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THE EULER CHARACTERISTIC OF A FAMILY OF ALGEBRAIC VARIETIES

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Abstract. In this paper we derive a formula for the l-adic Euler characteristic of a one-parameter family of algebraic varieties. We define an algebraic analog of the local monodromy of isolated singularities of algebraic hypersurfaces, defined in the complex case by Milnor. We discuss various conjectures connected with the definition of the conductor of a family of algebraic varieties.

Bibliography: 26 items.

Introduction

We consider a family of algebraic varieties $f: X \longrightarrow Y$ over a nonsingular complete curve Y, i.e. f is a proper flat morphism of algebraic varieties with connected fibers. Assume that the ground field is the field of complex numbers $\mathbf{C}$, and let F be a "typical" fiber of the morphism f. In this case we have the "well-known" formula

$$\chi(X) = \chi(F)\chi(Y) + \sum_{y \in Y} \chi(X_y) - \chi(F), \qquad (*)$$

expressing the deviation from multiplicativity of the Euler characteristic (topological) in terms of the analogous characteristic of degenerate fibers X_y of the morphism f. A proof of this formula for the case when X is a smooth variety can be found in [1] (the restriction on the dimension of X made in the statement of this theorem is inessential).

In this paper we shall prove a formula analogous to $(*)$ and valid for an arbitrary algebraically closed field k of characteristic $p \geq 0$. In this case it is natural to consider the l-adic Euler characteristics $EP(X)$, $EP(Y)$ and $EP(X_y)$, and we shall prove (see §1 for an explanation of the notation) the formula (l is a prime number different from p)

$$EP(X) = EP(X_{\bar{\eta}})EP(Y) + \sum_{y \in \overline{Y}} [EP(X_y) - EP(X_{\bar{\eta}}) + \alpha_y(f;\, l)], \qquad (**)$$

where $X_{\bar{\eta}}$ is the geometric fiber of the morphism f over the generic point η of the curve Y, and $\alpha_y(f;\, l)$ is a "higher ramification invariant" (equal to zero if $p = 0$ or if the

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fiber X_y is smooth). This formula is obtained by a simple application of a general formula of Grothendieck for the Euler characteristic of a constructible étale sheaf on a smooth algebraic curve (cf. [15]; [17], exposé X). In case X is a normal surface, and also for arbitrary varieties with isolated singular points over a field of characteristic zero, in §4 we obtain from formula (**) the independence of $EP(X)$ of the prime number $l \neq p$. Our proof is "elementary" and does not use the fundamental theorems of l-adic cohomology. With the help of these it is clear that one can obtain considerably stronger results. Namely, for $p = 0$ from Artin's comparison theorem for arbitrary algebraic schemes ([16], exposé XVI) we get this result for arbitrary X. In case $p > 0$, as P. Deligne explained to the author, this result can be derived for arbitrary proper k-schemes from the interpretation of $EP(X)$ as the difference of the degree of the denominator and the degree of the numerator of the ζ-function of an algebraic variety over a finite field.

In §5 we discuss some conjectures connected with the definition of the conductor of a family of algebraic varieties.

§1. Statement of the main theorem

For any proper scheme Z over an algebraically closed field k of characteristic $p \geq 0$, we define the *l-adic Euler characteristic* of the scheme Z (l is a prime number different from p) to be

$$EP(Z) = \sum_i (-1)^i b_i(Z; l),$$

where $b_i(Z; l) = \dim_{\mathbf{Q}_l} H^i(Z; \mathbf{Q}_l)$ is the dimension of the space of rational l-adic cohomology of the scheme Z (see [17], exposé VI). That this definition is legal follows from the finiteness theorem ([16], exposé XIV) and the theorem on finite cohomological dimension (loc. cit., exposé X). Moreover, as explained in the Introduction, $EP(Z)$ does not depend on l.

Everywhere in what follows Y denotes a smooth connected curve over a field k, and $\overline{Y}$ its set of closed points. For any closed point $y \in \overline{Y}$ we denote by $\widetilde{\eta}_y$ the spectrum of the field of fractions K_y of the henselization of the local ring $O^h_{Y,y}$. Let i_y: $\widetilde{\eta}_y \to Y$ be the corresponding canonical morphism. For any constructible étale sheaf F on Y, annihilated by multiplication by l, the sheaf $\widetilde{F}_y = i^*_y(F)$ on $\widetilde{\eta}_y$ is identified with a finite-dimensional G_y-$\mathbf{F}_l$-bimodule, where G_y is the Galois group of the separable closure $\overline{K}_y$ of the field K_y.

For any field K which is complete relative to a discrete valuation, with residue field k, and for a finite-dimensional G_k-$\mathbf{F}_l$-bimodule M, one defines "Serre's higher ramification measure" $\delta(K, M)$ ([20]; see also [13], [15]).

