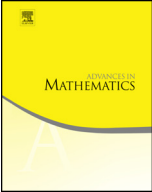




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Automorphism groups of rational elliptic and quasi-elliptic surfaces in all characteristics



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ABSTRACT

We study the groups of automorphisms of rational algebraic surfaces that admit a relatively minimal pencil of curves of arithmetic genus one over an algebraically closed field of arbitrary characteristic. In particular, we classify such surfaces that admit non-trivial automorphisms that act trivially on the Picard group. As an application, we classify classical Enriques surfaces in characteristic 2 that admit non-trivial numerically trivial automorphisms.

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1. Introduction

Let S be a rational surface over an algebraically closed field \mathbb{k} that admits a relatively minimal fibration over \mathbb{P}^1 whose generic fiber F_η , i.e. the fiber over the generic point η of \mathbb{P}^1 , is an irreducible curve of arithmetic genus one over the field $K = \mathbb{k}(\eta)$ of rational functions on \mathbb{P}^1 . We say that $f : S \rightarrow \mathbb{P}^1$ is an elliptic fibration if F_η is smooth, and a quasi-elliptic fibration otherwise. The latter case can happen only if \mathbb{k} is of characteristic 2 or 3. To combine both cases we call f a genus one fibration and call a surface S equipped with such a fibration a genus one surface, or an elliptic or quasi-elliptic surface, if we want to specify the type of the fibration. We recall that the *index* of S is the greatest common divisor of the multiplicities of fibers of f , and that S admits a section if and only if it has index 1. In this paper, we study the group $\text{Aut}(S)$ of biregular automorphisms of S , by decomposing $\text{Aut}(S)$ into subgroups that are easier to understand. Moreover, we give a classification of rational genus one surfaces and of classical Enriques surfaces in characteristic 2 which admit numerically trivial automorphisms, complementing the results of [9].

1.1. Subgroups of $\text{Aut}(S)$

It is well known that the jacobian variety of F_η admits a realization as the generic fiber of a genus one fibration $J(f) : J(S) \rightarrow \mathbb{P}^1$ on a rational surface $J(S)$ that admits a section. After fixing a section E_0 , the set of sections $J(\mathbb{P}^1)$ acquires the structure of a finitely generated abelian group with E_0 as the zero element. This group is called the *Mordell-Weil group* of the fibration and will be denoted by $\text{MW}(J(f))$. Via the restriction of sections to the generic fiber, the Mordell-Weil group becomes isomorphic to the group of rational points $J_\eta(K)$ of the generic fiber. The Mordell-Weil group acts as translations on F_η and, since f is relatively minimal, this action extends to a regular action on S . Since f is induced by the complete linear system associated to (a multiple of) the anti-canonical bundle on S , there is a unique action of $\text{Aut}(S)$ on \mathbb{P}^1 compatible with the action of $\text{Aut}(S)$ on S . In other words, there is a natural homomorphism

$$\beta : \text{Aut}(S) \rightarrow \text{Aut}(\mathbb{P}^1).$$

We let

$$\rho : \text{Aut}(S) \rightarrow \text{O}(\text{Pic}(S)), \quad g \mapsto g^*$$

be the natural homomorphism describing the action on the Picard group of the surface.

We define the following subgroups of $\text{Aut}(S)$.

- $\text{Aut}_{\text{ct}}(S) := \text{Ker}(\rho)$.
- $\text{Aut}(S)^\dagger := \text{Ker}(\beta)$.
- $\text{Aut}_{\text{ct}}(S)^\dagger := \text{Aut}_{\text{ct}}(S) \cap \text{Aut}(S)^\dagger$.
- $\text{Aut}_{\text{tr}}(S) := \text{MW}(J(f))$.
- $\text{Aut}_E(S) := \{g \in \text{Aut}(S) : g(E) = E\}$, where E is a (-1) -curve on S .

1.1.1. Action on the jacobian fibration

By Theorem 3.3, and since f is the only genus one fibration of S , the group $\text{Aut}(S)$ acts on $J(S)$ through a morphism $\varphi : \text{Aut}(S) \rightarrow \text{Aut}_0(J(S))$, where $\text{Aut}_0(J(S))$ is the subgroup of $\text{Aut}(J(S))$ preserving a fixed section E_0 . The following theorem is a consequence of Theorem 3.3 and Theorem 4.2.

Theorem 1.1. *Let $f : S \rightarrow \mathbb{P}^1$ be a rational genus one surface of index m . Then, there is a homomorphism $\varphi : \text{Aut}(S) \rightarrow \text{Aut}_0(J(S))$ such that:*

- (1) $\text{Ker}(\varphi) = \text{Aut}_{\text{tr}}(S)$.
- (2) $\text{Aut}_0(J(S)) = \text{Aut}(\mathcal{D})$ for some weak del Pezzo surface \mathcal{D} of degree 1.
- (3) $\text{im}(\varphi) \subseteq \text{Aut}_0(J(S))$ is contained in the stabilizer of an m -torsion point on the identity component of a fiber of $J(f)$.
- (4) If $m = 1$, then φ admits a section. In particular, $\text{Aut}(S) = \text{MW}(J(f)) \rtimes \text{Aut}(\mathcal{D})$.

Remark 1.2. The groups $\text{Aut}_{\text{tr}}(S)$ and $\text{Aut}(\mathcal{D})$ are well-studied:

- (1) In the case of elliptic (resp. quasi-elliptic) surfaces all possible groups $\text{Aut}_{\text{tr}}(S)$ are listed in [27, Table 8.2] (resp. [12], [13]).
- (2) The list of all possible finite groups acting on a del Pezzo surface of degree 1 was essentially known since the 19th century. It is contained in [6] and [7, 8.8.4] (under an assumption on the characteristic).
- (3) In Section 5, we give a formula for the action of $d \cdot \text{Aut}_{\text{tr}}(S)$ on $\text{Pic}(S)$, where d is the smallest positive integer such that all discriminant groups of fibers of f are d -torsion. This generalizes Gizatullin’s formula for the case $d = 1$ [10].

1.1.2. Action on the base curve

The following decomposition is well-known and we mention it here for the sake of completeness.

Proposition 1.3. *Let $f : S \rightarrow \mathbb{P}^1$ be a rational genus one surface of index m . Then, the natural homomorphism $\beta : \text{Aut}(S) \rightarrow \text{Aut}(\mathbb{P}^1) = \text{PGL}_2$ satisfies the following properties:*

- (1) $\text{Aut}(S)^\dagger := \text{Ker}(\beta) = \text{Aut}(F_\eta)$.

- (2) $\text{im}(\beta) \subseteq \text{PGL}_2$ preserves the image of the set of singular fibers of f .
- (3) The j -map of f is $\text{im}(\beta)$ -invariant.

Remark 1.4. Properties (2) and (3) of Proposition 1.3 could, in principle, be used to classify the groups $\text{im}(\beta)$, but the list would be too long and we decided not to pursue this computation. Some of these groups appear in Table 3.

1.1.3. *Action on the Picard lattice*

It is a basic fact that the Picard group $\text{Pic}(S)$ of a rational genus one surface S is a free abelian group and the intersection product equips it with a structure of an odd unimodular hyperbolic lattice. A choice of blowing down morphism $\pi : S \rightarrow \mathbb{P}^2$ allows one to construct an orthonormal basis (e_0, e_1, \dots, e_9) , called a geometric basis, that defines an isomorphism $\phi : \text{Pic}(S) \rightarrow \mathbb{I}^{1,9}$, called a *marking* of S . The image of the orthogonal complement of the canonical class K_S of S is a sublattice of $\mathbb{I}^{1,9}$ isomorphic to the root lattice E_9 of the affine root system of type \tilde{E}_8 . In this way, a marking ϕ and the homomorphism ρ define a homomorphism

$$\Phi : \text{Aut}(S) \rightarrow W(E_9),$$

where $W(E_9)$ is the Weyl group of the lattice E_9 , the index 2 subgroup of $O(E_9)$ generated by reflections in roots. The lattice E_9 has radical of rank 1 generated by a primitive isotropic vector \mathfrak{f} , which is the image of the canonical class K_S under any marking. It is known that the orthogonal complement \mathfrak{f}^\perp modulo $\mathbb{Z}\mathfrak{f}$ is isomorphic to the unimodular even negative lattice E_8 . This defines a homomorphism $r : W(E_9) \rightarrow W(E_8)$. Moreover, there is an inclusion $E_8 \hookrightarrow W(E_9)$ defined by the formula $v \mapsto \iota_v$, where $\iota_v(x) = x + \langle x, v \rangle \mathfrak{f}$.

The following theorem describes the action of $\text{Aut}(S)$ on $\text{Pic}(S)$. It is a consequence of Section 4 and Corollary 5.5.

Theorem 1.5. *Let $f : S \rightarrow \mathbb{P}^1$ be a rational genus one surface of index m . Then, there is a morphism $\Phi : \text{Aut}(S) \rightarrow W(E_9)$ such that:*

- (1) $\text{Ker}(\Phi) = \text{Aut}_{\text{ct}}(S)$.
- (2) A choice of a (-1) -curve $E \subseteq S$ defines a splitting $W(E_9) \cong E_8 \rtimes W(E_8)$.
- (3) For the splitting in (2) defined by E , we have $\Phi(\text{Aut}_E(S)) \subseteq W(E_8)$.
- (4) $\Phi(d \cdot \text{Aut}_{\text{tr}}(S)) \subseteq E_8$, where d is the smallest positive integer such that all the discriminant groups of fibers of f are d -torsion groups.

1.2. *Numerically trivial automorphisms of cohomologically flat genus one surfaces with rational jacobians*

The bulk of the paper is devoted to the classification of all rational genus one surfaces with non-trivial group $\text{Aut}_{\text{ct}}(S)$. Thanks to the availability of the Weierstrass equation, it

is probably not very hard to do it in the case when the fibration coincides with its jacobian fibration. However, we take another, more geometric approach to the classification in this case. The case of non-jacobian fibrations turns out to be much more technically involved. In Theorem 3.3, we describe an explicit relationship between the automorphism group of a cohomologically flat genus one surface and the automorphism group of its jacobian. This not only allows us to describe non-jacobian rational genus one surfaces with cohomologically trivial automorphisms, but it also allows us to give a classification of Enriques surfaces over a field of characteristic 2 with non-trivial canonical class that admit non-trivial numerically trivial automorphisms, i.e., automorphisms acting trivially on the group of divisor classes up to numerical equivalence. This classification was absent in [9].

1.2.1. *Rational genus one surfaces with cohomologically trivial automorphisms*

Recall that $\text{Aut}_{\text{ct}}(S)$ is the subgroup of $\text{Aut}(S)$ that acts trivially on $\text{Pic}(S)$.

Theorem 1.6. *Let $f : S \rightarrow \mathbb{P}^1$ be a rational genus one surface of index m .*

- (1) *If $m = 1$, then $\text{Aut}_{\text{ct}}(S)$ is non-trivial if and only if S is one of the surfaces in Table 3.*
- (2) *If $m > 1$ and $\text{Aut}_{\text{ct}}(S)$ is non-trivial, then S occurs in Table 4.*

1.2.2. *Classical Enriques surfaces in characteristic 2 with numerically trivial automorphisms*

An Enriques surface S is called *classical* if ω_S is non-trivial. In contrast to the case of rational surfaces, the Picard group $\text{Pic}(S)$ is not torsion-free. We let $\text{Aut}_{\text{nt}}(S) := \text{Ker}(\text{Aut}(S) \rightarrow \text{Pic}(S)/\text{Torsion})$ be the group of numerically trivial automorphisms. In characteristic different from 2, the classification of Enriques surfaces with numerically trivial automorphisms is known (see [9] and [5, Section 8.2]). We give a classification of classical Enriques surfaces with numerically trivial automorphisms in characteristic 2.

Theorem 1.7. *Let S be a classical Enriques surface in characteristic 2. Then, $\text{Aut}_{\text{nt}}(S)$ is non-trivial if and only if S contains one of the four configurations of (-2) -curves described in Theorem 7.4 and then $\text{Aut}_{\text{nt}}(S)$ is as in Theorem 7.6.*

Convention 1.8. Following the mathematical taste of both authors we work over an algebraically closed field of arbitrary characteristic.

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2. Generalities

2.1. Genus one fibrations

The proofs of all the facts about genus one fibrations on algebraic surfaces for which we do not supply a proof can be found in [4, Chapter 4]. One also finds there references to the original results.

A complete geometrically integral curve over a field K of arithmetic genus one is called elliptic (resp. quasi-elliptic) if it is smooth (resp. regular but non-smooth). We do not assume that it admits a rational point. An elliptic curve with a fixed rational point is an abelian curve, i.e. has a structure of an abelian variety of dimension one. A quasi-elliptic curve contains a unique non-smooth point defined over a radical extension of K of degree $p = 2$ or 3 . It is isomorphic to a plane cuspidal curve over the algebraic closure of K . If it has a rational point, then the complement of this point admits a structure of a connected unipotent algebraic group of dimension 1, a non-trivial inseparable form of the additive group $\mathbb{G}_{a,K}$. We refer to an elliptic or a quasi-elliptic curve E as a *genus one curve*. For any Dedekind scheme B over an algebraically closed field \mathbb{k} with field of fractions K , there exists a projective flat morphism $f : X \rightarrow B$ with generic fiber X_η (i.e. the fiber over the generic point $\eta \in B$ with residue field K) isomorphic to E . Using resolution of singularities of two-dimensional excellent schemes, one may assume that X is regular. The theory of birational transformations of such schemes due to Lichtenbaum and Shafarevich allows one to assume that X is relatively minimal over B , i.e. any birational morphism $X \rightarrow X'$ over B is an isomorphism.

Let X^\sharp denote the open subset of points on X where f is smooth. If X_η is smooth, then $X \setminus X^\sharp$ consists of singular points of fibers over closed points. If X_η is quasi-elliptic, then $X \setminus X^\sharp$ also contains the closure \mathfrak{C} of the cusp of X_η . If $X_\eta(K) \neq \emptyset$, the closure of a rational point on X_η defines a section $f : B \rightarrow X^\sharp$. It equips X^\sharp with a structure of a commutative group scheme over B . This group scheme coincides with the *Néron model* of $(X^\sharp)_\eta$. If $X(K) = \emptyset$, then the jacobian variety $J(X_\eta)$ (the Rosenlicht generalized jacobian variety if X_η quasi-elliptic) admits a relatively minimal model $J(f) : J(X) \rightarrow B$ with generic fiber $J(X_\eta)$. It is called the *jacobian fibration* of f . If one of the fibers is singular as it will be in our case, the group of rational points $X(K)$ is a finitely generated abelian group, called the *Mordell-Weil group* of f and denoted by $\text{MW}(f)$. Note that $\text{MW}(f)$ is a p -group if f is quasi-elliptic.

A choice of a group structure on J_η^\sharp equips X_η^\sharp with a unique structure of a *torsor* (i.e. a principal homogeneous space) under J_η^\sharp . The *index* of a genus one fibration $f : X \rightarrow B$ is the greatest common divisor of the degrees of extensions L/K such that $X_\eta^\sharp(L) \neq \emptyset$. The closure of such an L -valued point of X_η^\sharp in X is a (possibly inseparable) degree $m = [L : K]$ cover of the base B and we call it an m -section. It is known that the index coincides with the *period* of X_η^\sharp , i.e. the order of the isomorphism class of X_η^\sharp in the Weil-Châtelet group $H^1(K, J_\eta^\sharp)$ of isomorphism classes of torsors under J_η^\sharp .

A fibration that becomes isomorphic to the trivial fibration after a finite base change is called *isotrivial*. If the fibration is elliptic, this is equivalent to the condition that the j -invariant of the jacobian fibration is constant.

The fibers of f over closed points are curves of arithmetic genus one over \mathbb{k} . If f is elliptic they are elliptic curves over \mathbb{k} , except for finitely many singular fibers. If f is quasi-elliptic they are isomorphic to a cuspidal plane cubic over \mathbb{k} , except for finitely many which are reducible curves.

We are assuming that the reader is familiar with Kodaira’s classification of singular fibers of genus one fibrations. We use the dictionary between Kodaira’s notation and the notation from [4] based on the notations of extended Dynkin diagrams:

$$\left(\begin{array}{ccccccccccc} I_1 & II & III & IV & I_n (n > 1) & I_{n-4}^* (n \geq 4) & IV^* & III^* & II^* \\ \tilde{A}_0^* & \tilde{A}_0^{**} & \tilde{A}_1^* & \tilde{A}_2^* & \tilde{A}_{n-1} & \tilde{D}_n & \tilde{E}_6 & \tilde{E}_7 & \tilde{E}_8 \end{array} \right)$$

The fibers of types I_n with $n \geq 1$ are called of *multiplicative type* and the other singular fibers are called of *additive type*. Multiplicative fibers do not occur on quasi-elliptic fibrations. Any fiber X_t can be considered as a Cartier or Weil divisor on X . As a Cartier divisor its local equation is given by the pull-back of a local parameter at t . As a Weil divisor, it is given by the linear combination of its irreducible components taken with multiplicity equal to the order of the vanishing of the local equation at its generic point. It is known that $X_t = m_t \bar{X}_t$, where one of the irreducible components of X_t enters in \bar{X}_t with multiplicity 1. The number m_t is called the *multiplicity* of the fiber and a fiber with $m_t > 1$ is called an m_t -multiple fiber. Otherwise it is called a simple or non-multiple fiber. To denote that a fiber of a given type is an m_t -multiple fiber, we write m_t in front of its type notation.

If $X_\eta(K) \neq \emptyset$, all fibers are non-multiple, however the converse is not true in general. The isomorphism classes of torsors with no multiple fibers form a subgroup of the Weil-Châtelet group, called the *Tate-Shafarevich group*. In the case when $H^1(J(X), \mathcal{O}_{J(X)}) = H^2(J(X), \mathcal{O}_{J(X)}) = 0$ (e.g. if X is a rational surface), this group is trivial, and the Weil-Châtelet group is isomorphic to the direct sum of the Weil-Châtelet groups $H^1(K_t, J(X) \otimes_K K_t)$, where K_t/K is the extension of K equal to the field of fractions of the completion of the local ring of B at a closed point t . By Hensel’s lemma, the class of the torsor in one of the latter groups is trivial if and only if the fiber X_t is non-multiple.

If $X_t = m_t \bar{X}_t$ is a multiple fiber, the normal sheaf $\mathcal{N}_t = \mathcal{O}_{X_t}(\bar{X}_t)$ defines an m_t -torsion element in the Picard group $\text{Pic}(\bar{X}_t)$ of isomorphism classes of invertible sheaves on \bar{X}_t . If its order in this group is equal to m_t , the fiber is called *tame*, otherwise it is a *wild* fiber. A genus one fibration has no wild fibers if and only if it is *cohomologically flat* (in dimension 0), i.e. $h^0(\mathcal{O}_{X_t}) = 1$ for all $t \in B$. If m_t is prime to the characteristic p , then the fiber is tame. Also, it is tame if $H^1(X, \mathcal{O}_X) = 0$.

By the Ogg-Shafarevich theory for elliptic fibrations (see [4, Chapter 4]) the local invariants of tame m_t -multiple fibers in $H^1(K_t, X \otimes_K K_t)$ are in bijection with m_t -torsion line bundles on \bar{X}_t . We expect that the same is true for quasi-elliptic fibrations. Since this seems to be unknown at the moment, let us define the following.

Definition 2.1. Let $f : X \rightarrow B$ be a genus one fibration. We say that f satisfies condition (OS) if for every closed point $t \in B$, the subgroup of the local Weil-Châtelet group $H^1(K_t, J(X) \otimes_K K_t)$ consisting of tame torsors of index m_t is naturally isomorphic to the m_t -torsion subgroup of $J(X_t)^0$.

Finally, if X is a projective surface with no wild fibers, then its canonical sheaf ω_X is given by the formula

$$\omega_X = f^*(\mathcal{L} \otimes \omega_B) \otimes \mathcal{O}_X\left(\sum_{t \in B} (m_t - 1)\bar{X}_t\right), \tag{2.1}$$

where \mathcal{L} is an invertible sheaf of degree $-\chi(X, \mathcal{O}_X)$.

2.2. Rational genus one fibrations

Now we specialize to our case, where we consider X to be a smooth projective rational surface S over \mathbb{k} . Obviously, $B = \mathbb{P}^1$ in this case. It follows from the discussion above that f is cohomologically flat. First, let us describe some basic properties of multiple fibers on rational genus one surfaces.

Lemma 2.2. *Let $f : S \rightarrow \mathbb{P}^1$ be a rational genus one surface. Then,*

- (1) f has at most one multiple fiber mF_0 ,
- (2) the canonical bundle satisfies $K_S = -F_0$, and
- (3) if $m > 1$, then the following hold:
 - (a) If F_0 is smooth and supersingular, then $p \nmid m$.
 - (b) If F_0 is multiplicative, then $p \nmid m$.
 - (c) If F_0 is additive, then $m = p$.

