)i + \sum_{i=2}^{s} a_i(i-1) \\ &= -(s-1) + \sum_{i=2}^{s} a_i(i-1) = -s + 1 + \sum_{i=2}^{s} ia_i - \sum_{i=2}^{s} a_i. \end{aligned}$$

Now the standard incidence relation argument gives us

$$\sum_{i=2}^{s} ia_i = \sum_{p \in \mathrm{Sing}(C)} m_p = \sum_{i=1}^{s} k_i.$$

This allows us to rewrite the expression for d in the form

$$d = 1 + \sum_{i=1}^{s}(k_i - 1) - t.$$

Now $d \geq 1$ unless all lines pass though one point; by Theorem 4, $d = 1$ if and only if $s = 3$ and $t = 3$. □

REMARK. As explained to me by Hal Schenck, for a real arrangement of lines Corollary 3 follows easily from the Euler formula applied to the cellular subdivision of $\mathbb{RP}^2$ defined by the arrangement. Interpret the left-hand side as the number f_1 of edges and the right-hand side as the number f_0 of vertices; then use that $f_0 \geq s$ and $f_2 \geq f_0 + 1$ if the arrangement is not a pencil (see [Gr, pp. 10, 12]).

The argument used in the proof of Theorem 4 does not, unfortunately, apply to nonreduced polynomials. However, the following conjecture seems to be reasonable.

CONJECTURE. *Let $F = A_1^{m_1} \cdots A_s^{m_s}$ be the factorization of F into prime factors. Let $G = A_1 \cdots A_s$. Then the polar linear system $\mathcal{P}_F$ is homaloidal if and only if $\mathcal{P}_G$ is homaloidal.*

4. Arrangements of Hyperplanes in $\mathbb{P}^3$

Here we shall consider the special case when $F = \prod_{i=1}^{n} L_i$ is the product of linear polynomials in four variables without multiple factors. Its set of zeros is an arrangement of hyperplanes in $\mathbb{P}^3$.

Let $\mathcal{A} = \{H_1, \dots, H_N\}$ be the set of planes $\{L_i = 0\}$, let $\mathcal{L}$ be the set of lines that are contained in more than one plane H_i, and let $\mathcal{P}$ be the set of points that are contained in more than two planes H_i. For any $l \in \mathcal{L}$, set

$$k_l = \#\{i : l \subset H_i\}, \qquad a_l = \#\{p \in \mathcal{P} : p \in l\}.$$

For any $p \in \mathcal{P}$, set

$$k_p = \#\{i : p \in H_i\}.$$

We define $d_{\mathcal{A}}$ to be the degree of the polar linear system defined by F.

LEMMA 5.

$$d_{\mathcal{A}} = (N-1)^3 - \sum_{p \in \mathcal{P}} (k_p - 1) + \sum_{l \in \mathcal{L}} (k_l - 1)(a_l - 1).$$

Proof. We can resolve the points of indeterminacy of $\mathcal{P}_F$ by first blowing up each point $p \in \mathcal{P}$ followed by blowing up the proper transforms of each line $l \in \mathcal{L}$. Let

$$D = \sum_{p \in \mathcal{P}} (k_p - 1) E_p + \sum_{l \in \mathcal{L}} (k_l - 1) E_l,$$

where the notation is self-explanatory. We have (see [Fu]) that

$$d_{\mathcal{A}} = ((N-1)H - D)^3,$$

where H is the preimage of a general plane in the blow-up. Using the standard formulas for the blow-up a smooth subvariety, we have

$$E_l^3 = -c_1(N_{\bar{l}}) = -\left[\left(4H - 2 \sum_{l \in \mathcal{L}, p \in l} E_p\right) \cdot \bar{l} - 2\right] = 2a_l - 2.$$