We recall its definition. Let L/K be a finite Galois extension such that the Galois group G_L acts trivially on M. Thus M can be considered as a G-module, where $G = \mathrm{Gal}(L/K)$. Let $G_i = \{g \in G \mid \nu_\pi(g(\pi) - \pi) \geq i + 1\}$ (where π is the uniformizing parameter of the field K) be the higher ramification groups, $e_i = \#(G_i)$ and $e = \# G$ the orders of these groups. Then

$$\delta(K, M) = \sum_{i=0} \frac{e_i}{e} \dim_{\mathbf{F}_l}(M/M^{G_i}).$$

An equivalent definition is

$$\delta(K, M) = \dim_{\mathbf{F}_l} \mathrm{Hom}_{\mathbf{Z}_l[G]}(P, M),$$

where P is the Swan module associated to G.

Applying this definition to the G_y-module $\widetilde{F}_y$, we set $\delta_y(F) = \delta(K_y, \widetilde{F}_y)$.

By the definition of the constructibility of a sheaf F there exists an open set $U \subset Y$ such that the sheaf $F|U$ is locally constant. This means that there exists a finite étale morphism $\phi: V \longrightarrow U$ such that $\phi^*(F)$ is the constant sheaf. This condition is also equivalent to the fact that the function $y \mapsto \#(F_{\bar{y}})$ is constant on U (see [16], exposé IX). For any closed point $y \in U$ we obviously have $\delta_y(F) = 0$. But this is true for any point $y \in \overline{Y}$ if $\mathrm{char}(k) = 0$.

Now let $f: X \longrightarrow Y$ be a proper flat morphism with geometrically connected generic fiber X_η of dimension n. The sheaf $\mu_{l,X} = \mathrm{Ker}(\mathbf{G}_{m,X} \xrightarrow{l} \mathbf{G}_{m,X})$ on X is annihilated by multiplication by l, and so the sheaves $R^i f_* \mu_{l,X}$ on Y also are ([16], exposé X). Moreover, by virtue of the finiteness theorem (loc. cit., exposé XIV) these sheaves are constructible on Y. Thus we can set

$$\alpha_y^i(f; l) = \delta_y(R^i f_* \mu_{l,X}), \quad \alpha_y(f; l) = \sum_{i=1}^{2n} (-1)^i \alpha_y^i(f; l).$$

Now we are in a position to state the following theorem.

Theorem 1.1. *We have the formula*

$$EP(X) = EP(X_{\bar{\eta}})EP(Y) + \sum_{y \in \overline{Y}} [EP(X_y) - EP(X_{\bar{\eta}}) + \alpha_y(f; l)],$$

where $X_{\bar{\eta}} = X_\eta \otimes_{k(\eta)} \overline{k(\eta)}$ *is a geometric general fiber of the morphism* f.

This theorem will be proved in §3.

Corollary 1.2. *Assume that the morphism f is smooth. Then the l-adic Euler characteristic is multiplicative, i.e.* $EP(X) = EP(X_{\bar{\eta}})EP(Y)$.

In fact in this case the sheaves $R^i f_* \mu_{l^k,X}$, for $k > 0$, are locally constant (specialization theorem [16], exposé XVI). Thus the l-adic sheaves $\varprojlim_k R^i f_* \mu_{l^k,X} = R^i f_* \mathbf{Z}_l$ are locally constant, i.e. the function $y \mapsto b_i(X_{\bar{y}}; l)$ is constant. Hence $EP(X_y) = EP(X_{\bar{\eta}})$ for any point $y \in \overline{Y}$. Moreover, as we saw above, the invariant $\delta_y(R^i f_* \mu_{l,X}) = \alpha_y^i(f; l) = 0$. Now apply Theorem 1.1.

Corollary 1.3. *Suppose* $\mathrm{char}(k) = 0$. *Then*

$$EP(X) = EP(X_{\bar{\eta}})EP(Y) + \sum_{y \in \overline{Y}} (EP(X_y) - EP(X_{\bar{\eta}})).$$

If $k = \mathbf{C}$,

$$\chi(X) = \chi(F)\chi(Y) + \sum_{y \in \overline{Y}} (\chi(X_y) - \chi(F)),$$

where F *is an arbitrary fiber of* f *over an open set* $U \subset \overline{Y}$. *Here* χ *denotes the usual topological Euler characteristic.*

In fact, as we saw above, in case $\operatorname{char}(k) = 0$ the invariant $\alpha_y(f; l) = 0$. And if $k = \mathbf{C}$, by the comparison theorem ([16], exposé XVI) $H^i(X, (\mathbf{Z}/l^k)_X) \simeq H^i_{\mathrm{cl}}(X, \mathbf{Z}/l^k)$, whence

$$H^i(X, \mathbf{Z}_l) = \varprojlim_k H^i(X, (\mathbf{Z}/l^k)_X) \simeq \varprojlim_k H^i_{\mathrm{cl}}(X, \mathbf{Z}/l^k)$$
$$\simeq H^i_{\mathrm{cl}}(X, \mathbf{Z}) \otimes \mathbf{Z}_l \oplus \text{ finite group}$$