Proof. As f is cohomologically flat, the first two claims follow immediately from Equation (2.1). Since the multiple fiber is not wild, its normal bundle $\mathcal{O}_F(F_0)$ has order m . If F_0 is either supersingular or multiplicative, the p -torsion subgroup of $\text{Pic}(F_0)$ is trivial, which means that $p \nmid m$. If F is additive, then $\text{Pic}(F)$ is p -elementary and thus $m = p$. \square

The Noether formula $12\chi(S, \mathcal{O}_S) = c_2 + K_S^2$ implies that the second Chern number of S is equal to 12. It coincides with the topological Euler-Poincaré characteristic of S computed in étale topology if $\mathbb{k} \neq \mathbb{C}$. It is known that

$$c_2(S) = 12 = \sum_{t \in B} (e_t + \delta_t).$$

If b_t denotes the number of irreducible components of S_t , then $e_t = 0$ if S_t is smooth, $e_t = b_t$ if S_t is multiplicative, $e_t = b_t + 1$ if S_t is additive, and $e_t = b_t - 1$ if f is quasi-

Table 1
Some configurations of singular fibers on rational genus one surfaces.

F	Possible combinations of other singular (resp. non-cuspidal) fibers	Characteristic
\bar{E}_8	$(\bar{A}_0^{**}), (\bar{A}_0^*, \bar{A}_0^*)$	$\neq 2, 3$
	none, (\bar{A}_0^*)	2, 3
\bar{D}_8	$(\bar{A}_0^*, \bar{A}_0^*)$	$\neq 2$
	none	2
\bar{A}_8	$(\bar{A}_0^*, \bar{A}_0^*, \bar{A}_0^*)$	$\neq 3$
	(\bar{A}_0^{**})	3
\bar{E}_7	$(\bar{A}_0^*, \bar{A}_0^{**}), (\bar{A}_0^*, \bar{A}_0^*, \bar{A}_0^*), (\bar{A}_1^*), (\bar{A}_1, \bar{A}_0^*)$	$\neq 2, 3$
	$(\bar{A}_0^{**}), (\bar{A}_0^*, \bar{A}_0^*, \bar{A}_0^*), (\bar{A}_1^*), (\bar{A}_1, \bar{A}_0^*)$	3
	none, $(\bar{A}_0^*, \bar{A}_0^*), (\bar{A}_1), (\bar{A}_1^*)$	2
\bar{D}_7	$(\bar{A}_0^{**}, \bar{A}_0^*), (\bar{A}_0^*, \bar{A}_0^*, \bar{A}_0^*)$	$\neq 2, 3$
	$(\bar{A}_0^{**}), (\bar{A}_0^*, \bar{A}_0^*, \bar{A}_0^*)$	3
	none, (\bar{A}_0^*)	2
\bar{A}_7	$(\bar{A}_0^{**}, \bar{A}_0^*, \bar{A}_0^*), (\bar{A}_0^*, \bar{A}_0^*, \bar{A}_0^*, \bar{A}_0^*), (\bar{A}_1, \bar{A}_0^*, \bar{A}_0^*)$	$\neq 2, 3$
	$(\bar{A}_0^*, \bar{A}_0^*, \bar{A}_0^*, \bar{A}_0^*), (\bar{A}_1, \bar{A}_0^*, \bar{A}_0^*)$	3
	$(\bar{A}_0^{**}), (\bar{A}_0^*, \bar{A}_0^*, \bar{A}_0^*, \bar{A}_0^*), (\bar{A}_1^*)$	2
\bar{E}_6	$(\bar{A}_0^{**}, \bar{A}_0^{**}), (\bar{A}_0^{**}, \bar{A}_0^*, \bar{A}_0^*), (\bar{A}_0^*, \bar{A}_0^*, \bar{A}_0^*, \bar{A}_0^*), (\bar{A}_1, \bar{A}_0^{**}), (\bar{A}_1, \bar{A}_0^*, \bar{A}_0^*), (\bar{A}_1^*, \bar{A}_0^*), (\bar{A}_2^*), (\bar{A}_2, \bar{A}_0^*)$	$\neq 2, 3$
	none, $(\bar{A}_0^{**}), (\bar{A}_0^*, \bar{A}_0^*), (\bar{A}_0^*, \bar{A}_0^*, \bar{A}_0^*), (\bar{A}_1), (\bar{A}_1^*), (\bar{A}_1, \bar{A}_0^*)(\bar{A}_2), (\bar{A}_2^*)$	3
	$(\bar{A}_0^{**}), (\bar{A}_0^*, \bar{A}_0^*, \bar{A}_0^*, \bar{A}_0^*), (\bar{A}_1^*), (\bar{A}_1, \bar{A}_0^*, \bar{A}_0^*), (\bar{A}_2^*), (\bar{A}_2, \bar{A}_0^*)$	2
\bar{D}_6	$(\bar{A}_0^{**}, \bar{A}_0^{**}), (\bar{A}_0^{**}, \bar{A}_0^*, \bar{A}_0^*), (\bar{A}_0^*, \bar{A}_0^*, \bar{A}_0^*, \bar{A}_0^*), (\bar{A}_1^*, \bar{A}_0^*), (\bar{A}_1, \bar{A}_0^*, \bar{A}_0^*), (\bar{A}_1, \bar{A}_1)$	$\neq 2, 3$
	$(\bar{A}_0^{**}), (\bar{A}_0^*, \bar{A}_0^*), (\bar{A}_0^*, \bar{A}_0^*, \bar{A}_0^*, \bar{A}_0^*), (\bar{A}_1, \bar{A}_0^*, \bar{A}_0^*), (\bar{A}_1, \bar{A}_1)$	3
	none, $(\bar{A}_0^*, \bar{A}_0^*), (\bar{A}_1), (\bar{A}_1^*, \bar{A}_1^*)$	2

elliptic. The number $\delta_t \geq 0$ can be nonzero only if $p = 2, 3$, f is elliptic, and S_t is of additive type.

Recall that a genus one fibration on a rational surface is called *extremal* if one of the following equivalent conditions is satisfied:

- The Mordell-Weil group of the jacobian fibration is finite;
- $\sum_{t \in \mathbb{P}^1} (b_t - 1) = 8$.

Since the Mordell-Weil group of a quasi-elliptic surface is finite, $\sum_{t \in \mathbb{P}^1} e_t = 8$ holds for all quasi-elliptic surfaces. For extremal genus one fibrations, the structure of the Mordell-Weil group is determined by the structure of the groups of components of reducible fibers: $MW(J(f))^{\oplus 2}$ is isomorphic to the direct sum of these groups.

The classification of all possible singular fibers of an elliptic fibration is known in all characteristics (see [25] if $p \neq 2, 3$ and [18], [19] if $p = 2, 3$). Also known is the classification of singular fibers of quasi-elliptic fibrations (see [12], [13], [4, Chapter 4]). For the convenience of the reader, we list possible singular fibers on rational elliptic or quasi-elliptic surfaces in the cases that will occur in our work in Table 1.

Lemma 2.3. *Let $f : S \rightarrow \mathbb{P}^1$ be a rational genus one surface. If f admits a fiber F such that*

- F is of multiplicative type with at least 8 components, or

- F is of additive type with at least 7 components,

then the singular fibers of f (resp. the non-cuspidal fibers if f is quasi-elliptic) are as in one of the cases in Table 1.

2.3. Halphen pencils

A rational surface S with a genus one fibration $f : S \rightarrow \mathbb{P}^1$ is a basic rational surface, i.e. it admits a birational morphism $\pi : S \rightarrow \mathbb{P}^2$. The morphism π admits a factorization

$$\pi : S = S_{10} \xrightarrow{\pi_9} S_9 \xrightarrow{\pi_8} \dots \xrightarrow{\pi_2} S_2 \xrightarrow{\pi_1} S_1 = \mathbb{P}^2, \tag{2.2}$$

where each morphism $\pi_i : S_{i+1} \rightarrow S_i$ is the blow-up of one point $x_i \in S_i$. For any $j > i$, let $\pi_{j,i} := \pi_i \circ \dots \circ \pi_{j-1} : S_j \rightarrow S_i$. A point $x_j \in S_j$ with $\pi_{j,i}(x_j) = x_i \in S_i$ is called *infinitely near* to x_i (of order $j - i$). A curve of negative self-intersection on S is either a smooth rational curve of self-intersection -1 (a (-1) -curve), and in this case it is an m -section of f , or a smooth rational curve of self-intersection -2 (a (-2) -curve) in which case it is an irreducible component of some fiber. This implies that at most 3 of the points x_i are collinear, at most 6 of them lie on a conic and there are no cubics passing through all points with a singular point at one of them.

The image of the pencil of genus one curves defined by f under π is a pencil \mathcal{P} of curves of degree $3m$ with general member having singular points of multiplicity m at the points x_1, \dots, x_9 (appropriately interpreted if some of the x_i are infinitely near to each other). This pencil of curves in \mathbb{P}^2 is called an *Halphen pencil* of index m . If $m > 1$, the image of the multiple fiber is the unique cubic curve passing through the points x_1, \dots, x_9 taken with multiplicity m . The points $\pi_{j,1}(x_j), j = 1, \dots, 9$, are the intersection points of two general members of the Halphen pencil defined by the elliptic fibration on S . Their number can be less than 9. The points $\pi_{j,2}(x_j), j = 2, \dots, 9$, are the intersection points of the proper transforms of two general members of the pencil in S_2 , and so on, until the proper transforms of the general members of the pencil on S do not intersect.

The set $\text{Bs}(\mathcal{P}) = \{x_1, \dots, x_9\}$ is the set of base points of the Halphen pencil \mathcal{P} . We order the set $\text{Bs}(\mathcal{P})$ as follows:

$$(x_1, \dots, x_9) = (x_1^{(1)}, \dots, x_1^{(k_1)}, x_2^{(1)}, \dots, x_2^{(k_2)}, \dots, x_s^{(1)}, \dots, x_s^{(k_s)}),$$

where

$$\begin{aligned} x_1^{(k_1)} \succ x_1^{(k_1-1)} \succ \dots \succ x_1^{(1)} &= q_1, & x_2^{(k_2)} \succ \dots \succ x_2^{(2)} \succ x_2^{(1)} \\ &= q_2, & x_s^{(k_s)} \succ \dots \succ x_s^{(2)} \succ x_s^{(1)} = q_s. \end{aligned}$$

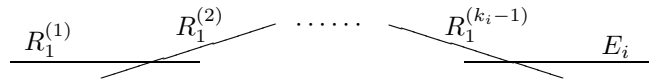
Here $x_j^{(i)} \succ x_j^{(i-1)}$ is infinitely near to $x_j^{(i-1)}$ of order 1. We have

$$k_1 + \dots + k_s = 9.$$

Let $\mathcal{E}_i := \pi_{10,i}^{-1}(x_i)$. It is called the *exceptional configuration* over the base point x_i . We have $\mathcal{E}_i \subseteq \mathcal{E}_{i-1}$ if $x_i \succ x_{i-1}$ and the difference is a (-2) -curve. In our ordering we have s chains

$$\mathcal{R}_i = R_i^{(1)} + \dots + R_i^{(k_i-1)}$$

of such curves of lengths $k_i - 1, i = 1, \dots, s$ and s irreducible exceptional configurations E_i (which are (-1) -curves) that intersect $R_i^{(k_i-1)}$ with multiplicity 1. All other exceptional configurations are of the form $E_i + R_i^{(k_i-1)} + \dots + R_i^{(j)}, i = 1, \dots, s, j = 1, \dots, k_i - 1$.



Of course, the birational morphism $\pi : S \rightarrow \mathbb{P}^2$ is not unique, so the shape of the corresponding Halphen pencil and its base points could be different. Two Halphen pencils corresponding to the same surface S are obtained from each other by a Cremona transformation of the plane.

A choice of $\pi : S \rightarrow \mathbb{P}^2$ defines a *geometric basis* (e_0, e_1, \dots, e_9) in $\text{Pic}(S)$. Here $e_0 = c_1(\pi^* \mathcal{O}_{\mathbb{P}^2}(1))$ and the other e_i 's are the divisor classes of the exceptional configurations \mathcal{E}_i . We have

$$e_0^2 = 1, \quad e_i^2 = -1, \quad i \neq 0, \quad e_i \cdot e_j = 0, \quad i \neq j$$

and

$$K_S = -3e_0 + e_1 + \dots + e_9.$$

As we explained in the introduction, a geometric basis defines a marking $\text{Pic}(S) \rightarrow \mathbb{I}^{1,9}$ with the image of K_S equal to the generator \mathfrak{f} of the radical of the orthogonal complement \mathfrak{f}^\perp in $\mathbb{I}^{1,9}$.

2.4. Some relevant fixed loci

The following Lemma is an elementary fact from linear algebra:

Lemma 2.4. *Let T be an automorphism of \mathbb{P}^2 of finite order n and let \mathcal{F} be its set of fixed points. Then one of the following cases occurs:*

- (a) $(p, n) = 1$ and \mathcal{F} is the union of a line L and a point outside L ;
- (b) $(p, n) = 1, n \geq 3$ and \mathcal{F} consists of 3 non-collinear isolated points q_1, q, q' ;
- (c) $n = pk, (k, p) = 1$ and the fixed locus of T^k is a point;

(d) $n = pk, (k, p) = 1$ and the fixed locus of T^k is a line L .

Every T -invariant line meets \mathcal{F} in at least 1 (if $(p, n) \neq 1$) or 2 (if $(p, n) = 1$) points.

Recall the structure of the fixed loci of irreducible curves of arithmetic genus one.

Lemma 2.5. *Let E be an irreducible curve of arithmetic genus one over an algebraically closed field. Choose a smooth point on E and let G be the group of automorphisms of E preserving this point. For $g \in G$, let E_{red}^g be the reduced fixed locus of g and let $E^\#$ be the smooth locus of E .*

(1) *If $p \neq 2, 3$ and E is elliptic*

j	G	$\text{ord}(g)$	E_{red}^g
$\neq 0, 1$	$\mathbb{Z}/2\mathbb{Z}$	2	$(\mathbb{Z}/2\mathbb{Z})^2$
1	$\mathbb{Z}/4\mathbb{Z}$	$\begin{cases} 2 \\ 4 \end{cases}$	$\begin{cases} (\mathbb{Z}/2\mathbb{Z})^2 \\ \mathbb{Z}/2\mathbb{Z} \end{cases}$
0	$\mathbb{Z}/6\mathbb{Z}$	$\begin{cases} 2 \\ 3 \\ 6 \end{cases}$	$\begin{cases} (\mathbb{Z}/2\mathbb{Z})^2 \\ \mathbb{Z}/3\mathbb{Z} \\ \{*\} \end{cases}$

(2) *If $p = 3$ and E is elliptic*

j	G	$\text{ord}(g)$	E_{red}^g
$\neq 0$	$\mathbb{Z}/2\mathbb{Z}$	2	$(\mathbb{Z}/2\mathbb{Z})^2$
0	$\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$	$\begin{cases} 2 \\ 3, 6 \\ 4 \end{cases}$	$\begin{cases} (\mathbb{Z}/2\mathbb{Z})^2 \\ \{*\} \\ \mathbb{Z}/2\mathbb{Z} \end{cases}$

(3) *If $p = 2$ and E is elliptic*

j	G	$\text{ord}(g)$	E_{red}^g
$\neq 0$	$\mathbb{Z}/2\mathbb{Z}$	2	$\mathbb{Z}/2\mathbb{Z}$
0	$Q_8 \rtimes \mathbb{Z}/3\mathbb{Z}$	$\begin{cases} 2, 4, 6 \\ 3 \end{cases}$	$\begin{cases} \{*\} \\ \mathbb{Z}/3\mathbb{Z} \end{cases}$

(4) *If E is cuspidal*

G	g	$ E_{red}^g $	$E_{red}^g \cap E^\#$
$\mathbb{G}_a \rtimes \mathbb{G}_m$	$\begin{cases} \in \mathbb{G}_a \\ \notin \mathbb{G}_a \end{cases}$	$\begin{cases} 1 \\ 2 \end{cases}$	$\begin{cases} \emptyset \\ \{*\} \end{cases}$

(5) If E is nodal

G	g	$ E_{red}^g $	$E_{red}^g \cap E^\#$
$\mathbb{G}_m \rtimes \mathbb{Z}/2\mathbb{Z}$	$\begin{cases} \in \mathbb{G}_m \\ \notin \mathbb{G}_m, p \neq 2 \\ \notin \mathbb{G}_m, p = 2 \end{cases}$	$\begin{cases} 1 \\ 3 \\ 2 \end{cases}$	$\begin{cases} \emptyset \\ \mathbb{Z}/2\mathbb{Z} \\ \{*\} \end{cases}$

3. Action of $\text{Aut}(S)$ on the jacobian surface

All the facts mentioned in this subsection without a reference to their proofs can be found in [26].

Let $f : X \rightarrow B$ be a genus one fibration as in Subsection 2.1. For any B -scheme $T \rightarrow B$ we denote by $f_T : X_T \rightarrow T$ the morphism obtained from f by base change along $T \rightarrow B$. We denote by $\mathcal{P}_{X/B}$ the relative Picard functor. It is a sheaf in the fppf-topology on B associated to the pre-sheaf $T \rightarrow \text{Pic}(X_T)$. It is automatically a sheaf in the étale topology and coincides with the sheaf $R^1 f_* \mathbb{G}_m$. There is a natural injective homomorphism $\text{Pic}(X_T)/f_T^*(\text{Pic}(T)) \rightarrow \mathcal{P}_{X/B}(T)$, which is bijective if $X(B) \neq \emptyset$ (or if the cohomological Brauer group $H^2(T, \mathbb{G}_m)$ of T is trivial).

For every point $t \in B$ including the generic point, the fiber of $\mathcal{P}_{X/B,t}$ is representable by a one-dimensional commutative group scheme locally of finite type $\text{Pic}_{X_t/t}$, the Picard scheme of X_t . Its connected component of the identity $\text{Pic}_{X_t/t}^0$ is the (generalized) jacobian of the curve X_t and denoted by $J(X_t)$. If $X_t = \sum m_t^{(i)} X_t^{(i)}$, then $J(X_t)(\mathbb{k})$ is the group of isomorphism classes of invertible sheaves \mathcal{L} on X_t such that $\text{deg } \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_t^{(i)}} = 0$ for all i . Recall that the degree of an invertible sheaf \mathcal{L} on a complete curve Z over a field is defined to be $\chi(Z, \mathcal{L}) - \chi(Z, \mathcal{O}_Z)$.

Lemma 3.1. *Suppose $f : X \rightarrow B$ is cohomologically flat. Let $X_t = m_t \bar{X}_t$. Then the restriction homomorphism*

$$\text{Pic}_{X_t/t} \rightarrow \text{Pic}_{\bar{X}_t/t}$$

is an isomorphism.

Proof. By one of the properties of cohomological flatness, we have $h^0(X_t, \mathcal{O}_{X_t}) = 1$. Since the Euler-Poincaré characteristic of the structure sheaves of geometric fibers is constant, we obtain $h^1(X_t, \mathcal{O}_{X_t}) = 1$. It is known that the linear space $H^1(X_t, \mathcal{O}_{X_t})$

has a structure of a Lie algebra isomorphic to the Lie algebra of the Picard scheme of X_t . Thus $\dim \text{Pic}_{X_t/t} = 1$. The kernel of the restriction homomorphism is a unipotent algebraic group equal to an extension of vector spaces of the cohomology groups of some nilpotent sheaves on X_t (see [24, Section 6, Corollaire]). Then, $\dim \text{Pic}_{X_t/t} = 1$ shows that the dimension of these spaces is zero, and hence the kernel is trivial. Since $\dim \text{Pic}_{\bar{X}_t/t} = 1$, this proves the claim. \square

We denote by $\mathcal{P}'_{X/B}$ (resp. $\mathcal{P}^0_{X/B}$) the subsheaf of $\mathcal{P}_{X/B}$ whose values on any $T \rightarrow B$ are elements of $\mathcal{P}_{X/B}(T)$ such that their restriction to any fiber of $X_T \rightarrow T$ are the isomorphism classes of invertible sheaves of degree 0 (resp. degree 0 on each irreducible component of the fiber). In particular, the fiber of $\mathcal{P}^0_{X/B}$ over any point $t \in B$ is representable by the jacobian variety $J(X_t)$.

It is a natural guess that the Picard functor $\mathcal{P}'_{X/B}$ (resp. $\mathcal{P}^0_{X/B}$) must be related to the group scheme $J^\sharp(X) \rightarrow B$ (resp. its connected component of identity). However, neither of these functors is representable by a separated group scheme over B , unless all fibers of f are non-multiple. The reason is very simple: If f admits a multiple fiber mF , then the restriction map $\mathcal{P}_{X/B}(B) \rightarrow \mathcal{P}_{X/B}(\text{Spec } k(B))$ is not injective, since the ideal sheaf of F is in its kernel.