Here $\bar{l}$ denotes the proper transform of the line l under the blowing up the points from $\mathcal{P}$, and $N_{\bar{l}}$ is the normal bundle of $\bar{l}$. Next, we have

$$E_l^2 \cdot E_p = -1, \qquad E_p^3 = 1.$$

Collecting this together yields

$$D^3 = \sum_{l\in\mathcal{L}} (k_l - 1)^3(2a_l - 2) + \sum_{p\in\mathcal{P}} (k_p - 1)^3 - 3 \sum_{l\in\mathcal{L}, p\in l} (k_l - 1)^2(k_p - 1),$$

$$H \cdot D^2 = \sum_{l\in\mathcal{L}} (k_l - 1)^2 E_l \cdot H = -\sum_{l\in\mathcal{L}} (k_l - 1)^2,$$

$$H^2 \cdot D = 0.$$

This gives

$$d_{\mathcal{A}} = (N-1)^3 - 3(N-1) \sum_{l\in\mathcal{L}} (k_l - 1)^2 - \sum_{l\in\mathcal{L}} (k_l - 1)^3(2a_l - 2) - \sum_{p\in\mathcal{P}} (k_p - 1)^3 + 3 \sum_{l\in\mathcal{L}, p\in l} (k_l - 1)^2(k_p - 1).$$

Observe now that

$$\sum_{p\in l} (k_p - 1) = \sum_{p\in l} k_p - a_l = (a_l k_l + N - k_l) - a_l = (a_l - 1)k_l + N - a_l.$$

This allows us to rewrite the expression for $d_{\mathcal{A}}$ as

$$\begin{aligned} d_{\mathcal{A}} &= (N-1)^3 - 3(N-1) \sum_{l\in\mathcal{L}} (k_l - 1)^2 - \sum_{l\in\mathcal{L}} (k_l - 1)^3(2a_l - 2) \\ &\quad - \sum_{p\in\mathcal{P}} (k_p - 1)^3 + 3 \sum_{l\in\mathcal{L}} (k_l - 1)^3(a_l - 1) + 3(N-1) \sum_{l\in\mathcal{L}} (k_l - 1)^2 \\ &= (N-1)^3 - \sum_{p\in\mathcal{P}} (k_p - 1) + \sum_{l\in\mathcal{L}} (k_l - 1)(a_l - 1), \end{aligned}$$

which proves the lemma. □

LEMMA 6. *Let*

$$t_s = \#\{p \in \mathcal{P} : k_p = s\}, \qquad t_q(1) = \#\{l \in \mathcal{L} : k_l = q\},$$

$$t_{sq} = \sum_{l\in\mathcal{L}: k_l = q} \#\{p \in l : k_p = s\}.$$

Then

$$\binom{N}{3} = \sum_s \binom{s}{3} t_s - \sum_{s,q} \binom{q}{3} (t_{sq} - t_q(1)).$$

Proof. This is a 3-dimensional analog of Corollary 2 to Lemma 4. It easily follows from the incidence relation count for triples of distinct planes and points and lines. □

COROLLARY 4.

$$d_{\mathcal{A}} = N - 1 - \sum_{p \in \mathcal{P}} (k_p - 1) + \sum_{l \in \mathcal{L}} (a_l - 1)(k_l - 1).$$

Proof. Combine the previous two lemmas. □

LEMMA 7. *Let $\mathcal{A}$ be an arrangement of N hyperplanes in $\mathbb{P}^3$ defined by a polynomial F. Then the following properties are equivalent*:

(i) *all planes pass through a point*;
(ii) *the partials of F are linearly dependent*;
(iii) $d_{\mathcal{A}} = 0$.

The proof is obvious.