(the last by the universal coefficient theorem). Tensoring by $\mathbf{Q}_l$, we get

$$H^i(X, \mathbf{Q}_l) = H^i(X, \mathbf{Z}_l) \otimes_{\mathbf{Z}} \mathbf{Q}_l \simeq H^i_{\mathrm{cl}}(X, \mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Q}_l,$$

whence $b_i(X; l) = \beta_i(X)$, and hence $EP(X) = \chi(X)$. The equalities $EP(X_y) = \chi(X_y)$ and $EP(Y) = \chi(Y)$ are proved analogously. It remains to observe that $EP(X_{\overline{\eta}}) = EP(X_y)$ for almost all $y \in \overline{Y}$, namely for any point $y \in U \cap \overline{Y}$, where U is an open set on which the sheaves $R^i f_* \mu_{l^k, X}$ are locally constant.

We also note the following facts about the invariants $\alpha^i_y(f; l)$.

Proposition 1.4. *The invariants* $\alpha^i_y(f; l)$ *depend only on the generic fiber* X_η *of the morphism* f. *More precisely, if* $f: X \to Y$ *and* $f': X' \to Y$ *are proper flat morphisms and* $X_\eta \simeq X'_\eta$ *over* η, *then* $\alpha^i_y(f; l) = \alpha^i_y(f'; l)$ *for any* $y \in \overline{Y}$ *and* $0 \le i \le 2 \dim X_\eta$.

Proof. Recall that

$$\alpha^i_y(f; l) = \delta_y(R^i f_* \mu_{l,X}) = \delta(K_y, (R^i f_* \mu_{l,X})^{\sim}_y).$$

Applying the base change theorem (see [16], exposé XI), we shall obtain

$$(R^i f_* \mu_{l,X})^{\sim}_y = i^*_y (R^i f_* \mu_{l,X}) = H^i(X \otimes_Y \overline{K}_y, \mu_l)$$
$$= H^i(X \otimes_Y \eta \otimes_\eta \operatorname{Spec} \overline{K}_y, \mu_l) = H^i(X_\eta \otimes_Y \overline{K}_y, \mu_l).$$

Analogously we have

$$(R^i f'_* \mu_{l,X'})^{\sim}_y = H^i(X'_\eta \otimes_Y \overline{K}_y, \mu_l),$$

from which it follows that

$$\alpha^i_y(f; l) = \delta(K_y, (R^i f_* \mu_{l,X})^{\sim}_y) = \delta(K_y, (R^i f'_* \mu_{l,X'})^{\sim}_y) = \alpha^i_y(f'; l),$$

which was required.

Proposition 1.5. *Assume that the generic fiber of the morphism f is smooth. Then for any point $y \in \overline{Y}$*

$$\alpha_y^i(f;\, l) = \alpha_y^{2n-i}(f;\, l), \quad 0 \leqslant i \leqslant 2n = 2\dim X_\eta.$$

Proof. As we saw above in the proof of Proposition 1.4,

$$(R^i f_* \mu_{l,X})_y^{\sim} = H^i(X_\eta \underset{Y}{\otimes} \overline{K}_y, \mu_l).$$

Applying Poincaré duality for étale cohomology ([22]), we will find that

$$(R^i f_* \mu_{l,X})_y^{\sim} = \mathrm{Hom}((R^{2n-i} f_* \mu_{l,X})_y^{\sim}, \mu_l).$$

Set $M_y^i = (R^i f_* \mu_{l,X})_y^{\sim}$. Recalling the definition of the invariants $\delta(K_y, M_y^i)$, we see that it suffices to prove that

$$\dim_{\mathbf{F}_l}(M_y^i)^{G_j} = \dim_{\mathbf{F}_l}(\mathrm{Hom}(M_y^{2n-i}, \mu_l)^{G_j}),$$

where G_j is a higher ramification group. But this fact, which asserts that the dimensions of invariant subspaces of representations and the conjugate representations are the same, is well known.

Corollary 1.6. *Under the assumptions of Proposition* 1.5,

$$\alpha_y^0(f;\, l) = \alpha_y^{2n}(f;\, l) = 0.$$

In fact, $\alpha_y^0(f;\, l) = \delta_y(f_* \mu_{l,X})$. But, since the base Y is normal, and the fibers of f are connected, $f_* O_X = O_Y$ ([5], Chapter III, 4.3). Hence $f_* \mathbf{G}_{m,X} = \mathbf{G}_{m,Y}$, which obviously gives $f_* \mu_{l,X} = \mu_{l,Y}$. The sheaf $\mu_{l,Y} \simeq (\mathbf{Z}/l)_Y$ (not canonically!) is constant on Y, so $\alpha_y^0(f;\, l) = \delta_y(\mu_{l,Y}) = 0$. Now use Proposition 1.5.