To remedy this situation, Raynaud considers the subsheaf $\mathcal{E}_{X/B}$ of $\mathcal{P}_{X/B}$ whose sections over an open subset U (in the étale topology) restrict to the identity section of $\mathcal{P}_{X/B}$ over the generic point of U . Then he proves that the quotient sheaf $\mathcal{Q}_{X/B} = \mathcal{P}_{X/B}/\mathcal{E}_{X/B}$ is the maximal separated quotient sheaf of $\mathcal{P}_{X/B}$ and it is representable by a separated group scheme, locally of finite type over B . We have an exact sequence of sheaves of abelian groups

$$0 \rightarrow \mathcal{E}_{X/B} \rightarrow \mathcal{P}_{X/B} \rightarrow \mathcal{Q}_{X/B} \rightarrow 0. \tag{3.1}$$

Let $\mathcal{Q}'_{X/B}$ (resp. $\mathcal{Q}^0_{X/B}$) be the image of $\mathcal{P}'_{X/B}$ (resp. $\mathcal{P}^0_{X/B}$). Then Raynaud proves that $\mathcal{Q}'_{X/B}$ is representable by the group scheme $J^\sharp(X)$ and we have $\mathcal{Q}^0_{X/B} = (J^\sharp(X))^0$.

For any closed point t of B , let $B(t) = \text{Spec } \mathcal{O}^h_{B,t}$ be the spectrum of the henselization of the local ring $\mathcal{O}_{B,t}$ (or its formal completion) and $f(t) : X(t) = X \times_B B(t) \rightarrow B(t)$ be the base change morphism. Since the base change is formally étale, the scheme $X(t)$ is regular and $X(t) \rightarrow B(t)$ is a genus one fibration with local base $B(t)$. The fiber over the closed point of $B(t)$ is isomorphic to the fiber X_t under the projection $X(t) \rightarrow X$. We denote by η_t the generic point of $B(t)$ and by K_t its residue field, the field of fractions of $\mathcal{O}^h_{B,t}$.

Let D_t be the free abelian group generated by the irreducible components $X_t^{(i)}$ of the fiber X_t over the closed point of $B(t)$. Let $X_t^{(i)} \cdot X_t^{(j)}$ be equal to the degree of $\mathcal{O}_{X_t^{(i)}}(j)$ on $X_t^{(i)}$. One shows that the matrix $(X_t^{(i)} \cdot X_t^{(j)})$ defines a symmetric bilinear form on D_t . Let $i : D_t \rightarrow \text{Pic}(X(t))$ be the natural map that sends a component $X_t^{(i)}$ to the isomorphism class of the invertible sheaf $\mathcal{O}_{X(t)}(X_t^{(i)})$. Its kernel is generated by $[X_t] = \sum m_t [X_t^{(i)}]$ and its image \bar{D}_t is a finitely generated subgroup of $\text{Pic}(X(t))$ with

torsion subgroup of order m_t generated by $\mathcal{O}_{X(t)}(\bar{X}_t)$. The symmetric bilinear form on D_t defines a symmetric bilinear form on \bar{D}_t/Tors and equips it with a structure of an even negative definite quadratic lattice, a root lattice of type A_n, D_n, E_n , where n is equal to the subscript in the notation of the type of the fiber.

Let Disc_t denote the discriminant group of this lattice. It is a finite abelian group isomorphic to the group of symmetries of the set of irreducible components of X_t preserving the incidence relation. For example, if X_t is of type \tilde{A}_n , the discriminant group is isomorphic to $\mathbb{Z}/(n+1)\mathbb{Z}$ and if X_t is of type \tilde{E}_8 , then Disc_t is trivial. The order of Disc_t is equal to the number of reduced components of \bar{X}_t .

Now we are ready to describe the sheaf $\mathcal{E}_{X/B}$.

Proposition 3.2. *Assume that $B = B(t)$ is local and $f : X = X(t) \rightarrow B$ is cohomologically flat. Then*

- (1) $(\mathcal{E}_{X/B})_t \cong \bar{D}_t$,
- (2) $(\mathcal{E}_{X/B})_t \cap (\mathcal{P}_{X/B}^0)_t = \text{Tors}(\bar{D}_t) \cong \mathbb{Z}/m_t\mathbb{Z}$,
- (3) $(\mathcal{Q}'_{X/B})_t / (\mathcal{Q}_{X/B}^0)_t \cong \text{Disc}_t$,
- (4) $\mathcal{Q}'_{X/B}$ is the Néron model of $J(X_\eta)$, i.e. it coincides with the sheaf $(i_\eta)_*(J(X_\eta))$, where $i_\eta : \eta \hookrightarrow B$ is the inclusion morphism of the generic point. In particular $\mathcal{Q}'_{X/B}(B) = J(X_\eta)(K)$.

In particular, there is a natural isomorphism

$$\Phi_t : \text{Disc}_t \rightarrow J(t)^\sharp / J(t)^0. \tag{3.2}$$

We note that one can prove the existence of the isomorphism Φ_t without the assumption that f is cohomologically flat [2, Theorem 9.6.1]. This yields the fundamental fact that the types of fibers of $f : X \rightarrow B$ and $J(f) : J(X) \rightarrow B$ over the same point are the same even for multiple fibers.

Suppose $g \in \text{Aut}(X)$ preserves the genus one fibration (this is automatic if B is local). Assume first that g acts identically on the base B , i.e. $g \in \text{Aut}(X/B)$. Then, for any $T \rightarrow B$, the automorphism g defines an automorphism $g \times \text{id}_T : X_T \rightarrow X_T$ and hence a homomorphism $g^* : \text{Pic}(X_T) \rightarrow \text{Pic}(X_T)$. Passing to the associated sheaf we obtain an automorphism of sheaves $g^* : \mathcal{P}_{X/B} \rightarrow \mathcal{P}_{X/B}$. If g does not act identically on B we modify the definition of g^* as follows. Let $h : B \rightarrow B$ be the image of g in $\text{Aut}(B)$. Then h defines an isomorphism of sheaves $\mathcal{P}_{X/B} \rightarrow h^*(\mathcal{P}_{X/B})$. On the other hand, we have a canonical isomorphism $h^*(\mathcal{P}_{X/B}) \rightarrow \mathcal{P}_{X'/B}$, where $X' \rightarrow B$ denotes the base change with respect to $h : B \rightarrow B$ ([17, Example 9.4.4]). Now, for any $T \rightarrow B$, the map $X_T \rightarrow X'_T$ defines a functorial isomorphism $g_T^* : \text{Pic}(X'_T) \rightarrow \text{Pic}(X_T)$, hence an isomorphism of sheaves $g' : \mathcal{P}_{X'/B} \rightarrow \mathcal{P}_{X/B}$. Composing it with h^* , we get an isomorphism of sheaves

$$g^* : \mathcal{P}_{X/B} \rightarrow \mathcal{P}_{X/B}.$$

Obviously, g^* leaves invariant the subsheaves $\mathcal{P}'_{X/B}$ and $\mathcal{E}_{X/B}$ and hence defines an isomorphism of group schemes $\varphi(g) : J^\sharp(X) = \mathcal{Q}'_{X/B} \rightarrow \mathcal{Q}'_{X/B} = J^\sharp(X)$. If f is quasi-elliptic, $\varphi(g)$ extends to an isomorphism at the generic point of the curve \mathcal{C} of cusps on the regular surface $J(X)$. Thus $\varphi(g)$ is an isomorphism on an open subset of $J(X)$ whose complement is supported in a finite set of singular fibers. Since f is a relatively minimal model, it is easy to see that $\varphi(g)$ extends to an automorphism of X .

Therefore, denoting by $\text{Aut}_f(X)$ the group of automorphisms of X preserving the fibration f , we obtain a homomorphism

$$\varphi : \text{Aut}_f(X) \rightarrow \text{Aut}_{J(f)}(J(X)). \tag{3.3}$$

The main tools for our analysis of automorphisms of non-jacobian elliptic surfaces are the following properties of the homomorphism φ . Recall that for a surface X , the notation $\text{Num}(X)$ denotes the group of numerical equivalence classes of divisors on X .

Theorem 3.3. *Let $f : X \rightarrow B$ be a genus one surface with jacobian fibration $J(f) : J(X) \rightarrow B$ and let $\text{Aut}_f(X)$ be the group of automorphisms of X preserving f . Assume that f is cohomologically flat. Then, there is a homomorphism $\varphi : \text{Aut}_f(X) \rightarrow \text{Aut}_{J(f)}(J(X))$ satisfying the following properties, where $g \in \text{Aut}_f(X)$:*

- (1) Both g and $\varphi(g)$ induce the same automorphism of B .
- (2) $\text{Ker}(\varphi) \cong \text{MW}(J(f))$.
- (3) $\varphi(g)$ preserves the zero section of $J(f) : J(X) \rightarrow B$.
- (4) If g acts trivially on $\text{Num}(X)$, then $\varphi(g)$ acts trivially on $\text{Num}(J(X))$.
- (5) Let mF_0 be a fiber of f of multiplicity m and let $(J_0^\sharp)^0$ be the identity component of the smooth part J_0^\sharp of the corresponding fiber J_0 of $J(f)$. If g preserves F_0 , then either $\varphi(g)$ acts trivially on $(J_0^\sharp)^0$ or one of the following holds, where $n = \text{ord}(\varphi(g)|_{(J_0^\sharp)^0})$:
 - (a) F_0 is smooth, $m = n = 3$, $p \neq 3$.
 - (b) F_0 is smooth, $m = 2, n \in \{2, 4\}, p \neq 2$.
 - (c) F_0 is smooth and ordinary, $m = n = p = 2$.
 - (d) F_0 is an irreducible nodal curve, $n = m = 2, p \neq 2$.
 - (e) F_0 is of type \tilde{A}_1 , $n = m = 2, p \neq 2$.

Proof. We define φ as explained right before the theorem. Then, Property (1) follows from the construction of φ .

To see (2), we use that $\text{Ker}(\varphi)$ acts trivially on B by Property (1). Moreover, $g \in \text{Ker}(\varphi)$ is an automorphism of X_η over η that acts identically on its jacobian J_η^\sharp . It is well-known that this action is trivial if and only if g comes from a translation, that is, if and only if $g \in \text{MW}(J(f))$.

To see Property (3) we use that the action of $\varphi(g)$ on $J^\sharp(X)$ is via the action of g on $\mathcal{P}'_{X/B}$ that obviously preserves the zero section. Now g preserves the subsheaf $\mathcal{E}_{X/B}$

and the action of g on $\mathcal{P}'_{X/B}(B)$ defines the action of $\varphi(G)$ on $\mathcal{Q}'_{X/B}(B) = J^\sharp(X)(B) = J_\eta(K) = \text{MW}(J(f))$.

Let us prove Property (4). We have $\text{MW}(J(f)) = J_\eta(K) = \mathcal{Q}'_{X/B}(B)$. Since the $H^i_{\text{et}}(B, \mathcal{E}_{X/B})$ are torsion groups, the short exact sequence (3.1) induces an isomorphism $\mathcal{P}'_{X/B}(B) \otimes \mathbb{Q} \cong \mathcal{Q}'_{X/B}(B) \otimes \mathbb{Q}$. Next, we use that the Brauer group of B is trivial by Tsen's Theorem, so that we have $\mathcal{P}'_{X/B}(B) \subseteq \mathcal{P}_{X/B}(B) = \text{Pic}(X)/f^* \text{Pic}(B)$ and hence $\text{MW}(J(f)) \otimes \mathbb{Q}$ is a subquotient of $\text{Num}(X) \otimes \mathbb{Q}$ in a natural way. In particular, if g acts trivially on $\text{Num}(X)$, then $\varphi(g)$ acts trivially on $\text{MW}(J(f)) \otimes \mathbb{Q}$. We also know that g preserves all reducible fibers X_t and hence $\varphi(g)$ acts on J_t^\sharp . Now, since g acts trivially on the discriminant group Disc_t , the bijectivity of the map (3.2) implies that $\varphi(g)$ acts trivially on the discriminant groups of reducible fibers of $J(f)$, and hence $\varphi(g)$ preserves all components of reducible fibers and acts trivially on $\text{MW}(J(f)) \otimes \mathbb{Q}$. Since these divisor classes generate $\text{Num}(J(X)) \otimes \mathbb{Q}$, we have shown that $\varphi(g)$ acts trivially on $\text{Num}(J(X)) \otimes \mathbb{Q}$ and hence also on $\text{Num}(J(X))$ itself.

To prove (5), we use that there is an isogeny $\lambda : J(F_0) = J(mF_0) = (\mathcal{P}_{X/B})_0^0 \rightarrow (\mathcal{Q}'_{X/B})_0^0 = (J_0^\sharp)^0$ with kernel $\mathbb{Z}/m\mathbb{Z}$ by Proposition 3.2 and Lemma 3.1, and that, by our construction of φ , the automorphism $\varphi(g)|_{(J_0^\sharp)^0}$ comes from an automorphism h of $J(mF_0)$. Moreover, since g preserves F_0 , the automorphism h fixes the m -torsion point on $J(F_0) = J(mF_0)$ corresponding to the sheaf $\mathcal{O}_{mF_0}(F_0)$, which generates the kernel of λ . Note that $\varphi(g)|_{(J_0^\sharp)^0}$ and h have the same order, because h is not a translation. Since h fixes an m -torsion point, we have $m \leq 3$ by Lemma 2.5. More precisely, by the same Lemma, we get the following cases in which $\varphi(g)$ can be non-trivial:

If $m = 3$, then $\text{ord}(h) = 3$, $p \neq 3$, and $J(mF_0)$ is an elliptic curve with absolute invariant 0. If $m = 2$ and F_0 is an elliptic curve, then either $\text{ord}(h) = 4$, $p \neq 2$, and $J(mF_0)$ is an elliptic curve with absolute invariant 1, or $\text{ord}(h) = 2$ and $J(mF_0)$ is an elliptic curve which satisfies $j \neq 0$ if $p = 2$. Finally, if $m = 2$ and F_0 is multiplicative, then $\text{ord}(h) = 2$, $p \neq 2$ and h interchanges the two boundary points in the regular compactification of $(J_0^\sharp)^0$. But since $\varphi(g)$ preserves the irreducible components of J_0^\sharp , the automorphism h can interchange the two boundary points only if J_0^\sharp has at most two components, that is, only if F_0 is of type \tilde{A}_0^* or \tilde{A}_1 . \square

4. The subgroup $\text{Aut}_E(S)$

From now on, unless specified otherwise, we will consider a rational genus one surfaces $f : S \rightarrow \mathbb{P}^1$ of index $m \geq 1$ and we let mF_0 be its multiple fiber (or any fiber if $m = 1$).

Fix a (-1) -curve E on S considered as an m -section and let $\text{Aut}_E(S)$ be the stabilizer subgroup of E in $\text{Aut}(S)$. If $m = 1$, then for any $g \in \text{Aut}(S)$ there exists a unique element E' in the Mordell-Weil group such $g(E) = t_{E'}(E)$, where $t_{E'}$ is the translation automorphism of S defined by the section E' . This shows that

$$\text{Aut}(S) = \text{MW}(f) \rtimes \text{Aut}_E(S). \tag{4.1}$$

Moreover, since $MW(f)$ acts transitively on the set of sections, we obtain that all subgroups of the form $\text{Aut}_E(S)$ for different choices of E are conjugate by an element from $MW(f)$. We refer the reader to [14] and [15] for a classification of automorphism groups of jacobian rational elliptic surfaces over the complex numbers based on the decomposition (4.1).

However, if $m > 1$, (4.1) may not be true anymore because $\text{Aut}(S)$ may not act transitively on the set of (-1) -curves on S . Nevertheless, by Proposition 3.3, we have an exact sequence

$$0 \rightarrow MW(J(f)) \cap \text{Aut}_E(S) \rightarrow \text{Aut}_E(S) \xrightarrow{\varphi} \text{Aut}_0(J(S)),$$

where $J(f) : J(S) \rightarrow \mathbb{P}^1$ is the jacobian fibration and $\text{Aut}_0(J)$ is the group of automorphisms fixing its zero section. It is not clear from Proposition 3.3 which conjugacy classes of subgroups of $\text{Aut}_0(J(S))$ are realized as $\varphi(\text{Aut}_E(S))$.

Choose a geometric marking on S and let $\text{Aut}(S)^*$ be the image of $\text{Aut}(S)$ in $W(\mathbb{E}_9)$. A choice of a (-1) -curve E defines a blow-down $S \rightarrow \mathcal{D}$ to a weak del Pezzo surface \mathcal{D} of degree 1. This gives an embedding of $K_{\mathcal{D}}^\perp = \mathbb{E}_8$ in \mathbb{E}_9 and gives a splitting of the surjection $W(\mathbb{E}_9) \rightarrow W(\mathbb{E}_8)$. This yields a well-known expression of $W(\mathbb{E}_9)$ as the semi-direct product

$$W(\mathbb{E}_9) = \mathbb{E}_8 \rtimes W(\mathbb{E}_8). \tag{4.2}$$

So, we see that there is a natural homomorphism $\text{Aut}_E(S) \rightarrow W(\mathbb{E}_8)$ whose kernel is the subgroup $\text{Aut}_{\text{ct}}(S)$. Since $W(\mathbb{E}_8)$ is finite, we have the following immediate corollary.

Proposition 4.1. *The group $\text{Aut}_E(S)$ is finite if and only if $\text{Aut}_{\text{ct}}(S)$ is finite. The quotient group $\text{Aut}_E(S)/\text{Aut}_{\text{ct}}(S)$ is isomorphic to a finite subgroup of $W(\mathbb{E}_8)$.*

As explained above, $\text{Aut}_E(S)$ can be realized as a subgroup of the group of automorphisms of a weak del Pezzo surface \mathcal{D} . In fact, we can say more about this surface \mathcal{D} :

Let G be a finite group. Recall that a G -surface is a pair (X, G) consisting of a smooth projective surface X with a faithful action of a finite group G . The G -surfaces form, naturally, a category with morphisms being G -equivariant morphisms of the underlying surfaces. A G -surface is called minimal if any birational morphism $(X, G) \rightarrow (X', G)$ is an isomorphism. If X is a rational surface, then there is a birational morphism $(X, G) \rightarrow (X_0, G)$, where X_0 is either a del Pezzo surface with $\text{Pic}(X_0)^G = \mathbb{Z}K_{X_0}$ or X_0 admits a structure of a conic bundle $X_0 \rightarrow \mathbb{P}^1$ with $\text{Pic}(X_0)^G \cong \mathbb{Z}^2$. This is all well-known and used, for example, in the classification of conjugacy classes of finite subgroups of the plane Cremona group (see [6]). Applying this to our case, we see that any finite subgroup G of $\text{Aut}_E(S)$ is a lift of a finite subgroup G of a minimal del Pezzo G -surface (\mathcal{D}, G) . Even more, we know that G preserves E , and hence it fixes the image of E in \mathcal{D} . If $p = 0$ or sufficiently large, all such groups were classified in [8].

Elaborating on the ideas above, we obtain the following.

Theorem 4.2. *Let $f : S \rightarrow \mathbb{P}^1$ be a rational genus one surface of index $m \geq 1$ and let E be a (-1) -curve on S . Then, the following hold:*

- (1) *There is a weak del Pezzo surface \mathcal{D} and a birational morphism $\phi : S \rightarrow \mathcal{D}$ such that $\text{Aut}_E(S)$ is a lift of a subgroup G of $\text{Aut}(\mathcal{D})$ that fixes a point on \mathcal{D} . If G is finite, then (\mathcal{D}, G) can be taken to be a minimal G -surface.*
- (2) *If $m = 1$, then one can choose the weak del Pezzo surface \mathcal{D} in (1) such that $\text{Aut}_E(S) = \text{Aut}(\mathcal{D})$.*
- (3) *If $m > 1$, let mF be the multiple fiber of f . Then, there is a jacobian rational genus one surface $f' : S' \rightarrow \mathbb{P}^1$ which admits a fiber F' of the same type as mF such that $\text{Aut}_E(S) \subseteq \text{Aut}_0(S')$ is the stabilizer of a point P of exact order m on the identity component of F' . Moreover, $\text{Aut}_{\text{ct}}(S)$ is the stabilizer of P for the action of $\text{Aut}_{\text{ct}}(S')$.*

Proof. Part (1) is immediate from the discussion before the theorem.

As for (2) and (3), let $\pi : S \rightarrow \mathcal{D}$ be the blow-down of E and let $\pi' : S' \rightarrow \mathcal{D}$ be the blow-up of the unique base point Q of $|-K_{\mathcal{D}}|$. In particular, the anticanonical system $|-K_{S'}|$ induces a jacobian genus one fibration $f' : S' \rightarrow \mathbb{P}^1$ on S' . Moreover, the group $\text{Aut}_E(S)$ is identified with the stabilizer of the image P of E in \mathcal{D} . Since Q is the base-point of $|-K_{\mathcal{D}}|$, the $\text{Aut}(\mathcal{D})$ -action on \mathcal{D} lifts to S' and the lifted action factors through $\text{Aut}_0(S')$. If $m = 1$, then $P = Q$ and we get Claim (2). Hence, we may assume $m > 1$ in the following.

Note that the lifting of the $\text{Aut}(\mathcal{D})$ -action to S' identifies $\text{Aut}_E(S)$ with the stabilizer of the point $\pi'^{-1}(P)$ with respect to the action of $\text{Aut}_0(S')$. Moreover, we have $F \in |-K_S|$ by the canonical bundle formula, hence the strict transform F' of $\pi(F)$ in S' is a member of the anti-canonical system and therefore F' is a fiber of f' .