LEMMA 8. *Let $\mathcal{A}$ be an arrangement of N planes, and let $\mathcal{A}'$ be a new arrangement obtained by adding one more plane to $\mathcal{A}$. Assume that $d_{\mathcal{A}} \neq 0$. Then*

$$d_{\mathcal{A}'} > d_{\mathcal{A}}.$$

Proof. Let

$$\mathcal{P}' = \{p \in \mathcal{P} : p \in H\}, \qquad \mathcal{L}' = \{l \in \mathcal{L} : l \subset H\},$$
$$\mathcal{L}'' = \{l \in \mathcal{L} : p \notin l \text{ for any } p \in \mathcal{P}'\},$$
$$\mathcal{N} = \{l \subset H \cap (H_1 \cup \cdots \cup H_N)\} \setminus \mathcal{L}.$$

Note that each line $l \in \mathcal{N}$ is a double line and that each line $l \in \mathcal{L}''$ contains one new singular point $H \cap l$ of multiplicity $k_l + 1$. Applying the previous corollary, we obtain

$$d_{\mathcal{A}'} = N - \sum_{p \in \mathcal{P}\setminus\mathcal{P}'} (k_p - 1) - \sum_{p \in \mathcal{P}'} k_p - \sum_{l \in \mathcal{L}''} k_l + \sum_{l \in \mathcal{L}'} k_l(a_l - 1)$$
$$+ \sum_{l \in \mathcal{L}\setminus\mathcal{L}'} (k_l - 1)a_l + \sum_{l \in \mathcal{N}} (a'_l - 1),$$

where $a_{l'}$ denotes the number a_l defined for the extended arrangement. Applying the corollary again yields

$$d_{\mathcal{A}'} - d_{\mathcal{A}} = 1 + \left(\sum_{l \in \mathcal{L}\setminus(\mathcal{L}'\cup\mathcal{L}''} (k_l - 1) - \#\mathcal{P}' \right)$$
$$+ \left(\sum_{l \in \mathcal{N}} (a'_l - 1) - \#\mathcal{L}'' \right) + \sum_{l \in \mathcal{L}'} (a_l - 1). \tag{4.1}$$

For each $p \in \mathcal{P}'$ there exists a line $l \in \mathcal{L} \setminus (\mathcal{L}' \cup \mathcal{L}'')$ passing through p. Since $k_l > 1$ for each line, we see that $\sum_{l \in \mathcal{L}\setminus(\mathcal{L}'\cup\mathcal{L}'')} (k_l - 1) - \#\mathcal{P}' \geq 0$. Now consider the arrangement of lines in the plane H formed by the lines $l \in \mathcal{N}$. Its multiple points are the points of intersection of H with lines in $\mathcal{L}''$. Applying Corollary 3 to Theorem 4, we see that $\sum_{l \in \mathcal{N}} (a'_l - 1) - \#\mathcal{L}'' \geq 0$ unless there is only one line in $\mathcal{L}''$ when this difference is equal to -1. But in this case H must contain at least one line from $\mathcal{L}$ and hence there is an additional term $\sum_{l\in\mathcal{L}'}(a_l-1)$. If it is zero, then each line $l\in\mathcal{L}'$ contains only one singular point of the arrangement. This implies that all planes except perhaps one contain l, which means that all planes pass through a point and $d_{\mathcal{A}}=0$. Hence the term is positive, and we have proved the inequality $d_{\mathcal{A}'}>d_{\mathcal{A}}$. □

THEOREM 5. *Let $\mathcal{A}$ be an arrangement of N planes in $\mathbb{P}^3$ with $d_{\mathcal{A}}=1$. Then $\mathcal{A}$ is the union of four planes in general linear position.*

Proof. According to Lemma 8, deleting any plane H from the arrangement $\mathcal{A}$ defines an arrangement $\mathcal{A}'$ with $d_{\mathcal{A}'}=0$. We may assume that H does not pass through the common point of the planes from $\mathcal{A}'$. In the notation of the proof of Lemma 8, where the new arrangement is our $\mathcal{A}$ and the old one is $\mathcal{A}\setminus\{H\}$, we have $\#\mathcal{L}''=N-1$. Now the term $\left(\sum_{l\in\mathcal{N}}(a_l'-1)-\#\mathcal{L}''\right)$ in (4.1) must be equal to zero, since otherwise $d_{\mathcal{A}}>1$. By Lemma 6, $N-1=3$; thus, $N=4$. Since $d_{\mathcal{A}}\neq 0$, the planes do not have a common point and hence the arrangement is as in the assertion of the theorem. □

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