§2. The Euler-Grothendieck formula

Let F be a constructible étale sheaf on a curve Y, annihilated by multiplication by l. In this case the cohomology groups $H^i(Y, F)$ are also annihilated by multiplication by l, and consequently possess a natural structure of a linear space over the finite field $\mathbf{F}_l$. In particular, the Euler-Poincaré characteristic of the sheaf F is defined:

$$\chi(Y, F) = \sum (-1)^i \dim_{\mathbf{F}_l}(H^i(Y, F)).$$

Analogously, the local cohomology groups $H_y^i(F)$ are $\mathbf{F}_l$-spaces, where y is a closed point of Y. Set

$$\chi_y(F) = \sum (-1)^i \dim_{\mathbf{F}_l}(H_y^i(F)).$$

A theorem of Grothendieck ([15], [17], exposé X) asserts that

$$\chi(Y, F) = \dim_{\mathbf{F}_l}(F_{\overline{\eta}}) \cdot EP(Y) - \sum_{y \in \overline{Y}} \varepsilon_y(F), \tag{2.1}$$

where

$$\varepsilon_y(F) = \delta_y(F) + \dim_{\mathbf{F}_l}(F_{\bar\eta}) - \chi_y(F).$$

We give a more explicit formula for the term $\chi_y(F)$, and together with this also one for the entire local invariant $\epsilon_y(F)$.

Lemma 2.1. $\chi_y(F) = \dim_{\mathbf{F}_l}(F_y)$.

Proof. Let $\widetilde{Y}_y = \operatorname{Spec}(O^h_{Y,y})$, and let $\tilde{i}_y\colon \widetilde{Y}_y \to Y$ be the canonical morphism. Then we have an "exact sequence of the pair" for local cohomology ([16], exposé XVII)

$$0 \to H^0_y(F) \to H^0(\widetilde{Y}_y, \tilde{i}^*_y(F)) \to H^0(\tilde{\eta}_y, \widetilde{F}_y) \to H^1_y(F) \to 0,$$
$$H^1(\tilde{\eta}_y, \widetilde{F}) \simeq H^2_y(F), \quad H^i_y(F) = 0, \ i > 2.$$

Here we used the fact that $H^i(\widetilde{Y}_y, i^*_y(F)) = 0$, $i > 0$, since the scheme $\widetilde{Y}_y$ is strictly henselian, and the fact that $H^i(\tilde{\eta}_y, \widetilde{F}_y) = 0$, since $\operatorname{cd}(K_y) \le 1$. Thus we have

$$\chi_y(F) = \dim_{\mathbf{F}_l}(F_y) - \dim_{\mathbf{F}_l}(\widetilde{F}_y^{G_y}) + \dim_{\mathbf{F}_l}(H^1(G_y, \widetilde{F}_y)),$$

where $G_y = \operatorname{Gal}(\overline{K}_y/K_y)$. By the local duality theorem ([17], exposé I; cf. also [12] and [23]), $H^1(G_y, \widetilde{F}_y) \simeq \operatorname{Hom}(\widetilde{F}_y^{G_y}, \mu_l)$. However, obviously (cf. the proof of Proposition 1.5)

$$\dim_{\mathbf{F}_l}(\widetilde{F}_y^{G_y}) = \dim_{\mathbf{F}_l}(\operatorname{Hom}(\widetilde{F}_y^{G_y}, \mu_l)),$$

from which we also get the assertion of the lemma.

Corollary 2.2.

$$\chi(Y, F) = \dim_{\mathbf{F}_l}(F_{\bar\eta}) EP(Y) - \sum_{y \in \overline{Y}} [\dim_{\mathbf{F}_l}(F_{\bar\eta}) - \dim_{\mathbf{F}_l}(F_y) + \delta_y(F)]. \tag{2.2}$$

§3. Proof of Theorem 1.1

In this section the notation is the same as in the preceding sections. Consider the Leray spectral sequence for the morphism $f\colon X \to Y$ and the sheaf $\mu_{l,X}$:

$$E_2^{p,q} = H^p(Y, R^q f_* \mu_{l,X}) \Rightarrow H^{p+q}(X, \mu_{l,X}).$$

By the invariance of the Euler characteristic in the spectral sequence we have

$$\chi(X, \mu_{l,X}) = \sum_i (-1)^i \chi(Y, R^i f_* \mu_{l,X}). \tag{3.1}$$

As was already explained in §1, the sheaves $R^i f_* \mu_{l,X}$ are constructible and are annihilated by multiplication by l. Thus we can apply the Euler-Grothendieck formula (2.2) to it. As a result we obtain

$$\chi(Y, R^if_*\mu_{l,X}) = (\dim_{\mathbf{F}_l}(R^if_*\mu_{l,X})_{\bar\eta})\,EP(Y)$$
$$-\sum_{y\in\bar Y}[\dim_{\mathbf{F}_l}(R^if_*\mu_{l,X})_{\bar\eta} - \dim_{\mathbf{F}_l}(R^if_*\mu_{l,X})_y + \delta_y(R^if_*\mu_{l,X})]. \tag{3.2}$$