Finally, we have the following identities of normal bundles, where we use that π and π' restrict to isomorphisms on F and F' , respectively, where $O = F' \cap \pi'^{-1}(Q)$, and where $m \odot \pi'^{-1}(P)$ is m times the point $\pi'^{-1}(P)$ in the group law on the identity component of F' with neutral element O :

$$\begin{aligned} \mathcal{O}_F &= N_{F/S}^{\otimes m} = \pi^*(N_{\pi(F)/\mathcal{D}} \otimes \mathcal{O}_{\pi(F)}(-P))^{\otimes m} = \pi^*(\pi'_*(N_{F'/S'} \otimes \mathcal{O}_{F'}(O - \pi'^{-1}(P))))^{\otimes m} \\ &= \pi^*(\pi'_*(\mathcal{O}_{F'}(mO - m\pi'^{-1}(P)))) = \pi^*(\pi'_*(\mathcal{O}_{F'}(O - m \odot \pi'^{-1}(P)))). \end{aligned}$$

This shows that $\pi'^{-1}(P)$ is an m -torsion point on the identity component of F' . Similarly, one shows that $\pi'^{-1}(P)$ has exact order m .

To finish the proof, we have to show that $g \in \text{Aut}_E(S)$ is cohomologically trivial if and only if it acts as a cohomologically trivial automorphism on S' . But this is clear, since $\text{Pic}(S) = \mathbb{Z} \cdot E \oplus \pi^* \text{Pic}(\mathcal{D})$ and $\text{Pic}(S') = \mathbb{Z} \cdot \pi'^{-1}(Q) \oplus \pi^* \text{Pic}(\mathcal{D})$, so g is cohomologically trivial on S and S' if and only if it acts trivially on $\text{Pic}(\mathcal{D})$. \square

Example 4.3. In particular, if $f : S \rightarrow \mathbb{P}^1$ is jacobian without reducible fibers, then Theorem 4.2 identifies $\text{Aut}_E(S)$ with the automorphism group $\text{Aut}(\mathcal{D})$ of a del Pezzo surface \mathcal{D} of degree 1. The list of all possible finite groups acting on a del Pezzo surface of degree 1 was essentially known since the 19th century. It is contained in [6] and [7, 8.8.4] (under an assumption on the characteristic).

Remark 4.4. We remark that Theorem 4.2 can be generalized to automorphism schemes. In particular, every rational genus one surface with global vector fields is a blow-up of one of the surfaces listed in [22, Table 6].

Remark 4.5. The situation in Theorem 4.2 (3) can be reversed, that is, if we start with a jacobian rational genus one surface, contract a section, and blow-up a point of exact order m on the image of a fiber F' , we get another rational genus one surface with a fiber of multiplicity m and of the same type as F' . This process is called an *Halphen transform* and we refer the reader to [11] for further details.

5. The subgroup $\text{Aut}_{\text{tr}}(S)$

Let $J(f) : J(S) \rightarrow \mathbb{P}^1$ be the jacobian fibration of the rational genus one surface $f : S \rightarrow \mathbb{P}^1$ of index m . Recall that F_0 is a curve on S such that mF_0 is a fiber of f . Let $\text{MW}(J(f))$ be the Mordell-Weil group of sections of $J(f)$. We consider the root lattice $L := F_0^\perp / \mathbb{Z}F_0 \cong E_8$ and its sublattice T generated by irreducible components of fibers of f . Note that $\text{Disc}(T) \cong \bigoplus_{t \in \mathbb{P}^1} \text{Disc}_t$. We identify $\text{MW}(J(f))$ with the group $\text{Pic}(S_\eta)^0$ of divisors of degree 0 of the generic fiber S_η defined over $K = \mathbb{k}(\eta)$ (i.e. the group of rational points of the jacobian variety J_η of S_η).

Lemma 5.1. *Let $\Phi : L \rightarrow \text{MW}(J(f))$ be the restriction map. Then, Φ induces an isomorphism $L/T \cong \text{MW}(J(f))$.*

Proof. Since the kernel of Φ is T by definition, it suffices to observe that Φ is surjective: Write $D \in \text{Pic}(S_\eta)^0$ as $D = D' - D''$ with D' and D'' effective. Then $\overline{D'} - \overline{D''} \in L$ and $\Phi(\overline{D'} - \overline{D''}) = D$. \square

Set $M := T^\perp \subseteq L$. Since $M \cap T = \{0\}$, we can consider M as a subgroup of $\text{MW}(J(f))$ via Φ . For $A \in M$, we let $\tau_A \in \text{Aut}_{\text{tr}}(S)$ be the automorphism of S induced by $\Phi(A)$.

Proposition 5.2. *Let $M' \subseteq M$ be the subgroup of elements A such that τ_A preserves all components of fibers of f . Let d be the smallest positive integer such that all the Disc_t are d -torsion. Then,*

$$(1) \quad d \cdot \text{MW}(J(f)) \subseteq M',$$

(2) for all $A \in M'$ and $D \in \text{Pic}(S)$, we have

$$\tau_A^*(D) = D + m(D.F_0)A - \left(\frac{1}{2}m^2(D.F_0)A^2 + mA.D\right)F_0 \tag{5.1}$$

Proof. Since the group of symmetries of the fiber over $t \in \mathbb{P}^1$ is Disc_t and multiplication by d kills every Disc_t , every element in $d \cdot \text{MW}(J(f))$ preserves all components of all fibers of f . Hence, it suffices to show $d \cdot \text{MW}(J(f)) \subseteq M$. Since $\text{Disc}(T)$ is d -torsion and L is unimodular, $\text{Disc}(M)$ is also d -torsion and thus so is $\text{Disc}(M \oplus T)$. This implies that $d \cdot L \subseteq M \oplus T$, hence $d \cdot L/T = d \cdot \text{MW}(J(f)) \subseteq M$. This proves Claim (1).

Now, if $A \in M'$ and $D \in \text{Pic}(S)$, then

$$\tau_A^*(D)|_{S_\eta} = D|_{S_\eta} + \text{deg}(D|_{S_\eta})A|_{S_\eta} = (D + m(D.F_0)A)|_{S_\eta},$$

hence

$$\tau_A^*(D) = D + m(D.F_0)A + \sum a_i R_i$$

for some a_i , where the R_i are components of fibers of f . Let R be a (-2) -curve on S . By definition of M' , we have $\tau_A^*(R) = R$ and $A.R = 0$, hence

$$D.R = \tau_A^*(D).\tau_A^*(R) = \tau_A^*(D).R = D.R + \sum a_i R_i.R,$$

so that $\sum a_i R_i.R = 0$. Since the radical of the span of the R_i is generated by F_0 , this implies $\sum a_i R_i = nF_0$ for some n . To calculate n , we compute

$$D^2 = (\tau_A^*(D))^2 = D^2 + m^2(D.F_0)^2 A^2 + 2m(D.F_0)A.D + 2n(D.F_0).$$

Hence, we have

$$n = -\left(\frac{1}{2}m^2(D.F_0)A^2 + mA.D\right),$$

which is what we wanted to prove. \square

Remark 5.3. In the case where f has only irreducible fibers, that is, if $d = 1$, then Formula (5.1) was first obtained by M. Gizatullin [10, Proposition 9] and its proof is also reproduced in [3, 2.4].

Remark 5.4. It follows from Formula (5.1) that the group $\text{MW}(J(f)) \cap \text{Aut}_{\text{ct}}(S)$ is d -torsion. We will refine this observation in Lemma 6.5.

Corollary 5.5. *The group $d \cdot \text{Aut}_{\text{tr}}(S)$ is contained in the kernel E_8 of the restriction map $W(E_9) \rightarrow W(E_8)$.*

Proof. It suffices to observe that $d \cdot \text{Aut}_{\text{tr}}(S)$ acts trivially on L . This follows immediately from Formula (5.1), since

$$\tau_A^*(D) = D + m(A.D)F_0$$

for all $\tau_A \in d \cdot \text{Aut}_{\text{tr}}(S)$ and $D \in F_0^\perp$. \square

Remark 5.6. Suppose $d = 1$ and take $\alpha \in M$ to be a root. Then $\alpha' = -F_0 - \alpha$ is also a root, and one checks that $\tau_\alpha = (s_\alpha \circ s_{\alpha'})^m$, where s_β is the reflection $x \mapsto x + (x, \beta)\beta$ with respect to a root β .

Remark 5.7. If $d = 1$, then Formula (5.1) shows that $\text{Aut}_{\text{tr}}(S)$ is identified with the m -congruence subgroup mE_8 of $W(E_9)$. The reader should compare this with the corresponding result for the stabilizer of a genus one fibration on a very general Enriques or Coble surface (see e.g. [1, p. 397]).

6. The subgroup $\text{Aut}_{\text{ct}}(S)$

In this section, we will study the group $\text{Aut}_{\text{ct}}(S)$ (whose elements are called cohomologically trivial automorphisms) for a rational genus one fibration $f : S \rightarrow \mathbb{P}^1$. We use the notation of Section 2.3.

Lemma 6.1. *Let $\pi : S \rightarrow \mathbb{P}^2$ be a realization of S as a blow-up of \mathbb{P}^2 . Assume that $\text{Aut}_{\text{ct}}(S)$ is non-trivial. Then, the following hold:*

- (1) *The action of $\text{Aut}_{\text{ct}}(S)$ on S descends to \mathbb{P}^2 .*
- (2) *The induced action on \mathbb{P}^2 preserves the irreducible components of the members of the Halphen pencil \mathcal{P} corresponding to f that are either reducible or singular at some q_i .*
- (3) *If q_1, \dots, q_s are the non-infinitely near base points of \mathcal{P} , then $s \leq 4$.*
- (4) *If $s = 4$, then we may assume that q_1, q_2, q_3 lie on a line ℓ whose strict transform $\tilde{\ell}$ on S is a (-2) -curve. Moreover, there is a realization $\pi' : S \rightarrow \mathbb{P}^2$ such that either $s \leq 3$ or $k_4 > \max\{k_1, k_2, k_3\}$, where k_i is the number of base points of \mathcal{P} infinitely near to q_i .*
- (5) *One of the fibers of f has at least four irreducible components.*

Proof. Claim (1) and (2) follow immediately from the fact that $\text{Aut}_{\text{ct}}(S)$ preserves all negative curves on S . As for Claim (3), note that by Lemma 2.4 at least $s - 1$ of the q_i lie on a line ℓ . Thus, $\tilde{\ell}$ has self-intersection at most $2 - s$. But $\tilde{\ell}^2 \geq -2$, hence $s \leq 4$ and if $s = 4$ then $\tilde{\ell}^2 = -2$.

As for the second part of Claim (4), assume $s = 4$ and $k_1 \leq k_2 \leq k_3$. Compose $\pi : S \rightarrow \mathbb{P}^2$ with the quadratic Cremona transformation $\iota : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ with fundamental

points q_1, q_2, q_4 , set $\pi' := \iota \circ \pi$, and let \mathcal{P}' be the induced Halphen pencil. Then, either \mathcal{P}' has less than four non-infinitely near base points or the multiplicity of the base point q_3 of \mathcal{P}' is $k_3 + 1$. Under the Cremona transformation we get the new base points q'_1, q'_2, q'_3, q'_4 with the first three aligned and $(k'_1, k'_2, k'_3, k'_4) = (k_1, k_2, k_4 - 1, k_3 + 1)$. We may thus assume

$$k_4 > \max\{k_1, k_2, k_3\}.$$

To prove Claim (5), observe first that if $k_i \geq 4$ for some k_i , then the (-2) -curves blown down to q_i form a chain of at least 3 curves that enter in a fiber with at least 4 irreducible components. We may thus assume that $s \geq 3$ and $k_i < 4$ for all i . Moreover, as f and $J(f)$ have the same types of singular fibers, we may assume that f admits a section.

If $s = 4$, then the (-2) -curve $\tilde{\ell}$ is an irreducible component of some fiber that contains also $k = k_1 + k_2 + k_3 - 3$ irreducible components that are blown down to the points q_1, q_2, q_3 on the line ℓ . Since $k_4 < 4$, we have $k \geq 3$, which yields the claim. The same argument applies if $s = 3$ and the three base points are collinear.

If $s = 3$, the q_i are not collinear, and $(k_1, k_2, k_3) = (3, 3, 3)$, we can apply Lemma 2.4 to deduce that for some q_i , say q_1 , the only $\text{Aut}_{\text{ct}}(S)$ -invariant lines through q_1 are $\ell_{1i} := \langle q_1, q_i \rangle$, $i = 2, 3$. Then, we may assume that the tangent line to a general member of the pencil at q_1 equals ℓ_{12} , so that the strict transform $\tilde{\ell}_{12}$ of ℓ_{12} on S is a (-2) -curve. The fiber of f containing $\tilde{\ell}_{12}$ has at least 4 components. \square

6.1. Cohomologically trivial automorphisms of jacobian rational genus one surfaces

In this section, we calculate the group $\text{Aut}_{\text{ct}}(S)$ for jacobian rational genus one surfaces $f : J \rightarrow \mathbb{P}^1$.

Proposition 6.2. *Let $f : J \rightarrow \mathbb{P}^1$ be a jacobian rational genus one surface. If the singular fibers of f are as in Column 2 of Table 3 and either f is extremal or $\text{Aut}_{\text{ct}}(J)$ is as in Column 5 and 6 of Table 3, then f admits one of the Weierstrass equations in Column 4.*

Conversely, a surface with the given Weierstrass equation admits the group $\text{Aut}_{\text{ct}}(J)$ and the number of moduli given in Column 5, 6, and 7. If f is extremal and does not occur in Table 3, then $\text{Aut}_{\text{ct}}(J)$ is trivial.

Proof. For the first statement, note that the equations for the extremal fibrations are well-known and we refer the reader to [23], [20], [21], [12], and [13]. For the non-extremal fibrations, it follows from the Table that $\text{Aut}_{\text{ct}}(J)^\dagger = \{\text{id}\}$. Then, in characteristic $p \neq 2, 3$, finding the Weierstrass equations is a simple exercise using Tate’s algorithm while guaranteeing that the discriminant of f admits the stated symmetries. The only case where this approach is complicated is Case 22. In this case, note that $J/\text{Aut}_{\text{ct}}(J)$ is an elliptic surface with singular fibers $\tilde{D}_4, \tilde{A}_3, \tilde{A}_0, \tilde{A}_0$. This fibration admits a 2-torsion

Table 2
Automorphisms in $\text{Aut}_{\text{ct}}(J) \setminus \text{Aut}_{\text{ct}}(J)^\dagger$.

Case	t	x	y	Action on fiber at $t = \infty$	
5, 8	$a^6 t + b^p$	$a^2 x + 2b$	$a^3 y + 3b$	$x \mapsto a^{-10} x$	$y \mapsto a^{-15} y$
4, 7	$t + b^p + b$	$x + 2b$	$y + 3b$	$x \mapsto x$	$y \mapsto y$
10	$t + b^2 + b$	x	$y + bx$	$x \mapsto x$	$y \mapsto y$
11	$t + b^2$	x	$y + bx + b$	$x \mapsto x$	$y \mapsto y$
1, 13, 23, 31	$a^i t$	$a^2 x$	$a^3 y$	$x \mapsto a^{2-2i} x$	$y \mapsto a^{3-3i} y$
2, 9	$-t$	$-x$	$\sqrt{-1}y$	$x \mapsto -x$	$y \mapsto -\sqrt{-1}y$
21, 22, 24, 25, 26, 27, 28, 29	$-t$	x	$-y$	$x \mapsto x$	$y \mapsto y$
12, 15, 18	$\zeta_3 t$	$\zeta_3^2 x$	y	$x \mapsto x$	$y \mapsto y$
19, 20	$\zeta_3 t$	$\zeta_3 x$	y	$x \mapsto \zeta_3^2 x$	$y \mapsto y$
17	$t + \lambda$	x	$y + x + \lambda$	$x \mapsto x$	$y \mapsto y$

section by [27, Table 8.2], hence so does f and we can find a Weierstrass equation of the form $y^2 = x^3 + a_2 x^2 + a_4 x$. From there, the computation is straightforward. For the Weierstrass equations in the case $p = 2, 3$, one can simplify the equations given in [20] and [21]. We leave the details to the reader.

Let us prove the converse statement. The group $\text{Aut}_{\text{ct}}(J)^\dagger$ is a subgroup of the automorphism group of the generic fiber of j and the action of this group on $\text{MW}(J(f))$ is well-known, which makes the calculation of $\text{Aut}_{\text{ct}}(J)^\dagger$ straightforward. For the calculation of $\text{Aut}_{\text{ct}}(J) / \text{Aut}_{\text{ct}}(J)^\dagger$ we describe some automorphisms of J in Table 2 and leave it to the reader to check that these automorphisms exhaust all elements in $\text{Aut}_{\text{ct}}(J)$. In Table 2, we have $a \in \mathbb{G}_m$, $b \in \mathbb{G}_a$ and $\zeta_3^3 = 1$ and $i = 6, 4, 3, 2$, respectively. \square

The following exceptional case, which corresponds to Case 17 in Table 3 will occur in the proof of Theorem 6.4.

Lemma 6.3. *Let $f : S \rightarrow \mathbb{P}^1$ be a jacobian rational elliptic surface whose only reducible fiber F_0 is of type \tilde{E}_7 and which admits an automorphism $g \in \text{Aut}_{\text{ct}}(S)$ with $\text{ord}(g) = p$. Then, $p = 2$, the singular fibers of f are F_0 and two fibers of type \tilde{A}_0^* , $\text{Aut}_{\text{ct}}(S) = \mathbb{Z}/2\mathbb{Z}$, and $\text{Aut}_{\text{ct}}(S)^\dagger = \{\text{id}\}$.*

Proof. First, we claim that $\text{Aut}_{\text{ct}}(S)^\dagger = \{\text{id}\}$. Indeed, all fibers of f are $\text{Aut}_{\text{ct}}(S)^\dagger$ -invariant. Since f is not extremal and $\text{Aut}_{\text{ct}}(S)^\dagger$ preserves all its sections, we have infinitely many fixed points on each fiber, so $\text{Aut}_{\text{ct}}(S)^\dagger = \{\text{id}\}$. In particular, $g \notin \text{Aut}_{\text{ct}}(S)^\dagger$.

If $p \neq 2$, then Lemma 2.3 shows that f admits three fibers of type \tilde{A}_0^* , so we have $p = 3$. By [19] the discriminant of f is of the form

$$\Delta = t^9 s(c_1^3 - t c_0^2 c_1^2 + t^2 c_0^3 d_1)$$

for homogeneous polynomials $c_i, d_i \in k[s, t]$ of degree i . The fiber F_0 is located at $t = 0$ and it is easy to check that there is no linear automorphism of $k[s, t]$ that permutes the other zeroes of Δ , so g does not exist. Hence, we must have $p = 2$. By Lemma 2.3, we

only have to exclude the case where F_0 is the only singular fiber of f . In this case, by [29, Table 2], f is given by the Weierstrass equation

$$y^2 + y = x^3 + tx$$

and one can easily check that it admits no involutions acting non-trivially on the base. \square

Theorem 6.4. *Let $f : S \rightarrow \mathbb{P}^1$ be a jacobian rational elliptic surface. Assume that $\text{Aut}_{\text{ct}}(S)$ is non-trivial. Then, the singular fibers of f are as in Table 3. Moreover, either f is extremal or the groups $\text{Aut}_{\text{ct}}(S)$ and $\text{Aut}_{\text{ct}}(S)^\dagger$ are as in Column 5 and 6 Table 3.*

Proof. We fix some realization $\pi : S \rightarrow \mathbb{P}^2$ as in Section 2.3 and use the same notation as in that section. By Lemma 6.1, there is a reducible fiber F_0 of f with at least 4 components and this F_0 is necessarily $\text{Aut}_{\text{ct}}(J)$ -invariant. We set $C_0 = f(F_0)$ and let $g \in \text{Aut}_{\text{ct}}(J)$ be a non-trivial automorphism of order n .

Case: $s = 1$

In this case, F_0 contains a chain of 8 (-2) -curves, so it is of type \tilde{A}_8 or \tilde{E}_8 . This leads precisely to the **Cases 1, ..., 8, and 12**.

In the remaining cases, we will write C_0 as in the following claim.

Claim. *If $s \geq 2$, then $C_0 = \ell + Q$ for a line ℓ and a conic Q .*

If C_0 is not of the stated form, then it is integral and singular. We may assume that the singularity of C_0 is at q_1 . Then, $F_0 = \tilde{C}_0 + R_1^{(1)} + \dots + R_1^{(k_1-1)}$, so C_0 cannot be cuspidal. Since $k_1 \geq 3$, the tangent lines to F_0 at q_1 are g -invariant, so they are not swapped by g and hence g has only one fixed point on C_0 by Lemma 2.5. Since the q_i are fixed points of g , this is impossible if $s \geq 2$. \square

Case: $s = 2$

Let $\ell_{12} = \langle q_1, q_2 \rangle$.