By the base change theorem ([16], exposé XII),

$$(R^if_*\mu_{l,X})_{\bar\eta} = H^i(X_{\bar\eta}, \mu_{l,X_{\bar\eta}}), \quad (R^if_*\mu_{l,X})_y = H^i(X_y, \mu_{l,X_y}).$$

Substituting (3.2) in (3.1) and using the notation of §§1 and 2 we will obtain

$$\chi(X, \mu_{l,X}) = \chi(X_{\bar\eta}, \mu_{l,X_{\bar\eta}})EP(Y) + \sum_{y\in\bar Y}[\chi(X_y, \mu_l) - \chi(X_{\bar\eta}, \mu_l) + \alpha_y(f; l)]. \tag{3.3}$$

We have used the fact that $\delta_y(R^0f_*\mu_{l,X}) = \alpha_y^0(f;\, l) = 0$ is equal to zero (cf. the proof of Corollary 1.6).

To prove Theorem 1.1 it remains to prove the following assertion.

Lemma 3.1. *Let Z be a proper k-scheme. For any prime $l \neq p$ we have $\chi(Z, \mu_{l,Z}) = EP(Z)$.*

Proof. Let μ_{l^∞} denote the sheaf $\varinjlim_k \mu_{l^k,Z}$ on Z. For any integer $k > 0$ we have an exact sequence

$$0 \to \mu_{l^k,Z} \to \mu_{l^\infty} \xrightarrow{l^k} \mu_{l^\infty} \to 0.$$

The corresponding exact cohomology sequence has the form

$$0 \to H^{i-1}(Z, \mu_{l^\infty})^{(l^k)} \to H^i(Z, \mu_{l^k,Z}) \to H^i(Z, \mu_{l^\infty})_{l^k} \to 0. \tag{3.4}$$

Here and later on for any abelian group A we let $A^{(n)}$ (respectively A_n) denote the cokernel (the kernel) of the homomorphism of multiplication by n: $A \to A$ $(a \mapsto na)$.

Passing to the projective limit over powers of l in exact sequence (3.4), we will obtain the exact sequence

$$0 \to H^{i-1}(Z, \mu_{l^\infty}) \underset{\mathbf{Z}_l}{\otimes} \mathbf{Q}_l/\mathbf{Z}_l \to H^i(Z, \mathbf{Z}_l(1)) \to T_l(H^i(Z, \mu_{l^\infty})) \to 0. \tag{3.5}$$

Since the groups $H^i(Z, \mu_{l^\infty})$ are of finite type (this follows from the finiteness of the groups $H^i(Z, \mu_{l^k})$ and an exact sequence (3.4)), they have the form

$$H^i(Z, \mu_{l^\infty}) = (\mathbf{Q}_l/\mathbf{Z}_l)^{\beta_i} \oplus t^i(Z; l),$$

where $t^i(Z;\, l)$ are finite groups.

On the other hand, it follows from (3.5) that

$$\mathbf{Q}_l^{b_i(Z;\,l)} = H^i(Z, \mathbf{Q}_l) \simeq H^i(Z, \mathbf{Z}_l(1)) \otimes \mathbf{Q}_l \simeq T_l(H^i(Z, \mu_{l^\infty})) \otimes \mathbf{Q}_l = \mathbf{Q}_l^{\beta_i}.$$

Hence $b_i(Z; l) = \beta_i$ and

$$H^i(Z, \mu_{l^\infty}) = (\mathbf{Q}_l/\mathbf{Z}_l)^{b_i(Z;l)} \oplus t^i(Z; l).$$

Now applying (3.4) for $k = 1$ we will obtain

$$\dim_{\mathbf{F}_l}(H^i(Z, \mu_{l,Z})) = b_i(Z; l) + \dim_{\mathbf{F}_l}(t^{i-1}(Z; l)^{(l)}) + \dim_{\mathbf{F}_l}(t^i(Z; l)_l).$$

Hence

$$\chi(Z, \mu_{l,Z}) = EP(Z) + \sum_{i=0}^{\infty} (-1)^i (\dim_{\mathbf{F}_l}(t^i(Z; l)^{(l)}) - \dim_{\mathbf{F}_l}(t^i(Z; l)_l)].$$

Since the groups $t^i(Z; l)$ are finite, the expressions in the square brackets are equal to zero.

Replacing $\chi(\ , \mu_l)$ by $EP(\ )$ in (3.3), we will obtain the formula of Theorem 1.1.

§4. An application

In this section we let IS denote the class of projective k-schemes X such that all singular points of the corresponding reduced scheme X_{red} are isolated.