First, assume that Q is smooth and that $\ell \cap Q$ consists of two distinct points q, q' . Since F_0 has more than 3 components, we may assume $q = q_2$. If $q_1 \notin \ell$, then $q', q_1, q_2 \in Q$ are 3 fixed points of g , which is impossible by Lemma 2.4. So, $\ell = \ell_{12}$. We have the following subcases:

- If $q' = q_1$, then all $R_i^{(j)}$ with $j < k_i$ are components of F_0 , so F_0 is of type \tilde{A}_8 . This is **Case 12**.
- If $q' \neq q_1$, then $(k_1, k_2) = (2, 7)$, since we have to blow up 6 points on Q and 3 points on ℓ . Then, F_0 is of type \tilde{A}_7 and there is another reducible fiber of f containing $R_1^{(1)}$. This is **Case 21**.

Next, assume that Q is smooth and $\ell \cap Q$ consists of a single point Q . As above, we may assume that $q = q_2$. We have the following subcases:

Table 3

Weierstrass equations for jacobian rational elliptic surfaces with cohomologically trivial automorphisms.

	Singular fibers	e or qe	Weierstrass equation	$\text{Aut}_{ct}(J)^\dagger$	$\text{Aut}_{ct}(J)$	Moduli	p
1	$\tilde{E}_8, \tilde{A}_0^{**}$	e	$y^2 = x^3 + t$	$\mathbb{Z}/6\mathbb{Z}$	\mathbb{G}_m	0	$\neq 2, 3$
2	$\tilde{E}_8, \tilde{A}_0^*, \tilde{A}_0^*$	e	$y^2 = x^3 + x + t$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	0	$\neq 2, 3$
3	$\tilde{E}_8, \tilde{A}_0^*$	e	$y^2 = x^3 + x^2 + t$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	3
4	\tilde{E}_8	e	$y^2 = x^3 + x + t$	$\mathbb{Z}/6\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{G}_a$	0	3
5	\tilde{E}_8	qe	$y^2 = x^3 + t$	$\mathbb{Z}/2\mathbb{Z}$	Aff	0	3
6	$\tilde{E}_8, \tilde{A}_0^*$	e	$y^2 + xy = x^3 + t$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	2
7	\tilde{E}_8	e	$y^2 + y = x^3 + t$	Q_8	$(\mathbb{Z}/2\mathbb{Z})^2 \cdot \mathbb{G}_a$	0	2
8	\tilde{E}_8	qe	$y^2 = x^3 + t$	$\mathbb{Z}/3\mathbb{Z}$	Aff	0	2
9	$\tilde{D}_8, \tilde{A}_0^*, \tilde{A}_0^*$	e	$y^2 = x^3 + tx^2 + x$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	0	$\neq 2$
10	\tilde{D}_8	e	$y^2 + xy = x^3 + tx^2 + \lambda, \lambda \neq 0$	$\mathbb{Z}/2\mathbb{Z}$	\mathbb{G}_a	1	2
11	\tilde{D}_8	qe	$y^2 = x^3 + tx^2 + t$	{id}	\mathbb{G}_a	0	2
12	$\tilde{A}_8, \tilde{A}_0^*, \tilde{A}_0^*, \tilde{A}_0^*$	e	$y^2 + txy - y = x^3$	{id}	$\mathbb{Z}/3\mathbb{Z}$	0	$\neq 3$
13	$\tilde{E}_7, \tilde{A}_1^*$	e/qe	$y^2 = x^3 + tx$	μ_4	\mathbb{G}_m	0	any
14	$\tilde{E}_7, \tilde{A}_1, \tilde{A}_0^*$	e	$y^2 = x^3 + x^2 + tx$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	$\neq 2$
15	$\tilde{E}_7, \tilde{A}_0^*, \tilde{A}_0^*, \tilde{A}_0^*$	e	$y^2 = x^3 + tx + 1$	{id}	$\mathbb{Z}/3\mathbb{Z}$	0	$\neq 2, 3$
16	\tilde{E}_7, \tilde{A}_1	e	$y^2 + xy = x^3 + tx$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	2
17	$\tilde{E}_7, \tilde{A}_0^*, \tilde{A}_0^*$	e	$y^2 + xy = x^3 + tx + \lambda t, \lambda \neq 0$	{id}	$\mathbb{Z}/2\mathbb{Z}$	1	2
18	\tilde{E}_7	e	$y^2 + y = x^3 + tx$	{id}	$\mathbb{Z}/3\mathbb{Z}$	0	2
19	$\tilde{D}_7, \tilde{A}_0^*, \tilde{A}_0^*, \tilde{A}_0^*$	e	$y^2 = x^3 + tx^2 + 1$	{id}	$\mathbb{Z}/3\mathbb{Z}$	0	$\neq 2, 3$
20	\tilde{D}_7	e	$y^2 + y = x^3 + tx^2$	{id}	$\mathbb{Z}/3\mathbb{Z}$	0	2
21	$\tilde{A}_7, \tilde{A}_1, \tilde{A}_0^*, \tilde{A}_0^*$	e	$y^2 = x^3 + 2(t^2 - 1)x^2 - x$	{id}	$\mathbb{Z}/2\mathbb{Z}$	0	$\neq 2$
22	$\tilde{A}_7, \tilde{A}_0^*, \tilde{A}_0^*, \tilde{A}_0^*, \tilde{A}_0^*$	e	$y^2 = x^3 + 2(t^2 - 1)x^2 - \lambda x, \lambda \neq 0, 1$	{id}	$\mathbb{Z}/2\mathbb{Z}$	1	$\neq 2$
23	$\tilde{E}_6, \tilde{A}_2^*$	e/qe	$y^2 + ty = x^3$	μ_3	\mathbb{G}_m	0	any
24	$\tilde{E}_6, \tilde{A}_0^*, \tilde{A}_0^*, \tilde{A}_0^*, \tilde{A}_0^*$	e	$y^2 = x^3 - 3x + t^2 + \lambda, \lambda \neq 0, 2$	{id}	$\mathbb{Z}/2\mathbb{Z}$	1	$\neq 2, 3$
25	$\tilde{E}_6, \tilde{A}_0^{**}, \tilde{A}_0^{**}$	e	$y^2 = x^3 - 3x + t^2 + 2$	{id}	$\mathbb{Z}/2\mathbb{Z}$	0	$\neq 2, 3$
26	$\tilde{E}_6, \tilde{A}_1, \tilde{A}_0^*, \tilde{A}_0^*$	e	$y^2 = x^3 - 3x + t^2$	{id}	$\mathbb{Z}/2\mathbb{Z}$	0	$\neq 2, 3$
27	$\tilde{E}_6, \tilde{A}_0^*, \tilde{A}_0^*$	e	$y^2 = x^3 + x^2 + t^2 + \lambda, \lambda \neq 0$	{id}	$\mathbb{Z}/2\mathbb{Z}$	1	3
28	\tilde{E}_6, \tilde{A}_1	e	$y^2 = x^3 + x^2 + t^2$	{id}	$\mathbb{Z}/2\mathbb{Z}$	0	3
29	\tilde{E}_6	e	$y^2 = x^3 + x + t^2$	{id}	$\mathbb{Z}/2\mathbb{Z}$	0	3
30	$\tilde{D}_6, \tilde{A}_1, \tilde{A}_1$	e	$y^2 = x^3 + tx^2 + x + t$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	$\neq 2$
31	\tilde{D}_4, \tilde{D}_4	e/qe	$y^2 = x^3 + \lambda tx^2 + t^2 x, \lambda \neq 2$	μ_2	\mathbb{G}_m	1	any

- If $\ell \neq \ell_{12}$, then $q_1 \in Q$, $(k_1, k_2) = (4, 5)$ and F_0 is of type \tilde{D}_5 . On the one hand, g fixes the two adjacent components of multiplicity 2 of F_0 pointwise, so $(p, n) \neq 1$, and on the other hand, g acts non-trivially on Q , so $(p, n) = 1$. This contradiction shows that this case does not occur.
- If $\ell = \ell_{12}$, then $(k_1, k_2) = (1, 8)$ and F_0 is of type \tilde{D}_8 . This yields **Cases 9, 10, and 11**.

Now, assume that $C_0 = \ell + \ell_1 + \ell_2$ for distinct lines ℓ_1 and ℓ_2 . It is easy to check that the lines cannot all be distinct, so we may assume $\ell = \ell_1$. Possibly interchanging the ℓ_i , we may assume that $q_i \in \ell_i$.

- If $\ell_{12} \notin \{\ell, \ell_1\}$, then $(k_1, k_2) = (6, 3)$ and F_0 is of type \tilde{E}_6 . There is another fiber F_1 of f containing $R_2^{(1)} + R_2^{(2)}$. Hence, this is **Case 23**.
- If $\ell_1 = \ell_{12}$, then $(k_1, k_2) = (4, 5)$ and F_0 is of type \tilde{D}_8 . This leads to **Cases 9, 10, and 11**.
- If $\ell_2 = \ell_{12}$, then $(k_1, k_2) = (7, 2)$ and F_0 is of type \tilde{E}_7 . There is another fiber F_1 of f containing $R_2^{(1)}$. Hence, this leads to **Cases 13, 14, and 16**.

Finally, assume that $C_0 = 3\ell_{12}$. A general member of f meets C_0 at the q_i with multiplicities 3 and 6, so we may assume $(k_1, k_2) = (3, 6)$ and $x_2^{(2)} \in C_0$. Then, F_0 is of type \tilde{E}_7 . Let t_1 be the line with $x_1^{(2)} \in t_1$ and let Q_1 be the conic with $x_2^{(1)}, \dots, x_2^{(5)} \in Q_1$. We have the following subcases:

- If $x_1^{(3)} \in t_1$, then \tilde{t}_1 is a (-2) -curve and we arrive at **Cases 13, 14, and 16**. A similar argument works if $x_2^{(6)} \in Q_1$.
- If $x_1^{(3)} \notin t_1, x_2^{(6)} \notin Q_1$ and $t_1 \cap Q_1 = \{q\}$ for a single point q , then q is a fixed point of g , and \tilde{t}_1 and \tilde{Q}_1 are (-1) -curves. By Lemma 2.4, we have $(p, n) = 1$. Moreover, note that g acts non-trivially on Q_1 , so $\text{Aut}_{\text{ct}}(S)^\dagger = \{\text{id}\}$. Let F_1 be the second g -invariant fiber of f and note that F_1 is irreducible since $\pi(F_1)$ does not contain any line whose proper transform is a (-2) -curve. Since f admits the 3 disjoint sections \tilde{t}_1, E_1 , and E_2 , Lemma 2.5 shows that F_1 has to be smooth and since $q, q_1, q_2 \in F_1$, we have $n \leq 3$. If $p \neq 2$, then Lemma 2.3 shows that f admits 3 fibers of type \tilde{A}_0^* , so $n = 3$. This is **Case 15**. If $p = 2$, then $n = 3$ is implied by $(p, n) = 1$. Then, Lemma 2.3 shows that F_0 is the only singular fiber of f , so this is **Case 18**.
- If $x_1^{(3)} \notin t_1, x_2^{(6)} \notin Q_1$ and $t_1 \cap Q_1 = \{q, q'\}$ for two distinct points q, q' , then \tilde{t}_1 and \tilde{Q}_1 are again (-1) -curves. Then, g swaps q and q' , for otherwise it would fix t_1 and Q_1 pointwise. In particular, we have $\text{Aut}_{\text{ct}}(S)^\dagger = \{\text{id}\}$, and $\text{Aut}_{\text{ct}}(S) = \mathbb{Z}/2\mathbb{Z}$. If $p \neq 2$, then g has another fixed point $q'' \in t_1$ and, by Lemma 2.4, g fixes one of the lines ℓ_{12}, t_1 , and $\langle q'', q_2 \rangle$ pointwise. In every case, this would imply that g fixes Q_1 pointwise, which is absurd. Hence, $p = 2$ and we can apply Lemma 6.3 to deduce that we are in **Case 17**.

Case: $s = 3$ and q_1, q_2, q_3 collinear

Let ℓ_{123} be the line through the q_i .

First, assume that Q is smooth and $\ell \cap Q$ consists of two distinct points q, q' . Since F_0 has more than 3 components, we may assume $q = q_3$. Note that Q can contain at most 2 of the q_i by Lemma 2.4, so we may assume $\ell = \ell_{123}$, $q_1 \notin Q$ and $q_2 \in Q$. In particular, we have $k_1 = 1$. The $R_i^{(j)}$ with $i \in \{2, 3\}$ and $j < k_i$ are in F_0 , so F_0 is of type \tilde{A}_7 . Let t_i be the tangent line to Q at q_i . We have the following two subcases:

- If $x_i^{(3)} \in t_i$ for some $i \in \{2, 3\}$, then \tilde{t}_i is a (-2) -curve, so f admits another reducible fiber apart from F_0 . Thus, we are in **Case 21**.
- If $x_i^{(3)} \notin t_i$ for $i = 2, 3$, then \tilde{t}_i is a (-1) -curve on which g acts non-trivially by Lemma 2.4. So, we have $\text{Aut}_{\text{ct}}(S)^\dagger = \{\text{id}\}$. Let F_1 be the fiber of f such that $C_1 := \pi(F_1)$ passes through the intersection $\tilde{t}_2 \cap \tilde{t}_3$. This C_1 is g -invariant, irreducible, and contains 4 fixed points of g , so by Lemma 2.5 it is smooth, $p \neq 2$, and $\text{Aut}_{\text{ct}}(S) = \mathbb{Z}/2\mathbb{Z}$. This is **Case 22**.

Next, assume that Q is smooth and $\ell \cap Q$ consists of a single point q . As in the previous case, we may assume $q = q_3$ and $\ell = \ell_{123}$. Then, we have $(k_1, k_2, k_3) = (1, 1, 7)$, but since Q and ℓ are tangent at q_3 , we have $\tilde{\ell}^2 \leq -3$, which is impossible. So, this case does not occur.

Assume that $C_0 = \ell_1 + \ell_2 + \ell_3$ is the sum of three distinct lines. Since F_0 has more than 3 components, one of the lines has to be ℓ_{123} , so we may assume $\ell_1 = \ell_{123}$. It is easy to check that each of the ℓ_i has to contain some q_j , so we may assume $q_i \in \ell_i$. Then, we have $(k_1, k_2, k_3) = (1, 4, 4)$ and F_0 is of type \tilde{A}_8 . This is **Case 12**.

Now, assume that $C_0 = \ell_{123} + 2\ell$ for some line $\ell \neq \ell_{123}$. We may assume $q_3 \in \ell_{123}$. Then, $(k_1, k_2, k_3) = (1, 1, 7)$ and F_0 is of type \tilde{E}_7 . Since g acts non-trivially on E_1 , we have $\text{Aut}_{\text{ct}}(S)^\dagger = \{\text{id}\}$. If $(p, n) = 1$, then g admits another fixed point $q \in \ell \setminus \ell_{123}$. Let $F_1 \neq F_0$ be a g -invariant fiber and note that $C_1 := \pi(F_1)$ is irreducible, since the only lines whose proper transforms are (-2) -curves are ℓ_{123} and ℓ . Since $q \in \ell \setminus \ell_{123}$, F_1 does not contain q , so g has precisely the 3 smooth fixed points q_i on F_1 . By Lemma 2.5, this implies that F_1 is smooth and $n = 3$, so the number of singular fibers of f apart from F_0 is divisible by 3. These are **Cases 15 and 18**. If $(p, n) \neq 1$, we can apply Lemma 6.3 to deduce that we are in **Case 17**.

Next, assume that $C_0 = 2\ell_{123} + \ell$ for some line $\ell \neq \ell_{123}$ with $q_3 \in \ell$. Then, we have $(k_1, k_2, k_3) = (2, 2, 5)$ and F_0 is of type \tilde{D}_7 . Let t_i with $i = 1, 2$ be the line through q_i with $x_i^{(2)} \in t_i$. Then, t_1, t_2 , and ℓ meet in a single point q which is a fixed point of g , so $(p, n) = 1$ by Lemma 2.4. Since \tilde{t}_1 is a (-1) -curve and g acts non-trivially on it, we have $\text{Aut}_{\text{ct}}(S)^\dagger = \{\text{id}\}$. Let $F_1 \neq F_0$ be a g -invariant fiber. By the same argument as in the previous paragraph, F_1 is smooth, $n = 3$, and the number of singular fibers of f apart from F_0 is divisible by 3, so this leads to **Cases 19 and 20**.

Finally, assume that $C_0 = 3\ell_{123}$. We must have $(k_1, k_2, k_3) = (3, 3, 3)$ and then F_0 is of type \tilde{E}_6 . We let t_i be the line through q_i with $x_i^{(2)}$. These lines meet in a single point q which is a fixed point of g , so $(p, n) = 1$ by Lemma 2.4. If for some i we have $x_i^{(3)} \in t_i$,

then \tilde{t}_i is a (-1) -curve on which g acts non-trivially, so $\text{Aut}_{\text{ct}}(S)^\dagger = \{\text{id}\}$. We have the following subcases:

- If $x_i^{(3)} \in t_i$ for two of the i , then the corresponding \tilde{t}_i are part of the same fiber of f . This is only possible if all three of the \tilde{t}_i are (-2) -curves and then we are in **Case 23**.
- If $x_i^{(3)} \in t_i$ for precisely one i , say for $i = 1$, then the fiber F_1 of f containing \tilde{t}_1 also contains the strict transform of a smooth conic Q_{23} through q_2 and q_3 . Since g acts non-trivially on Q_{23} , we have $Q_{23} \cap t_1$ consists of 2 points, both different from q . Hence, we have $n = 2$ and the number of singular fibers of f apart from F_0 and F_1 is divisible by 2, so we obtain **Cases 26 and 28**.
- If $x_i^{(3)} \notin t_i$ for all i , let F_1 be the fiber of f passing through q and let $C_1 = \pi(F_1)$. Note that C_1 is irreducible, since the only line whose proper transform is a (-2) -curve is ℓ_{123} . Then, C_1 contains 4 fixed points of g , so, by Lemma 2.5, C_1 is smooth, $n = 2$, and $p \neq 2$. In particular, the number of singular fibers of f apart from F_0 is divisible 2, so this leads to **Cases 24, 25, 27, and 29**.

Case: $s = 3$ and q_1, q_2, q_3 are not collinear

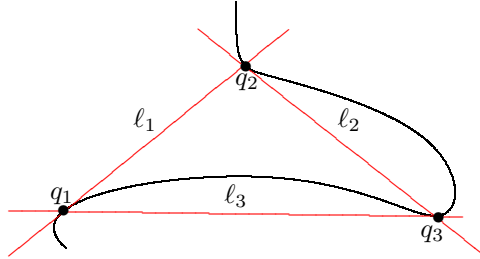
Let $\ell_{ij} = \langle q_i, q_j \rangle$ and note that, by Lemma 2.4, we have $(p, n) = 1$.

First, assume that Q is smooth. Since F_0 has more than 3 components, we may assume $q = q_3$. Note that Q can contain at most 2 of the q_i by Lemma 2.4, so we may assume $q_1 \in \ell \setminus Q$. Then, $\ell = \ell_{13}$ and $Q \cap \ell = \{q_3\}$. Since F_0 has more than 3 components, we have $k_3 \geq 4$ and F_0 is of type \tilde{D}_{k_3} . Also, as Q and ℓ are tangent at q_3 , this implies $k_1 = 1$. Note that the tangent to Q at q_2 is g -invariant, so it coincides with ℓ_{12} , for otherwise it would produce a third fixed point on ℓ_{13} forcing ℓ_{13} and Q to be fixed pointwise. So, we have the following subcases:

- If $k_2 \geq 2$, then $\tilde{\ell}_{12}$ is a (-2) -curve and there exists a fiber F_1 of f and a smooth conic Q' such that $C_1 = \pi(F_1) = \ell_{12} + Q'$. By the same argument as in the previous paragraph, ℓ_{12} is tangent to Q' at q_2 . Computing the proper transforms of Q' and ℓ_{12} , we see that F_1 can only be a fiber of f if $(k_1, k_2, k_3) = (1, 4, 4)$, in which case both F_0 and F_1 are of type \tilde{D}_4 . This is **Case 31**.
- If $(k_1, k_2, k_3) = (1, 1, 7)$, note that g acts non-trivially on one of the sections $\tilde{\ell}_{12}$ or $\tilde{\ell}_{23}$, so $\text{Aut}_{\text{ct}}(S)^\dagger = \{\text{id}\}$. Let F_1 be the second g -invariant fiber and set $C_1 := \pi(F_1)$. Note that C_1 is irreducible, since the only line that becomes a (-2) -curve on S is ℓ_{13} . None of the ℓ_{ij} is fixed pointwise by g , so $n \neq 2$ by Lemma 2.4 and C_1 contains precisely 3 fixed points. Thus, Lemma 2.5 shows that C_1 is smooth and $n = 3$. In particular, the number of singular fibers of f apart from F_0 is divisible by 3. This leads to **Cases 19 and 20**.

Next, assume that C_0 consists of 3 distinct lines. Since F_0 has at least 4 components, at least two of these lines have to coincide with some ℓ_{ij} , so we may assume $C_0 = \ell_{12} + \ell_{23} + \ell$ for some line ℓ . We have the following subcases:

- If $\ell = \ell_{13}$, we may assume that a general member of the pencil is tangent to ℓ_{12} at q_1 , and then it must be tangent to ℓ_{23} at q_3 and tangent to ℓ_{12} at q_2 . We have the following picture.