We let $\mathbf{P}^r$ denote projective space of dimension r over k, and $\check{\mathbf{P}}^r$ the dual projective space of hyperlanes of $\mathbf{P}^r$. A line D in $\check{\mathbf{P}}^r$, considered as a closed point of $\mathrm{Gr}(l, r)$, is called a *pencil of hyperplanes* in $\mathbf{P}^r$. We shall denote by H_t a hyperplane corresponding to a point $t \in D$. The intersection of two hyperplanes $H_{t_1} \cdot H_{t_2}$ $(t_1 \neq t_2)$ will be called an *axis* of the pencil.

Definition. Let i: $X \hookrightarrow \mathbf{P}^r$ be a projective imbedding of a scheme of class IS. A pencil $D \in \mathrm{Gr}(1, r)$ will be called *good* for the imbedding i if

a) an axis of D intersects X transversally (cf. [5], Chapter IV, 17.3),

b) there exists an open set $U \subset D$ such that H_t intersects X transversally for all $t \in U$, and

c) H_{t_0} intersects X transversally for all $t_0 \in D \backslash U$ except for a finite number of points.

In the case where X is smooth and if in condition c) H_{t_0} has only ordinary double points, the definition of a good pencil turns into the definition of a Lefschetz pencil in the sense of Katz' lecture ([18], exposé VII).

An imbedding i: $X \longrightarrow \mathbf{P}^r$ will be called good if there exists a good pencil relative to i.

Let $Y \subset X \times \check{\mathbf{P}}^r$ be the subscheme of "incidences" of the $\check{\mathbf{P}}^r$-scheme $X \times \mathbf{P}^r$ whose fiber at the point $H \in \check{\mathbf{P}}^r(k)$ is $X \cdot H$. An equivalent definition is that Y is the graph of a rational map $X \longrightarrow \check{\mathbf{P}}^r$ defined by a complete linear system of hyperplane sections of X. Let $S(Y)$ be the subscheme of singular points (i.e. points where it is not smooth) of the morphism f: $Y \longrightarrow \check{\mathbf{P}}^r$ induced by the projection $X \times \check{\mathbf{P}}^r \longrightarrow \check{\mathbf{P}}^r$. Let $\check{X} = f(S(Y))$ be the projection of $S(Y)$ onto $\check{\mathbf{P}}^r$. In case X is smooth, $\check{X}$ is the variety "dual to X". The points of $S(Y)$ are interpreted as pairs (x, H), where either x is singular on X, or H is tangent to X at x (i.e. $H \cdot X$ is singular at x).

Lemma 4.1. *Assume that the morphism $f\colon S(Y) \longrightarrow \check{X}$ is quasi-finite over every maximal point of $\check{X}$. Then the imbedding $X \hookrightarrow \mathbf{P}^r$ is good.*

Proof. Let $F \subset \check{X}$ be a subset of points $H \in \check{X}$ such that $\dim f^{-1}(H) > 0$. By Chevalley's theorem (see [5], Chapter IV, 13.1) this set is closed in each irreducible component of $\check{X}$. Since $\dim \check{X} \leq r-1$ (Bertini's theorem for hyperplane sections [11]), we have $\dim F \leq r-2$. Now we note that the pencil D in $\check{\mathbf{P}}^r$ is a good pencil if and only if

a) an axis of D intersects X transversally,

b) D is not contained in $\check{X}$, and

c) D does not intersect F.

We choose a point $H \in \check{\mathbf{P}}^r$ outside of $\check{X}$. Then $H \cdot X$ is a smooth subscheme of X, imbedded in $H \simeq \mathbf{P}^{r-1}$, and $\dim \widecheck{H\cdot X} \leq r-2$. We choose a line D passing through H and not intersecting $\widecheck{H\cdot X}$ and F. This can be done since the codimension of the varieties $\widecheck{H\cdot X}$ and F are ≥ 2 (see, for example, [10], p. 88). Obviously the line D will also be the desired pencil.

The proof of the following proposition was suggested by F. Zak.

Proposition 4.2. *For any imbedding $i\colon X \hookrightarrow \mathbf{P}^r$ of a scheme of class IS the composition*

$$s_d \circ i : X \hookrightarrow \mathbf{P}^r \hookrightarrow \mathbf{P}^{\binom{r+d}{d}-1},$$

where s_d is the Segre imbedding, $(d, p) = 1$, $d > 1$, is a good imbedding.

Proof. Let $\check{X}$ be the variety dual to X relative to the imbedding $s_d \circ i$. We shall show that for any nonsingular point $x_0 \in X$ there exists a point $H \in \check{X}$ such that $H \cdot X$ has an isolated singularity at x_0. From this it will follow that for any irreducible component S_i of the scheme $S(Y)$ there exists a fiber of the morphism $f_i\colon S_i \longrightarrow \check{X}$ consisting of a finite number of points. From this we obtain that the morphism f_i is quasifinite over a general point of $f_i(S_i)$. After this we use Lemma 4.1. Having chosen suitable homogeneous coordinates $t_0, \cdots, t_r$ in $\mathbf{P}^r$, we may assume that x_0 is the point $(1, 0, \cdots, 0)$ and that the functions t_i/t_0, $i = 1, \cdots, k$, are local coordinates on X in a neighborhood of x_0. Consider a hypersurface Γ with the equation $t_1^d + \cdots + t_k^d = 0$. However, the intersection $\Gamma \cdot X$ is the intersection $H \cdot X$, where H is the corresponding hypersurface in $\mathbf{P}^{\binom{r+d}{d}-1}$ and the point x_0 is the only singular point of $H \cdot X$. This proves the proposition.