It is immediate to see that F_0 is of type \tilde{A}_8 and thus this is **Case 12**.

- If $\ell \neq \ell_{13}$ and $q_2 \in \ell$, then $(k_1, k_2, k_3) = (2, 5, 2)$ and F_0 is of type \tilde{D}_6 . Moreover, there are two additional reducible fibers of f containing $R_1^{(1)}$ and $R_3^{(1)}$, respectively. Lemma 2.3 shows that we are in **Case 30**.
- If $\ell \neq \ell_{13}$ and $q_2 \notin \ell$, then we may assume $q_3 \in \ell$. Then, we have $(k_1, k_2, k_3) = (2, 3, 4)$ and F_0 is of type \tilde{A}_7 . There is another reducible fiber F_1 containing $R_1^{(1)}$, which is necessarily of type \tilde{A}_1 . Thus, this is **Case 21**.

Now, assume that $C_0 = 2\ell_{12} + \ell$ for some line $\ell \neq \ell_{12}$. Clearly, we must have $q_3 \in \ell$. If $\ell = \ell_{ij}$ for some i and j , then we assume $\ell = \ell_{23}$. We have the following subcases:

- If $\ell \neq \ell_{ij}$ for all i and j , then ℓ meets ℓ_{12} at some point $q \neq q_1, q_2$ and thus g fixes ℓ_{12} pointwise. Moreover, we have $k_3 = 3$. Since C_0 has multiplicity 2 at q_1 and q_2 , we may assume that $(k_1, k_2, k_3) = (4, 2, 3)$. Then, $x_1^{(2)} \in \ell_{12}$ and $x_3^{(2)} \in \ell_{23}$, since the latter is the unique g -invariant line different from ℓ_{12} through q_2 . So, F_0 is of type \tilde{D}_5 . Since $(p, n) = 1$ and g fixes both $\tilde{\ell}_{12}$ and $R_1^{(2)}$ pointwise, this case does not occur.
- If $\ell = \ell_{23}$ and $k_3 = 1$, then we must have $k_1, k_2 \geq 2$, $x_1^{(2)} \in \ell_{12}$, and $x_2^{(2)} \in \ell_{23}$. Computing the proper transform of C_0 , we get $(k_1, k_2, k_3) = (4, 4, 1)$ and F_0 is of type \tilde{D}_7 . As in the case above where Q is smooth and $(k_1, k_2, k_3) = (1, 1, 7)$, one can show that $n = 3$ and $\text{Aut}_{\text{ct}}(S)^\dagger = \{\text{id}\}$, so the number of singular fibers of f apart from F_0 is divisible by 3 and we are in **Cases 19 and 20**.
- If $\ell = \ell_{23}$, $k_3 = 2$, and $x_1^{(2)} \in \ell'$ for some line $\ell' \neq \ell_{12}, \ell_{13}$, then $(k_1, k_2, k_3) = (2, 5, 2)$ and F_0 is of type \tilde{E}_6 . Since g acts non-trivially on the (-1) -curve $\tilde{\ell}$ we have $\text{Aut}_{\text{ct}}(S)^\dagger = \{\text{id}\}$. Let F_1 be the fiber of f containing $\tilde{\ell}$. Since the only lines that become (-2) -curves on S are ℓ_{12} and ℓ_{13} , $C_1 = \pi(F_1)$ is irreducible and by Lemma 2.5 C_1 is nodal and $n = 2$, so F_1 is of type \tilde{A}_1 . In particular, the number of singular fibers of f apart from F_0 and F_1 is divisible by 2. This leads to **Cases 26 and 28**.

- If $\ell = \ell_{23}$, $k_3 = 2$, and $x_1^{(2)} \in \ell_{13}$, then again $(k_1, k_2, k_3) = (2, 5, 2)$ and F_0 is of type \tilde{E}_6 . This time, F_1 also contains $\tilde{\ell}_{23}$, so it consists of three components. This is **Case 23**.
- If $\ell = \ell_{23}$, $k_3 = 2$, and $x_1^{(2)} \in \ell_{12}$, then $(k_1, k_2, k_3) = (4, 3, 2)$ and F_0 is of type \tilde{D}_6 . Additionally, there are two distinct fibers containing $R_1^{(1)}$ and $R_3^{(1)}$, respectively. Thus, this is **Case 30**.

Finally, assume that $C_0 = \ell_{12} + 2\ell$ for some line ℓ different from the ℓ_{ij} . Then, we must have $q_3 \in \ell$ and we may assume $(k_1, k_2, k_3) = (1, 2, 6)$. The fiber F_0 is of type \tilde{E}_6 and there is another reducible fiber F_1 of f containing $R_2^{(1)}$. The only g -invariant lines that become (-2) -curves on S are ℓ_{12} and ℓ' , so $C_1 = \pi(F_1)$ is irreducible. Thus, we have $n = 2$ and C_1 is nodal by Lemma 2.5. Since g acts non-trivially on the (-1) -curve ℓ_{13} , we have $\text{Aut}_{\text{ct}}(S)^\dagger = \{\text{id}\}$ and thus the number of singular fibers of f apart from F_0 and F_1 is divisible by 2. These are **Cases 26 and 28**.

Case: $s = 4$

By Lemma 6.1 (4), we may assume $k_4 > k_i, i = 1, 2, 3$. Let ℓ_{123} be the line through q_1, q_2, q_3 and set $\ell_{i4} := \langle q_i, q_4 \rangle$ for $i = 1, 2, 3$. Note that ℓ_{123} is fixed pointwise by g and it follows from Lemma 2.4 that $(p, n) = 1$.

First, assume that Q is smooth. Then, ℓ contains two of the q_i and Q contains the other two, while $Q \cap \ell = \{q, q'\}$ for 2 distinct points q and q' which are interchanged by g . This implies that F_0 is of type \tilde{A}_1 , which contradicts our assumption that F_0 has at least 4 components.

Next, assume that C_0 consists of 3 distinct lines. We may assume that $C_0 = \ell_{123} + \ell_{34} + \ell$ for some line $\ell \neq \ell_{14}$. Then, we have $k_1 = 1$, so $k_4 > 3$ and thus $q_4 \in \ell$. We have the following subcases:

- If $\ell \neq \ell_{24}$, then $(k_1, k_2, k_3, k_4) = (1, 1, 3, 4)$ and F_0 is of type \tilde{A}_7 . Also, g acts non-trivially on E_1 , so $\text{Aut}_{\text{ct}}(S)^\dagger = \{\text{id}\}$. Let F_1 be the second g -invariant fiber and let $C_1 := \pi(F_1)$. All lines that become (-2) -curves on S are part of C_0 , so C_1 is irreducible. Lemma 2.5 shows that $n = 2$ and F_1 is smooth. In particular, the number of singular fibers of f apart from F_0 is divisible by 2, so this is **Case 22**.
- If $\ell = \ell_{24}$, then $(k_1, k_2, k_3, k_4) = (1, 2, 2, 4)$. Since F_0 is of type \tilde{A}_m for some m , we have $x_i^{(2)} \notin \ell_{i4}$ for $i = 2, 3$. But $x_4^{(2)}$ cannot be contained in both ℓ_{24} and ℓ_{34} , so one of them does not become a (-2) -curve on S . This contradiction shows that this case does not occur.

Now, assume that $C_0 = \ell_{123} + 2\ell$ for some line ℓ through q_4 . We may assume that ℓ does not pass through q_1, q_2 , so $k_1 = k_2 = 1$. We have the following subcases:

- If $\ell \neq \ell_{34}$, then $(k_1, k_2, k_3, k_4) = (1, 1, 1, 6)$, then F_0 is of type \tilde{E}_6 . Since g acts non-trivially on the (-1) -curve $\tilde{\ell}_{14}$, we have $\text{Aut}_{\text{ct}}(S)^\dagger = \{\text{id}\}$. Let F_1 be the second

g -invariant fiber, set $C_1 := \pi(F_1)$, and observe that C_1 is irreducible. By Lemma 2.5, C_1 is smooth and $n = 2$, hence the number of singular fibers of f apart from F_0 and F_1 is divisible by 2 and we obtain **Cases 24, 25, 27, and 29**.

- If $\ell = \ell_{34}$, then $k_3 \geq 2$ since $\tilde{\ell}_{34}$ and $\tilde{\ell}_{123}$ are both part of the same fiber. Then, the line with $x_3^{(2)}$ is g -invariant, so it coincides with ℓ_{34} . Thus, we must have $(k_1, k_2, k_3, k_4) = (1, 1, 5, 2)$, contradicting our assumption that $k_4 > k_3$.

Finally, assume that $C_0 = 2\ell_{123} + \ell$ for some line ℓ through q_4 . Since $k_4 \geq 3$, ℓ has to be different from the ℓ_{i4} . Then, we have $(k_1, k_2, k_3, k_4) = (2, 2, 2, 3)$ and F_0 is of type \tilde{D}_4 . There is another fiber F_1 of f of type \tilde{D}_4 with $\pi(F_1) = \ell_{14} + \ell_{24} + \ell_{34}$. So, this is **Case 31**. \square

6.2. Cohomologically trivial automorphisms of non-jacobian rational genus one surfaces

Let us now classify non-jacobian rational elliptic or quasi-elliptic surfaces $f : S \rightarrow \mathbb{P}^1$ with non-trivial group $\text{Aut}_{\text{ct}}(S)$. Recall that by the results of Section 3, we have exact sequences

$$0 \rightarrow \text{MW}(J(f)) \cap \text{Aut}_{\text{ct}}(S) \rightarrow \text{Aut}_{\text{ct}}(S)^\dagger \rightarrow \text{Aut}_{\text{ct}}(J(S))^\dagger \tag{6.1}$$

$$0 \rightarrow \text{Aut}_{\text{ct}}(S) / \text{Aut}_{\text{ct}}(S)^\dagger \rightarrow \text{Aut}_{\text{ct}}(J(S)) / \text{Aut}_{\text{ct}}(J(S))^\dagger \tag{6.2}$$

Let us first study the cases where $\text{MW}(J(f)) \cap \text{Aut}_{\text{ct}}(S)$ is non-trivial.

Lemma 6.5. *Let $f : S \rightarrow \mathbb{P}^1$ be a genus one fibration on a rational surface of index $m > 1$ and assume that $\text{Aut}_{\text{ct}}(S) \cap \text{MW}(J(f))$ is non-trivial. Then, one of the following cases occurs:*

- (1) f is elliptic, $2 \mid m$, $(p, m) = 1$, F_0 is of type \tilde{A}_7 and $\text{Aut}_{\text{ct}}(S) = \mathbb{Z}/2\mathbb{Z}$.
- (2) f is elliptic, $3 \mid m$, $(p, m) = 1$, F_0 is of type \tilde{A}_8 and $\text{Aut}_{\text{ct}}(S) = \mathbb{Z}/3\mathbb{Z}$.
- (3) $m = p = 2$, F_0 is of type \tilde{D}_8 , $\text{Aut}_{\text{ct}}(S) \cap \text{MW}(J(f)) = \mathbb{Z}/2\mathbb{Z}$, and $\text{Aut}_{\text{ct}}(S) = \mathbb{G}_a$. If f is elliptic, then $\text{Aut}_{\text{ct}}(S)^\dagger \subseteq (\mathbb{Z}/2\mathbb{Z})^2$ and if f is quasi-elliptic, then $\text{Aut}_{\text{ct}}(S)^\dagger = \mathbb{Z}/2\mathbb{Z}$.

Proof. If F_t is a simple fiber of f , then f is isomorphic to $J(f) \rightarrow \mathbb{P}^1$ in an étale neighborhood of t in such a way that the action of $\text{MW}(J(f))$ on $J(f)$ is identified with the action of $\text{MW}(J(f))$ on f . Hence, if $g \in \text{Aut}_{\text{ct}}(S) \cap \text{MW}(J(f))$, then it must be a translation automorphism by a section Σ of $J(f)$ which meets every fiber corresponding to a simple fiber of f in the identity component. In particular, the section Σ meets only one fiber of $J(f)$ in a non-identity component. On the other hand, since g preserves every m -section of f , the order n of Σ divides m and in particular Σ is an n -torsion section of $J(f)$.

Assume first that f is elliptic. We apply the theory of Mordell-Weil lattices [27]. The height $h(P)$ of a torsion section Σ is equal to zero, hence Theorem 6.24 from [27] shows that the sum of the local contributions is greater than or equal to 2. Since only one fiber defines a non-zero local contribution, Table 6.1 in [27] shows that this can happen only in the case when the type of the fiber is \tilde{A}_7, \tilde{A}_8 , or \tilde{D}_8 . It also shows that the torsion section does not intersect the zero section and has order 2, 3, or 2, respectively. In the first two cases, we have $(p, m) = 1$, since F_0 is multiplicative. In the third case, we have $m = p$, since F_0 is additive. But, as explained above, n divides m , hence $m = p = n = 2$.

Next, if f is quasi-elliptic, then the classification of Mordell-Weil groups of quasi-elliptic surfaces (see [12], [13]) shows that Σ as above exists if and only if $p = n = 2$ and f admits a fiber of type \tilde{D}_8 . As in the previous paragraph, we also have $m = 2$.

Finally, by Theorem 4.2 (3), the group $\text{Aut}_{\text{ct}}(S) \cap \text{MW}(J(f))$ is a subgroup of $\text{Aut}_{\text{ct}}(S')$ for some jacobian rational genus one surface $f' : S' \rightarrow \mathbb{P}^1$ which admits a fiber F'_0 of the same type as F_0 . Thus, in Case (1) and (2), Table 3 shows that $\text{Aut}_{\text{ct}}(S)$ is the stated group. In Case (3), we know from Table 2, that $\text{Aut}_{\text{ct}}(S') = \mathbb{G}_a$ acts trivially on the identity component of F'_0 . Thus, by Theorem 4.2 (3), we have $\text{Aut}_{\text{ct}}(S) = \text{Aut}_{\text{ct}}(S') = \mathbb{G}_a$. Moreover, we have $\text{MW}(J(f)) = \text{Aut}_{\text{ct}}(S) \cap \text{MW}(J(f)) = \mathbb{Z}/2\mathbb{Z}$ in this case. If f is quasi-elliptic, then $\text{Aut}_{\text{ct}}(J(S))^\dagger$ is trivial, so $\text{Aut}_{\text{ct}}(S) \cap \text{MW}(J(f)) = \text{Aut}_{\text{ct}}(S)^\dagger = \mathbb{Z}/2\mathbb{Z}$. If f is elliptic, then $\text{Aut}_{\text{ct}}(J(S))^\dagger = \mathbb{Z}/2\mathbb{Z}$, so $\text{Aut}_{\text{ct}}(S)^\dagger$ has order at most 4. As $\text{Aut}_{\text{ct}}(S)^\dagger$ is contained in \mathbb{G}_a , this shows $\text{Aut}_{\text{ct}}(S)^\dagger \subseteq (\mathbb{Z}/2\mathbb{Z})^2$. \square

Next, we are interested in the group $\text{Aut}_{\text{ct}}(S) / \text{Aut}_{\text{ct}}(S)^\dagger$. In Theorem 3.3 (5), we have seen that $\text{Aut}_{\text{ct}}(S)$ acts trivially on the identity component of the fiber J_0 of $J(f)$ corresponding to the multiple fiber mF_0 of f as soon as F_0 has more than two components. Using the results of the previous subsection, we can describe precisely all the cases where such cohomologically trivial automorphisms of $J(S)$ exist. In the following lemma, we are referring to the cases in Table 3.

Lemma 6.6. *Let $f : J \rightarrow \mathbb{P}^1$ be a jacobian rational genus one surface and let J_0 be a fiber. Let $G \subseteq \text{Aut}_{\text{ct}}(J)$ be the subgroup of automorphisms preserving J_0 and acting trivially on the identity component J_0^\sharp of J_0 . Then, the following hold:*

- (1) *If F_0 is additive, then $G \cap \text{Aut}_{\text{ct}}(J)^\dagger$ is the p -torsion subgroup of $\text{Aut}_{\text{ct}}(J)^\dagger$.*
- (2) *If F_0 is not additive, then $G \cap \text{Aut}_{\text{ct}}(J)^\dagger$ is trivial.*
- (3) *$G \not\subseteq \text{Aut}_{\text{ct}}(J)^\dagger$ holds precisely in the following cases:*
 - (a) $p \neq 5$, J is as in Case 1, F_0 is of type \tilde{E}_8 and $G = \mathbb{Z}/5\mathbb{Z}$.
 - (b) $p \neq 3$, J is as in Case 13, F_0 is of type \tilde{E}_7 and $G = \mathbb{Z}/3\mathbb{Z}$.
 - (c) $p \neq 2$, J is as in Case 23, F_0 is of type \tilde{E}_6 and $G = \mathbb{Z}/2\mathbb{Z}$.
 - (d) J is as in Case 4 or 7, F_0 is of type \tilde{E}_8 and $G = \mathbb{G}_a$.
 - (e) J is as in Case 5 or 8, F_0 is of type \tilde{E}_8 and $G = \mathbb{G}_a \times \mathbb{Z}/5\mathbb{Z}$.
 - (f) J is as in Case 10 or 11, F_0 is of type \tilde{D}_8 and $G = \mathbb{G}_a$.
 - (g) J is as in Case 12, 15, or 18, F_0 is of type \tilde{A}_8 or \tilde{E}_7 and $G = \mathbb{Z}/3\mathbb{Z}$.

- (h) J is as in Case 21, 22, 24, 25, 26, 27, 28, or 29, F_0 is of type \tilde{A}_7 or \tilde{E}_6 and $G = \mathbb{Z}/2\mathbb{Z}$.
- (i) J is as in Case 17, F_0 is of type \tilde{E}_7 and $G = \mathbb{Z}/2\mathbb{Z}$.

Proof. To prove the first two claims we use that the fixed locus of $g \in G$ contains the zero section and J_0^\sharp , hence it is not smooth. This implies that $G \cap \text{Aut}_{\text{ct}}(S)^\dagger$ is p^n -torsion. If J_0 is additive, then every automorphism of order p^n in $\text{Aut}_{\text{ct}}(J)^\dagger$ fixes J_0 pointwise (by Lemma 2.5 and the known structure of fixed loci on \mathbb{P}^1). So, $G \cap \text{Aut}_{\text{ct}}(J)^\dagger$ coincides with the p -torsion subgroup of $\text{Aut}_{\text{ct}}(J)^\dagger$. If J_0 is multiplicative or smooth, one easily checks that $G \cap \text{Aut}_{\text{ct}}(J)^\dagger$ is trivial, since it does not contain the inversion involution. The third claim follows from the explicit description of $\text{Aut}_{\text{ct}}(J(S)) \setminus \text{Aut}_{\text{ct}}(J(S))^\dagger$ in Table 2. \square

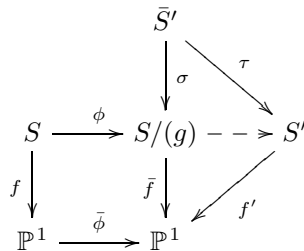
It turns out that an automorphism in $\text{Aut}_{\text{ct}}(J) \setminus \text{Aut}_{\text{ct}}(J)^\dagger$ which acts non-trivially on J_0 rarely comes from an automorphism of S .

Lemma 6.7. *Let $f : S \rightarrow \mathbb{P}^1$ be a rational genus one surface of index $m > 1$. Let mF_0 be the multiple fiber of f and let J_0 be the corresponding fiber of the jacobian fibration. Assume that there exists $g \in \text{Aut}_{\text{ct}}(S) \setminus \text{Aut}_{\text{ct}}(S)^\dagger$ such that $\varphi(g)$ acts non-trivially on the identity component J_0^\sharp of J_0 . Then, $J(f)$ is as in Case 2 or 9, $m = 2$, F_0 is smooth, $\text{Aut}_{\text{ct}}(S) = \mathbb{Z}/4\mathbb{Z}$ and $\text{Aut}_{\text{ct}}(S)^\dagger = \mathbb{Z}/2\mathbb{Z}$.*

Proof. Since $\varphi(g)$ acts non-trivially J_0^\sharp , we know that F_0 is of type $\tilde{A}_0, \tilde{A}_0^*$ or \tilde{A}_1 by Theorem 3.3 (5). Thus, by Lemma 6.6 (3), g acts faithfully on J_0^\sharp . In particular, we have $n := \text{ord}(g) \leq 4$.

If some non-trivial power of g acts trivially on \mathbb{P}^1 , then $n = 4$, $m = 2$ and $\varphi(g)^2 \in \text{Aut}_{\text{ct}}(J(S))^\dagger$. Using Table 2, we see that this can happen only in Case 2 or 9 and if F_0 is smooth. This is exactly the situation described in the statement of the lemma. We may thus assume that g acts faithfully on \mathbb{P}^1 .

By Lemma 6.1 (5), f admits a reducible fiber different from F_1 and since g preserves such a fiber, the action of g on \mathbb{P}^1 has two fixed points, so we have $p \nmid n$. Now, as g acts faithfully on \mathbb{P}^1 , the quotient surface $S/(g)$ admits a genus one fibration $\bar{f} : \bar{S} := S/(g) \rightarrow \mathbb{P}^1$. Let $\sigma : \bar{S}' \rightarrow \bar{S}$ be its minimal resolution and let $\tau : \bar{S}' \rightarrow S'$ be a birational morphism onto a relatively minimal genus one fibration $f' : S' \rightarrow \mathbb{P}^1$. We have the following commutative diagram



Let $\Phi = \tau \circ \sigma^{-1} \circ \phi$, so that we get a commutative diagram of rational maps

$$\begin{array}{ccc} S & \xrightarrow{\Phi} & S' \\ \downarrow f & & \downarrow f' \\ \mathbb{P}^1 & \xrightarrow{\bar{\phi}} & \mathbb{P}^1. \end{array}$$

Let E be an m -section of S . Since E and F_0 are invariant and meet transversally in a single point on S , it is easy to check that $\Phi(E)$ and $F'_0 := F_{\bar{\phi}(f(F_0))}$ meet transversally in a single point on S' . On the other hand, $\Phi(E)$ is an m -section of f' . This shows that f' has index m and F'_0 is the multiple fiber of f' . Now, by the known behavior of singular fibers of elliptic fibrations under tame base change (see e.g. [28, Section 5.2]) and since F'_0 is reducible, the fiber F'_0 is of additive type. This implies $m = p$ and since $m \mid n$ in every case of Theorem 3.3 (5), this contradicts $p \nmid n$. \square

Now we are able to go to the classification of groups $\text{Aut}_{\text{ct}}(S)$ for non-jacobian genus one surfaces.

Theorem 6.8. *Let $f : S \rightarrow \mathbb{P}^1$ be a non-jacobian rational genus one surface.*

- (1) *If $\text{Aut}_{\text{ct}}(S)$ is non-trivial, then the singular fibers and the multiple fiber are as in Table 4.*
- (2) *In every case, Table 4 also gives an upper bound on $\text{Aut}_{\text{ct}}(S)^\dagger$ and on the group $\text{Aut}_{\text{ct}}(S)$.*
- (3) *In every case, if f is elliptic or satisfies condition (OS) from Definition 2.1, then Table 4 gives a bound on the number of moduli.*

Proof. Let mF_0 be the multiple fiber of f and let J_0 be the corresponding fiber of $J(f)$ and let J_0^0 be its identity component.

If $\text{Aut}_{\text{ct}}(S) \cap \text{MW}(J(f))$ is non-trivial we apply Lemma 6.5 and we get Cases 10'', 11', 12', 21', and 22'.

In the other cases, $\text{Aut}_{\text{ct}}(S)$ acts faithfully on $J(S)$. We will distinguish several cases according to the type of F_0 .

If F_0 is of additive type and reducible, then $m = p$. By Theorem 3.3 (5), $\text{Aut}_{\text{ct}}(S)$ acts trivially on J_0^0 . Using Lemma 6.6 (1) and (3), one easily checks that we get the following Cases: 1''', 4'', 5', 6'', 7', 8', 10'', 11', 13'', 15', 16'', 17', 18', 23'', 24', 25', 26', 27', 28', and 29'. The same lemma also gives bounds on $\text{Aut}_{\text{ct}}(S)^\dagger$ and $\text{Aut}_{\text{ct}}(S)$.

If F_0 is of additive type and irreducible, then we also have $m = p$. By Theorem 3.3 (5), $\text{Aut}_{\text{ct}}(S)$ acts trivially on J_0^0 , so by Lemma 6.6, we have $\text{Aut}_{\text{ct}}(S)^\dagger = \text{Aut}_{\text{ct}}(S)$ and both groups are p -torsion groups contained in $\text{Aut}_{\text{ct}}(J(S))^\dagger$. In all cases in Table 3 that admit a fiber of type \tilde{A}_0^{**} , the group $\text{Aut}_{\text{ct}}(J(S))^\dagger$ is tame, so this case does not occur.

Table 4

Non-jacobian rational genus one surfaces with cohomologically trivial automorphisms.

	Singular fibers	Multiple fiber	e or qe	$\text{Aut}_{\text{ct}}(S)^\dagger$	$\text{Aut}_{\text{ct}}(S)$	Moduli	p
1'	$\tilde{E}_8, \tilde{A}_0^{**}$	$2\tilde{A}_0$	e	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	$\neq 2, 3$
1''	$\tilde{E}_8, \tilde{A}_0^{**}$	$3\tilde{A}_0$	e	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	0	$\neq 2, 3$
1'''	$\tilde{E}_8, \tilde{A}_0^{**}$	$p\tilde{E}_8$	e	{id}	$\mathbb{Z}/5\mathbb{Z}$	0	$\neq 0, 2, 3, 5$
2'	$\tilde{E}_8, \tilde{A}_0^*, \tilde{A}_0^*$	$2\tilde{A}_0$	e	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	1	$\neq 2, 3$
2''	$\tilde{E}_8, \tilde{A}_0^*, \tilde{A}_0^*$	$2\tilde{A}_0^*$	e	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	$\neq 2, 3$
3'	$\tilde{E}_8, \tilde{A}_0^*$	$2\tilde{A}_0$	e	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	1	3
3''	$\tilde{E}_8, \tilde{A}_0^*$	$2\tilde{A}_0^*$	e	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	3
4'	\tilde{E}_8	$2\tilde{A}_0$	e	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	3
4''	\tilde{E}_8	$3\tilde{E}_8$	e	$\mathbb{Z}/3\mathbb{Z}$	\mathbb{G}_a	1	3
5'	\tilde{E}_8	$3\tilde{E}_8$	qe	{id}	$\mathbb{G}_a \times \mathbb{Z}/5\mathbb{Z}$	0	3
6'	$\tilde{E}_8, \tilde{A}_0^*$	$2\tilde{A}_0$	e	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	1	2
6''	$\tilde{E}_8, \tilde{A}_0^*$	$2\tilde{E}_8$	e	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	1	2
7'	\tilde{E}_8	$2\tilde{E}_8$	e	Q_8	$(\mathbb{Z}/2\mathbb{Z})^2 \cdot \mathbb{G}_a$	1	2
8'	\tilde{E}_8	$2\tilde{E}_8$	qe	{id}	$\mathbb{G}_a \times \mathbb{Z}/5\mathbb{Z}$	0	2
9'	$\tilde{D}_8, \tilde{A}_0^*, \tilde{A}_0^*$	$2\tilde{A}_0$	e	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	0	$\neq 2$
9''	$\tilde{D}_8, \tilde{A}_0^*, \tilde{A}_0^*$	$2\tilde{A}_0^*$	e	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	$\neq 2$
10'	\tilde{D}_8	$2\tilde{A}_0$	e	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	2	2
10''	\tilde{D}_8	$2\tilde{D}_8$	e	$(\mathbb{Z}/2\mathbb{Z})^2$	\mathbb{G}_a	2	2
11'	\tilde{D}_8	$2\tilde{D}_8$	qe	$\mathbb{Z}/2\mathbb{Z}$	\mathbb{G}_a	1	2
12'	$\tilde{A}_8, \tilde{A}_0^*, \tilde{A}_0^*, \tilde{A}_0^*$	$m\tilde{A}_8, (m, p) = 1$	e	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	0	$\neq 3$
13'	$\tilde{E}_7, \tilde{A}_1^*$	$2\tilde{A}_0$	e	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	0	$\neq 2$
13''	$\tilde{E}_7, \tilde{A}_1^*$	$p\tilde{E}_7$	e/qe	{id}	$\mathbb{Z}/3\mathbb{Z}$	0	$\neq 0, 3$
14'	$\tilde{E}_7, \tilde{A}_1, \tilde{A}_0^*$	$2\tilde{A}_0$	e	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	1	$\neq 2$
14''	$\tilde{E}_7, \tilde{A}_1, \tilde{A}_0^*$	$2\tilde{A}_0^*$	e	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	$\neq 2$
14'''	$\tilde{E}_7, \tilde{A}_1, \tilde{A}_0^*$	$2\tilde{A}_1$	e	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	$\neq 2$
15'	$\tilde{E}_7, \tilde{A}_0^*, \tilde{A}_0^*, \tilde{A}_0^*$	$p\tilde{E}_7$	e	{id}	$\mathbb{Z}/3\mathbb{Z}$	1	$\neq 0, 2, 3$
16'	\tilde{E}_7, \tilde{A}_1	$2\tilde{A}_0$	e	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	1	2
16''	\tilde{E}_7, \tilde{A}_1	$2\tilde{E}_7$	e	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	1	2
17'	$\tilde{E}_7, \tilde{A}_0^*, \tilde{A}_0^*$	$2\tilde{E}_7$	e	{id}	$\mathbb{Z}/2\mathbb{Z}$	2	2
18'	\tilde{E}_7	$2\tilde{E}_7$	e	{id}	$\mathbb{Z}/3\mathbb{Z}$	1	2
21'	$\tilde{A}_7, \tilde{A}_1, \tilde{A}_0^*, \tilde{A}_0^*$	$m\tilde{A}_7, (m, p) = 1$	e	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	$\neq 2$
22'	$\tilde{A}_7, \tilde{A}_0^*, \tilde{A}_0^*, \tilde{A}_0^*, \tilde{A}_0^*$	$m\tilde{A}_7, (m, p) = 1$	e	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	1	$\neq 2$
23'	$\tilde{E}_6, \tilde{A}_2^*$	$3\tilde{A}_0$	e	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	0	$\neq 3$
23''	$\tilde{E}_6, \tilde{A}_2^*$	$p\tilde{E}_6$	e/qe	{id}	$\mathbb{Z}/2\mathbb{Z}$	0	$\neq 0, 2$
24'	$\tilde{E}_6, \tilde{A}_0^*, \tilde{A}_0^*, \tilde{A}_0^*, \tilde{A}_0^*$	$p\tilde{E}_6$	e	{id}	$\mathbb{Z}/2\mathbb{Z}$	2	$\neq 0, 2, 3$
25'	$\tilde{E}_6, \tilde{A}_0^*, \tilde{A}_0^*$	$p\tilde{E}_6$	e	{id}	$\mathbb{Z}/2\mathbb{Z}$	1	$\neq 0, 2, 3$
26'	$\tilde{E}_6, \tilde{A}_1, \tilde{A}_0^*, \tilde{A}_0^*$	$p\tilde{E}_6$	e	{id}	$\mathbb{Z}/2\mathbb{Z}$	1	$\neq 0, 2, 3$
27'	$\tilde{E}_6, \tilde{A}_0^*, \tilde{A}_0^*$	$3\tilde{E}_6$	e	{id}	$\mathbb{Z}/2\mathbb{Z}$	2	3
28'	\tilde{E}_6, \tilde{A}_1	$3\tilde{E}_6$	e	{id}	$\mathbb{Z}/2\mathbb{Z}$	1	3
29'	\tilde{E}_6	$3\tilde{E}_6$	e	{id}	$\mathbb{Z}/2\mathbb{Z}$	1	3
30'	$\tilde{D}_6, \tilde{A}_1, \tilde{A}_1$	$2\tilde{A}_0$	e	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	1	$\neq 2$
30''	$\tilde{D}_6, \tilde{A}_1, \tilde{A}_1$	$2\tilde{A}_1$	e	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	$\neq 2$
31'	\tilde{D}_4, \tilde{D}_4	$2\tilde{A}_0$	e	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	1	$\neq 2$

If F_0 is of multiplicative type with at least 3 components, then $(m, p) = 1$. By Theorem 3.3 (5), $\text{Aut}_{\text{ct}}(S)$ acts trivially on J_0^0 , so by Lemma 6.6 (3), either F_0 is of type \tilde{A}_8 and we get Case 12', or F_0 is of type \tilde{A}_7 and we get Cases 21' and 22'.

If F_0 is of type \tilde{A}_1 , then we also have $(m, p) = 1$. By Lemma 6.6, $\text{Aut}_{\text{ct}}(S)$ acts faithfully on J_0^0 , so Theorem 3.3 (5) shows that $\text{Aut}_{\text{ct}}(S) = \mathbb{Z}/2\mathbb{Z}$ and $p \neq 2$. By Lemma 6.7, we have $\text{Aut}_{\text{ct}}(S)^\dagger = \text{Aut}_{\text{ct}}(S)$. The only fibrations in Table 3 with $\text{Aut}_{\text{ct}}(J(S))^\dagger = \mathbb{Z}/2\mathbb{Z}$ and a fiber of type \tilde{A}_1 in characteristic $p \neq 2$ are Cases 14 and 30. Thus, we get Cases 14''' and 30''.

If F_0 is of type \tilde{A}_0^* , we have $(m, p) = 1$. By the same argument as in the previous paragraph, we have $\text{Aut}_{\text{ct}}(S)^\dagger = \text{Aut}_{\text{ct}}(S) = \mathbb{Z}/2\mathbb{Z}$. This leads to Cases $2'', 3'', 9''$, and $14''$.

Finally, if F_0 is smooth, then Lemma 6.6 shows that $\text{Aut}_{\text{ct}}(S)$ acts faithfully on J_0^0 and we can apply Theorem 3.3 (5) to get restrictions on m , $\text{Aut}_{\text{ct}}(S)^\dagger$, and $\text{Aut}_{\text{ct}}(S)$. By Lemma 6.7, we have $\text{Aut}_{\text{ct}}(S)^\dagger = \text{Aut}_{\text{ct}}(S)$ unless we are in Case $2'$ or $9'$. If $m = 3$, we obtain Cases $1''$ and $23'$. If $m = 2$, we obtain Cases $1', 3', 4', 6', 10', 13', 14', 16', 30'$, and $31'$.

To obtain a bound on the number of moduli we use the Ogg-Shafarevich theory of elliptic fibrations (see the discussion before Definition 2.1). According to this theory, a tame rational torsor of index m is determined uniquely by a choice of a multiple fiber F_{t_0} and a choice of a point P of order m in the connected component of the identity of $J_{t_0}^\sharp$. So, the number of moduli for the surfaces in Table 4 is the number of moduli of its jacobian $J(S)$ plus the number of moduli for choices of F_{t_0} and P up to automorphisms of $J(S)$. We leave it to the reader to calculate this number in every case. \square

Example 6.9. Assume $p \neq 2, 3$. Let $J(f) : J(S) \rightarrow \mathbb{P}^1$ be the jacobian rational elliptic surface with singular fibers of type $\tilde{E}_8, \tilde{A}_0^*, \tilde{A}_0^*$ and zero section E . Let $g \in \text{Aut}_{\text{ct}}(J(S))$ be a cohomologically trivial automorphism of order 4. Let J_0 be the smooth fiber of $J(f)$ preserved by g and let $P \in J_0$ be the fixed point of g which does not lie on E . Note that P is a 2-torsion point on J_0 . As mentioned in Remark 4.5, blowing down E and blowing up P , we obtain a rational genus one surface $f' : S' \rightarrow \mathbb{P}^1$ of index 2 together with a 2-section E' , the inverse image of P , and a cohomologically trivial automorphism g' of order 4. Using the description of g in Table 2, one checks that g' restricts to an automorphism of order 4 on E' and thus g' acts non-trivially on \mathbb{P}^1 . Therefore, by Table 4, the fibration f' is of type $2'$ or $9'$. In particular, some of the exceptional cases described in Lemma 6.7 exist.

Example 6.10. If $p \neq 0, 2, 3$ and $J(f) : J(S) \rightarrow \mathbb{P}^1$ is a jacobian rational elliptic surface with singular fibers of type $\tilde{E}_6, \tilde{A}_0^*, \tilde{A}_0^*, \tilde{A}_0^*$ and with a cohomologically trivial involution g , then g acts trivially on the identity component of the fiber of type \tilde{E}_6 by Table 2. Thus, similarly to Example 6.9, one can use any p -torsion point on the fiber of type \tilde{E}_6 to construct a non-jacobian rational elliptic surface with a multiple fiber of type \tilde{E}_6 and index p . Moreover, the resulting surface admits a cohomologically trivial involution, hence it is of type $24', 25'$, or $26'$.

Remark 6.11. We do not claim that all cases in Table 4 exist. We leave it to the interested reader to try to realize all cases using Halphen transforms as in Examples 6.9 and 6.10.

7. Numerically trivial automorphisms of classical Enriques surfaces in characteristic 2

We refer the reader to [4] for the basic facts on Enriques surfaces which we will use here without proof. Every Enriques surface S admits at least one genus one fibration $f : S \rightarrow \mathbb{P}^1$ and the jacobian fibration $J(f)$ of f is a rational genus one surface.

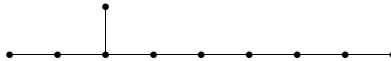
Moreover, the fibration f is cohomologically flat if and only if the Enriques surface S is classical, that is, if $\omega_S \neq \mathcal{O}_S$ (which always holds if the characteristic is not 2). In this case, f has precisely two double fibers. In this section, we give an application of Theorem 3.3 and the calculations of Section 6.1 to classical Enriques surfaces. This recovers some of the results of [9] and extends them to characteristic 2. We let $\text{Aut}_{\text{nt}}(S)$ be the subgroup of $\text{Aut}(S)$ of *numerically trivial automorphisms*, which are those automorphisms of S acting trivially on $\text{Num}(S)$. In the case $p \neq 2$, all Enriques surfaces with non-trivial $\text{Aut}_{\text{nt}}(S)$ have been classified in [5, Theorem 8.2.22]. So, we assume that $p = 2$.

As we shall see, Theorem 3.3 allows us to achieve a classification of classical Enriques surfaces with numerically trivial automorphisms in characteristic 2. Applying this theorem and $\text{Aut}_{\text{nt}}(J(S)) = \text{Aut}_{\text{ct}}(J(S))$, we get the following exact sequences

$$0 \rightarrow \text{MW}(J(f)) \cap \text{Aut}_{\text{nt}}(S) \rightarrow \text{Aut}_{\text{nt}}(S)^\dagger \rightarrow \text{Aut}_{\text{ct}}(J(S))^\dagger \tag{7.1}$$

$$0 \rightarrow \text{Aut}_{\text{nt}}(S) / \text{Aut}_{\text{nt}}(S)^\dagger \rightarrow \text{Aut}_{\text{ct}}(J(S)) / \text{Aut}_{\text{ct}}(J(S))^\dagger. \tag{7.2}$$

Before we start, we recall that an Enriques surface is called \tilde{E}_8 -extra-special if it contains the following configuration of (-2) -curves:



Note that the only genus one fibration on a \tilde{E}_8 -extra-special surface is a quasi-elliptic fibration with a double fiber of type \tilde{E}_8 . We will first prove the following refinement of [9, Theorem 8.5].

Theorem 7.1. *Let S be a classical Enriques surface in characteristic 2. Then, $\text{Aut}_{\text{nt}}(S) \subseteq (\mathbb{Z}/2\mathbb{Z})^b$ with $b \leq 2$.*

Proof. This was proved in [9, Theorem 8.5] if S is not \tilde{E}_8 -extra-special, so we may assume that we are in this special case and we let $f : S \rightarrow \mathbb{P}^1$ be the unique genus one fibration of this surface. Note that $J(f)$ corresponds to Case 8 in Table 3. Since $\text{MW}(J(f))$ is trivial, the group $\text{Aut}_{\text{nt}}(S)$ acts faithfully on $J(S)$ by (7.1). Let F_0 and F_1 be the double fibers of f , say with F_1 irreducible. Since $\text{Aut}_{\text{nt}}(S)$ preserves the F_i , Table 3 shows that $p \nmid |\text{Aut}_{\text{nt}}(S)|$. Then, Table 2 shows that $\text{Aut}_{\text{nt}}(S)$ acts faithfully on F_1 . By Theorem 3.3, this is only possible if $\text{Aut}_{\text{nt}}(S)$ is trivial. \square

Now, we follow the strategy we used in the previous subsection to calculate cohomologically trivial automorphisms of non-jacobian rational genus one surfaces.

Lemma 7.2. *Let $f : S \rightarrow \mathbb{P}^1$ be a genus one fibration on a classical Enriques surface in characteristic $p = 2$ with double fibers F_0, F_1 . Assume that $G := \text{Aut}_{\text{nt}}(S) \cap \text{MW}(J(f)) \neq$*

{1}. Then, one of the following cases occurs.

- (1) $G = (\mathbb{Z}/2\mathbb{Z})^2$, f is quasi-elliptic, and $(F_0, F_1) = (\tilde{D}_4, \tilde{D}_4)$.
- (2) $G = \mathbb{Z}/2\mathbb{Z}$, and f is elliptic and $(F_0, F_1) = (\tilde{D}_8, \tilde{A}_0)$, or f is quasi-elliptic and $(F_0, F_1) \in \{(\tilde{D}_4, \tilde{D}_4), (\tilde{D}_6, \tilde{A}_1^*), (\tilde{E}_7, \tilde{A}_1^*), (\tilde{D}_8, \tilde{A}_0^{**})\}$.