Now let X be an arbitrary reduced k-scheme of class *IS*. By the preceding proposition we may assume that X is well imbedded in $\mathbf{P}^r$. We choose a good pencil of hyperplane sections on X and let $\phi\colon X \longrightarrow \mathbf{P}^1$ be the corresponding rational map.

Proposition 4.3. *Assume that* $\dim X \leq 3$ *in case* $p > 0$. *There exists a commutative diagram of rational maps*

$$\begin{array}{ccc} X & \overset{\pi}{\longleftarrow} & X \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle f} & \\ & \mathbf{P}^1 & \end{array},$$

where π is a birational morphism which is a composition of monoidal transforms with nonsingular centers, and f a projective morphism with fibers of class IS with geometrically connected smooth general fiber.

Proof. The morphism π is none other than the resolution of points where the rational map is undefined, whose existence was proved by Hironaka [6] ($p = 0$) and Abhyankar [2] ($p > 0$ and $\dim X \leq 3$). The fibers of the morphism will belong to the class *IS*; and moreover, by the definition of a good pencil, all the fibers of f are smooth, and consequently the general fiber is also smooth. It is obvious that it is geometrically connected.

Proposition 4.4. *Let $\psi: X' \to X$ be a monoidal transformation (with nonsingular center Y of codimension d) of algebraic k-schemes. Assume that the imbedding $i: Y \hookrightarrow X$ is regular and the scheme is normal. Then*

$$EP(X') = EP(X) + (d-1)EP(Y).$$

Proof. Since the imbedding $i: Y \hookrightarrow X$ is regular, the structure of the scheme X' is well known (see [5], Chapter IV, 19.4), and also [9], 12.2). Let $Y' = X' \times Y$, and let $g: Y' \to Y$ be the restriction of the morphism ψ to Y'. Then a) ψ is an isomorphism outside Y', and b) g is the canonical projection $\mathbf{P}(N) \to Y$, where N is the projectivized conormal bundle to Y. By virtue of the invariance of the Euler characteristic in the spectral sequence

$$E_2^{p,q} = H^p(X, R^q\psi_*\mu_{l,X'}) \Rightarrow H^{p+q}(X', \mu_{l,X'}),$$

we have

$$\chi(X', \mu_{l,X'}) = \chi(X, \psi_*\mu_{l,X'}) + \sum_{i=1}^{2d-2} (-1)^i \chi(X, R^i\psi_*\mu_{l,X'}).$$

Since X is normal, $\psi_*\mu_{l,X'} = \mu_{l,X}$ (cf. the proof of Corollary 1.6); and, moreover, the sheaves $R^i\psi_*\mu_{l,X'}$, $i > 0$, are concentrated on Y and, by the base change theorem, coincide with the sheaves $R^i g_*\mu_{l,Y'}$. Hence

$$\chi(X', \mu_{l,X'}) = \chi(X, \mu_{l,X}) + \sum_{i=1}^{2d-2} (-1)^i \chi(Y, R^i g_*\mu_{l,Y'}). \tag{4.1}$$

The sheaves $R^i g_*\mu_{l,Y'}$ are easily calculated (see [18] for example). We have

$$R^i g_*\mu_{l,Y'} = \begin{cases} (\mathbf{Z}/l)_Y, & i = 2k \leqslant 2d-2, \\ 0, & i = 2k+1 < 2d-2. \end{cases}$$

Substituting this in (4.1), we will obtain

$$\chi(X', \mu_{l,X'}) = \chi(X, \mu_{l,X}) + (d-1)\chi(Y, (\mathbf{Z}/l)_Y).$$

Now we use Lemma 3.1.

Theorem 4.5. *Let X be an algebraic scheme of class IS over a field k of characteristic zero. Then $EP(X)$ does not depend on the prime number l.*

Proof. By [16] (exposé IX) we may assume that X is reduced. We shall argue by induction on dim X. In the case dim $X = 1$ the assertion follows from direct computations (a spectral sequence applied to the normalization morphism; cf. [3], §2). Apply Proposition 4.3 to X. Let

$$\begin{array}{ccc} X & \leftarrow & X \\ & \searrow \; \swarrow & \\ & \mathbf{P}^1 & \end{array}$$

be the diagram whose existence is asserted in Proposition 4.3. By the induction hypothesis and Proposition 4.4 it suffices to prove the assertion for the scheme $\overline{X}$. For this we use Corollary 1.3 and the induction hypothesis again. The theorem is proved.