Proof. Since both double fibers of f are tame, they must be either of additive type or ordinary elliptic curves. As we mentioned earlier, the fibers of the jacobian fibration $J(f) : J(S) \rightarrow \mathbb{P}^1$ are of the same type. By the same argument as in Lemma 7.2, the action of $\text{Aut}_{\text{nt}}(S) \cap \text{MW}(J(f))$ is induced by translations by sections of $J(f)$ which meet every fiber of $J(f)$ except possibly the two fibers corresponding to F_0 and F_1 in the identity component. Since $\text{Aut}_{\text{nt}}(S)$ is 2-torsion by Theorem 7.1, every non-trivial translation automorphism from $\text{Aut}_{\text{nt}}(S)$ is of order 2. If f is an elliptic fibration, as before, we apply the theory of Mordell-Weil lattices. We have now at most two local contributions with sum a positive integer ≥ 2 . Using Table 6.1 from [27] we get that the only possibility for the types of (F_0, F_1) is $(\tilde{D}_8, \tilde{A}_0)$.

Suppose f is quasi-elliptic. Although quasi-elliptic fibrations were not considered in [27], the same arguments apply without change. Alternatively, one can use the explicit description of the Mordell-Weil groups of rational quasi-elliptic surfaces in characteristic 2 given in [13] to obtain the possible types of F_0 and F_1 . \square

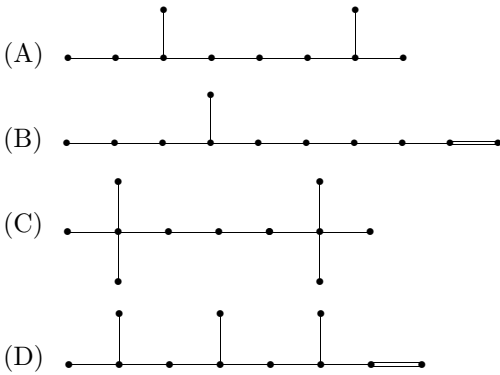
Proposition 7.3. *Let $f : S \rightarrow \mathbb{P}^1$ be a genus one fibration on a classical Enriques surface S in characteristic 2. If $\text{Aut}_{\text{nt}}(S) \neq \{\text{id}\}$, then one of the following cases occurs, where F_0 and F_1 are the double fibers of f :*

- (1) f is as in Lemma 7.2.
- (2) f is extremal, f admits a fiber of type \tilde{E}_8, \tilde{D}_8 , or \tilde{E}_7 and $J(f)$ occurs in Table 3,
- (3) f admits a fiber of type \tilde{E}_7 , $J(f)$ corresponds to Case 17 in Table 3, F_0 and F_1 are smooth, and $\text{Aut}_{\text{nt}}(S) = \mathbb{Z}/2\mathbb{Z}$.

Proof. We may assume that f does not occur in Lemma 7.2, so that $\text{Aut}_{\text{nt}}(S) \cap \text{MW}(J(f))$ is trivial and thus, by Sequence (7.2), $\text{Aut}_{\text{ct}}(J(S))$ is not. By Theorem 7.1, we know that $\text{Aut}_{\text{nt}}(S)$ is 2-torsion. In particular, $J(f)$ occurs in Table 3 and by checking which of the cases there admit cohomologically trivial involutions in characteristic 2, we obtain the stated list of fibers.

The only non-extremal fibration in Table 3 that admits a cohomologically trivial involution is Case 17. In this case, we have $\text{Aut}_{\text{nt}}(S) = \text{Aut}_{\text{ct}}(J(S)) = \mathbb{Z}/2\mathbb{Z}$, both acting non-trivially on the base. Hence, the F_i are interchanged by $\text{Aut}_{\text{nt}}(S)$. Since they cannot be nodal, they have to be smooth and ordinary. \square

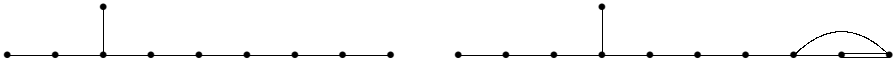
Theorem 7.4. *Let S be a classical Enriques surface in characteristic 2. If $\text{Aut}_{\text{nt}}(S)$ is non-trivial, then S contains one of the following configurations of (-2) -curves:*



In Cases (A), (B), and (C), these are in fact all (-2) -curves on S .

Proof. Throughout the proof, we assume that $\text{Aut}_{\text{nt}}(S)$ is non-trivial.

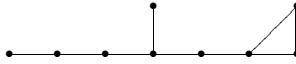
Assume that S does not admit a genus one fibration as in Proposition 7.3 (3). Then, all genus one fibrations on S are extremal. By [16], this implies that S has finite automorphism group. Comparing the genus one fibrations on Enriques surfaces with finite automorphism group given in [16, Section 14.1] with the ones that are allowed by Proposition, we see that we get configurations (A), (B), (C), or one of the following two configurations:



The first configuration belongs to the \tilde{E}_8 -extra-special surfaces, which we have already proved to have trivial group $\text{Aut}_{\text{nt}}(S)$ in the proof of Theorem 7.1. A surface containing the second configuration admits a quasi-elliptic fibration whose jacobian $J(S)$ is as in Case 13 of Table 3. Since $\text{Aut}_{\text{nt}}(S)$ is 2-torsion by Theorem 7.1 and $\text{Aut}_{\text{ct}}(J(S))$ has odd order, we have $\text{Aut}_{\text{nt}}(S) = \text{Aut}_{\text{nt}}(S) \cap \text{MW}(J(f))$. By Lemma 7.2, this group is trivial in this case. Hence, we are left with configurations (A), (B), and (C).

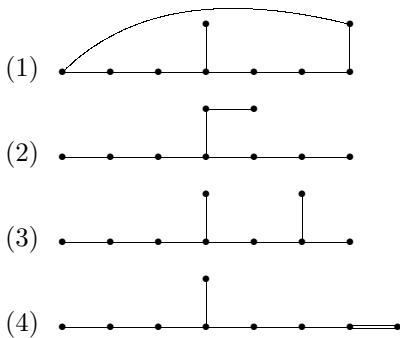
Next, since every Enriques surface admits some genus one fibration, we may assume that S admits a genus one fibration f as in Proposition 7.3 (3). By [5, Theorem 6.1.10] there is a second genus one fibration f' on S whose general fiber meets F with multiplicity 4. Let F' be a reducible fiber of f' with at least 5 irreducible components which exists by Proposition 7.3 and let E be an irreducible component of F' that meets F . Since F is a simple fiber, we have $E.F \in \{2, 4\}$.

First, assume that $E.F = 4$. Then, F' has to be a simple fiber of f' , E is a simple component of F' , and we may assume that $F' - E \subseteq F$ because of $(F' - E).F = 0$. Using the list of Proposition 7.3, we see that F' is of type \tilde{E}_7 and we get the following configuration:



The two right-most curves meet in two distinct points that are interchanged by $\text{Aut}_{\text{nt}}(S)$, for otherwise $\text{Aut}_{\text{nt}}(S)$ would fix these two curves pointwise, which is impossible since $\text{Aut}_{\text{nt}}(S)$ acts non-trivially on the base of f (see the proof of Proposition 7.3). Hence, these two curves define a (necessarily simple) fiber of type \tilde{A}_1 of some genus one fibration f'' of S . By Proposition 7.3, f'' admits a fiber F'' of type \tilde{E}_7 and it is easy to check that some component E'' of F'' satisfies $E''.F = 2$.

Hence, we may assume that $E.F = 2$. Then, we are in one of the following situations:



In Case (1), we find a configuration of type \tilde{A}_7 , which is impossible by Proposition 7.3. Similarly, in Case (2), we find a configuration of type \tilde{E}_6 , which is also impossible by Proposition 7.3. In Case (3), there is a configuration F'' of type \tilde{D}_6 , which is a double fiber of a quasi-elliptic fibration f'' on S , since there is a (-2) -curve E' on S with $E'.F'' = 1$. By Proposition 7.3 and Lemma 7.2, the fibration f'' also admits a double fiber of type \tilde{A}_1^* . This leads to configuration (D).

Finally, in Case (4), we find a genus one fibration f'' with a double fiber F''_0 of type \tilde{A}_1^* such that f'' also admits a fiber F''_1 containing a configuration of type D_6 . Here, we used again that there is a (-2) -curve E' on S with $E'.F''_0 = 1$. Note that F''_1 cannot be of type \tilde{E}_7 , for then there would be a (-2) -curve as in Case (2), which we have already excluded. Therefore, by Proposition 7.3, the fiber F''_1 is a double fiber of type \tilde{D}_6 . So, this also leads to configuration (D). \square

Remark 7.5. A closer inspection of Enriques surfaces of type (D) shows that every genus one fibration which admits a (-2) -curve as a 2-section on such a surface is of one of the following types:

- (1) a quasi-elliptic fibration with a simple fiber of type \tilde{D}_8 ,
- (2) an elliptic fibration with simple singular fibers of type $\tilde{E}_7, \tilde{A}_0^*, \tilde{A}_0^*$, or
- (3) a quasi-elliptic fibration with double fibers of type $\tilde{D}_6, \tilde{A}_1^*$ and a simple reducible fiber of type \tilde{A}_1^* .

Theorem 7.6. *Let S be a classical Enriques surface in characteristic 2. If S contains one of the configurations of (-2) -curves in Theorem 7.4, then $\text{Aut}_{\text{nt}}(S)$ and $\text{Aut}_{\text{ct}}(S)$ are as follows:*

- (A) $\text{Aut}_{\text{nt}}(S) = \text{Aut}_{\text{ct}}(S) = \mathbb{Z}/2\mathbb{Z}$.
- (B) $\text{Aut}_{\text{nt}}(S) = \mathbb{Z}/2\mathbb{Z}$, $\text{Aut}_{\text{ct}}(S) = \{\text{id}\}$.
- (C) $\text{Aut}_{\text{nt}}(S) = (\mathbb{Z}/2\mathbb{Z})^2$, $\text{Aut}_{\text{ct}}(S) = \{\text{id}\}$.
- (D) $\text{Aut}_{\text{nt}}(S) = \mathbb{Z}/2\mathbb{Z}$, $\text{Aut}_{\text{ct}}(S) = \{\text{id}\}$.

Proof. The claims about $\text{Aut}_{\text{ct}}(S)$ were proven in [9, Theorem 7.1], so we can focus on $\text{Aut}_{\text{nt}}(S)$.

First, assume that S is of type (A). By [16, Section 14.1], there are three genus one fibrations on S : A quasi-elliptic fibration f with a double fiber F_0 of type \tilde{D}_8 , an elliptic fibration f_1 with a double fiber of type \tilde{E}_8 , and an elliptic fibration f_2 with a simple fiber of type \tilde{E}_8 . By [13], we have $\text{MW}(J(f)) = \mathbb{Z}/2\mathbb{Z}$ and since this group cannot interchange the two \tilde{E}_8 -configurations (since they are fibers of non-isomorphic fibrations), the group $\text{MW}(J(f))$ acts trivially on the configuration of (-2) -curves. Since these curves generate $\text{Num}(S) \otimes \mathbb{Q}$, we deduce $\text{MW}(J(f)) \subseteq \text{Aut}_{\text{nt}}(S)$. By Lemma 7.2, we have $\text{Aut}_{\text{nt}}(S) \cap \text{MW}(J(f)) \subseteq \mathbb{Z}/2\mathbb{Z}$, so it suffices to show that $\text{Aut}_{\text{nt}}(S)$ acts trivially on $J(S)$. By Table 3, $\text{Aut}_{\text{nt}}(S)$ acts on $J(S)$ via automorphisms with only one fixed point on the base curve \mathbb{P}^1 . Since f has two double fibers, one of which is reducible, this implies that $\text{Aut}_{\text{nt}}(S)$ acts trivially on $J(S)$.

Next, assume that S is of type (B). Consider the fibration f with double fibers F_0 and F_1 of type \tilde{E}_7 and \tilde{A}_1^* . Note that the diagram of (-2) -curves has no symmetries and these (-2) -curves generate $\text{Num}(S)$, so $\text{MW}(J(f)) = \mathbb{Z}/2\mathbb{Z}$ acts trivially on $\text{Num}(S) \otimes \mathbb{Q}$. Thus, as in Case (A), it suffices to show that $\text{Aut}_{\text{nt}}(S)$ acts trivially on $J(S)$. Seeking a contradiction, assume that it does not. Then, by Table 2, $\text{Aut}_{\text{nt}}(S)$ acts non-trivially on the identity component of the fiber of $J(f)$ corresponding to F_1 . This is impossible by Theorem 3.3 (5).

Now, assume that S is of type (C). Consider the fibration f with double fibers F_0 and F_1 of type \tilde{D}_4 and \tilde{D}_4 . By [16, Section 14], the fibration f is quasi-elliptic and by [13], we have $\text{MW}(J(f)) = (\mathbb{Z}/2\mathbb{Z})^2$. A careful study of the fibrations f' with fibers of type \tilde{D}_8 shows that $\text{MW}(J(f))$ preserves all (-2) -curves, showing that $(\mathbb{Z}/2\mathbb{Z})^2 \subseteq \text{Aut}_{\text{nt}}(S)$. To prove the claim, it suffices to show that these are all numerically trivial automorphisms on S . Assume that there exists $g \in \text{Aut}_{\text{nt}}(S)$ acting non-trivially on $J(S)$. Then, by Table 3, the order of g is not a power of 2. On the other hand, consider the action of g on the jacobian $J(f') : J(S)' \rightarrow \mathbb{P}^1$ of f' . Since $\text{MW}(J(f')) = \mathbb{Z}/2\mathbb{Z}$, the action of g on $J(S)'$ has order equal to a power of 2. This is impossible by Table 3, so g does not exist.

Finally, assume that S is of type (D). Consider a quasi-elliptic fibration f with double fibers F_0 and F_1 of type \tilde{D}_6 and \tilde{A}_1^* . By [13], there is a 2-torsion section of $J(f)$ that meets only the two fibers corresponding to the F_i in a non-identity component. Let g

Table 5
Moduli of classical Enriques surfaces with numerically automorphisms in characteristic 2.

Type	Double fibers of f	Number of moduli
(A)	$2\tilde{D}_8, 2\tilde{A}_0^{**}$	2
(B)	$2\tilde{E}_7, 2\tilde{A}_1^*$	1
(C)	$2\tilde{D}_4, 2\tilde{D}_4$	2
(D)	$2\tilde{D}_6, 2\tilde{A}_1^*$	2

be the induced involution of S . Let $U \subseteq \text{Num}(S)$ be the hyperbolic plane spanned by F_0 and the curve of cusps C of f . Note that $U^\perp = E_8$. Let E_0 and E_1 be the two irreducible components of the reducible simple fiber of f . By construction, g^* preserves the sublattice $L := \langle E_0 - E_1 \rangle \oplus A_1 \oplus D_6 \subseteq U^\perp = E_8$ spanned by $E_0 - E_1$ and the irreducible components of F_0 and F_1 which do not meet C . Since g^* acts trivially on the first two summands, it acts trivially on the discriminant of D_6 , so $g^* \in W(D_6)$. Since D_6 is spanned by effective curves and g^* is induced by an automorphism of S , this shows that g^* is trivial, hence $g \in \text{Aut}_{\text{nt}}(S)$. Finally, since $\text{Aut}_{\text{ct}}(J(S))$ is trivial by Table 3, Lemma 7.2 shows that $\text{Aut}_{\text{nt}}(S) = \mathbb{Z}/2\mathbb{Z}$. \square

Remark 7.7. Note that the proof of Theorem 7.6 shows that every numerically trivial automorphism of a classical Enriques surface S in characteristic 2 is induced by translations by sections of the jacobian of some quasi-elliptic fibration on S .

Finally, we can give the following characterization of Enriques surfaces with non-trivial $\text{Aut}_{\text{nt}}(S)$ in terms of the existence of certain quasi-elliptic fibrations on the Enriques surface. Assuming condition (OS) from Definition 2.1, this also gives a way of calculating the number of moduli.

Corollary 7.8. *Let S be a classical Enriques surface in characteristic 2. Then, the following hold:*

- (1) *The group $\text{Aut}_{\text{nt}}(S)$ is non-trivial if and only if S admits a quasi-elliptic fibration f with double fibers as described in Table 5.*
- (2) *If f satisfies condition (OS) of Definition 2.1, Table 5 also gives the number of moduli of Enriques surfaces of a given type. In particular, assuming (OS), all types exist.*

Proof. Let us first prove the statement about the quasi-elliptic fibrations. We have already seen in the proof of Theorem 7.6 that Enriques surfaces of type (A), (B), (C), and (D) admit the stated genus one fibrations. Conversely, since the curve of cusps of a quasi-elliptic fibration f on S is a (-2) -curve which is a 2-section of f , it is easy to see that an Enriques surface admitting one of the stated fibrations admits a configuration of (-2) -curves as in Theorem 7.4.

To count the number of moduli, we recall that by Condition (OS), giving f is the same as giving $J(f)$ and choosing a 2-torsion point on each of the fibers of $J(f)$ that we want to be double fibers of f . By [13], the number of moduli for $J(f)$ in the four cases is 0, 0, 1, and 0, respectively. In Case (A), the choice of the cuspidal double fiber gives an additional parameter. Since the double fibers are additive, we get two additional parameters in each case for the choice of 2-torsion point. Now, we use that $\text{Aut}_{J(S)}^0$ is of dimension 1, 1, 1, and 0, respectively, and that $\text{Aut}_{J(S)}^0$ acts transitively on the set of cuspidal fibers of $J(f)$ in Case (A) and transitively on the set of 2-torsion points on the identity component of the fiber of type \tilde{E}_7, \tilde{D}_4 , and \tilde{D}_6 in Cases (B), (C), and (D), respectively. Putting everything together yields the stated number of moduli. \square

Since it is not known whether condition (OS) always holds, we will describe explicit examples of each of these four types. The explicit constructions of surfaces of Types (A), (B), and (C) can be found in [16]. We will give here an explicit construction of surfaces of Type (D). It follows a construction of surfaces of type (A) from [5, Example 6.2.11].

Example 7.9 (*Type (D)*). Consider two quasi-elliptic fibrations defined by the pencils $|2F_1|$ and $|2F_2|$, where F_1 is the divisor of type \tilde{D}_6 described by the subdiagram containing the first two vertical edges of the diagram for type (D) and $|2F_2|$ has a simple fiber of type \tilde{D}_8 described by the subdiagram containing the first and the third vertical edges. We check that $F_1 \cdot F_2 = 1$, so the linear system $|2F_1 + 2F_2|$ defines a bielliptic map $\phi : S \rightarrow D_4$, where D_4 is a 4-nodal anti-canonical del Pezzo surface of degree 4 in \mathbb{P}^2 (see [4, Section 3.3]). The morphism ϕ blows down (-2) -curves C with $C.F_1 = C.F_2 = 0$. Checking the diagram, we see that there are precisely 8 such curves, namely all curves contained in the fiber of type \tilde{D}_8 except for the curve of cusps of $|2F_1|$. It is known that the bielliptic map ϕ defines a birational model of S isomorphic to the inseparable double plane

$$w^2 + xyF_6(x, y, z) = 0,$$

where $V(F_6(x, y, z))$ is a curve of degree 6 with a double point at the point $[1 : 0 : 0]$ and tacnodes at $[0 : 1 : 0], [0 : 0 : 1]$ with tangent directions $V(x)$ and $V(y)$. The singular points of the double plane lie over the zeros of the differential $dx y F_6$. We take

$$F_6(x, y, z) = (x + y)(z^4(x + y) + az^3xy + z^2xy(bx + (b + 1)y) + x^2y^2(x + y)),$$

where $a \neq 0$ and b are parameters. The pencil $|2F_1|$ is equal to the pre-image of the pencil of conics $\lambda z^2 + \mu xy = 0$ and the pencil $|2F_2|$ is equal to the pre-image of the pencil of lines $\lambda x + \mu y = 0$. We check that the singularity of the double plane over the point $(x : y : z) = (1 : 1 : 0)$ is formally isomorphic to the singular point $r^2 + uv(u^2 + v) = 0$. This is a rational double point of type D_6^0 . It has also an ordinary double point over a point in the blow-up of the point $(0 : 0 : 1)$ and an ordinary double point at $(a : a : 1)$.

The pre-image of the line $x + y = 0$ that contains the point $(0 : 0 : 1)$, $(1 : 1 : 0)$ and $(a : a : 1)$ is the simple fiber of type \tilde{D}_8 of $|2F_2|$. The pre-image of the line $z = 0$ is a double fiber of type \tilde{D}_6 of the quasi-elliptic pencil $|2F_1|$. The pre-image of the exceptional curve of the blow-up of \mathbb{P}^2 at the point $(0 : 0 : 1)$ is a double fiber of type \tilde{A}_1^* of the same fibration. The pre-image of the conic $a^2z + xy = 0$ is a simple fiber of type \tilde{A}_1^* of $|2F_1|$. We leave it to the reader to verify that the minimal resolution of singularities of the double plane contains the configuration of (-2) -curves Type (D). In particular, by Theorem 7.6, we have $\text{Aut}_{\text{nt}}(S) = \mathbb{Z}/2\mathbb{Z}$. Using our explicit model, the numerically trivial involution can be described as $x \leftrightarrow y$, $w \mapsto w + zxy(x + y)$.

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