Theorem 4.6. *Let X be a scheme of class IS over a field k of characteristic $p > 0$. Assume that* $\dim X \leq 2$. *Then $EP(X)$ does not depend on the prime number l.*

Proof. Arguing as in the last proof, we reduce everything to the case of a surface $\overline{X}$ for which there exists a proper morphism $f: X \longrightarrow \mathbf{P}^1$ with geometrically connected smooth general fiber. By Theorem 1.1 it suffices to prove that the higher ramification invariants $\alpha_y(f; l)$ do not depend on l. In view of Corollary 1.6, we have $\alpha_y(f; l) = \delta_y(R^1 f_* \mu_{l,\overline{X}})$. However $\delta_y(R^1 f_* \mu_{l,\overline{X}}) = \delta(K_y, A(K_y)_l)$, where A is the Jacobian of the general fiber of f. The last invariant does not depend on the prime number l (in case $\dim A = 1$ see [13]; in the general case see [18], exposé IX).

§5. The conductor of a family of algebraic varieties

In this section the notation is the same in the preceding sections. Let $f: X \longrightarrow Y$ be a proper flat morphism of a scheme X onto a smooth complete curve Y with geometrically connected generic fiber X_η of dimension n. We have (Theorem 1.1)

$$EP(X) - EP(Y)\, EP(X_{\bar{\eta}}) = \sum_{y \in \overline{Y}} [EP(X_y) - EP(X_{\bar{\eta}}) + \alpha_y(f; l)]. \tag{5.1}$$

It is natural to conceive of the right side of this formula as an invariant of the degeneration of the morphism f.

Definition. The *exponent of the conductor of the morphism f* at a point $y \in \overline{Y}$ is the number

$$c_y(f; l) = EP(X_y) - EP(X_{\bar{\eta}}) + \alpha_y(f; l).$$

Conjecture 1. *$c_y(f; l)$ does not depend on the prime number $l \neq p$.*

Proposition 5.1. *Conjecture 1 holds in the following cases:*

a) $p = 0$.

b) *The generic fiber of f is a geometrically irreducible algebraic curve.*

Proof. In case a) the assertion is obvious, since $\alpha_y(f;\, l) = 0$. In case b) we must prove that $\alpha_y(f;\, l)$ does not depend on l. Since the fibers of f must be geometrically irreducible (by hypothesis), we have $R^2 f_* \mu_{l,X} = (\mathbf{Z}/l)_Y$, whence $\alpha_y^2(f;\, l) = 0$. Thus it remains to prove that $\alpha_y^1(f;\, l) = \delta_y(R^1 f_* \mu_{l,X})$ does not depend on l for any $y \in \overline{Y}$. Let J be the generalized Jacobian of the curve $X_{\overline{\eta}}$ (or, what is the same, the connected component of the Picard scheme $\mathrm{Pic}(X_{\overline{\eta}}/\overline{k(\eta)})$). The group J is an extension of an abelian variety A by a linear commutative group L. The latter, in turn, is a direct product of a torus $T \simeq \mathbf{G}_m^s$ by a unipotent group U (see [19]). Since the group $U(\overline{K}_y)$ is uniquely divisible by l, we have an exact sequence

$$0 \to (\mu_{l,\overline{K}_y})^s \to J\,(\overline{K}_y)_l \to A\,(\overline{K}_y)_l \to 0.$$

Since the invariant $\delta(K_y)$ is additive, we will obtain $\delta(K_y,\, J(\overline{K}_y)_l) = \delta(K_y,\, A(\overline{K}_y)_l)$. The last number does not depend on l by virtue of the result of Grothendieck already cited (see [18], exposé IX). The proposition is proved.

Definition. The divisor

$$C\,(f;\, l) = \sum_{y \in \overline{Y}} c_y\,(f;\, l)\, y$$

on Y will be called the *l-conductor of the morphism f*.

Conjecture 1 implies the independence of the l-conductor on l, which would allow us to have a good definition of conductor.

In any case, by formula (5.1),

$$c\,(f) \overset{\mathrm{dfn}}{=} \deg C\,(f;\, l) = \sum_{y \in \overline{Y}} c_y\,(f;\, l) = EP(X) - EP(X_{\overline{\eta}})\,EP(Y)$$

does not depend on l.

In the case where the scheme X is smooth and the fibers of f are reduced, we can give an invariant interpretation of the number $c(f) = \deg C(f;\, l)$ by means of the local invariants of singularities of the morphism f (see [7]). We have

$$c\,(f) = \sum_{i=0}^{n} (-1)^{i+n-1} \dim_k \mathrm{Ext}^i\,(O_X, K_\cdot\,(f)), \tag{5.2}$$

where $K\,(f)$ is some complex of sheaves associated to the morphism f (see loc.cit.). If all the fibers of the morphism f have only isolated singularities, then formula (5.2) degenerates into the following:

$$c\,(f) = (-1)^{n-1} \dim_k \mathrm{Hom}\,(O_X, K_\cdot\,(f)) = (-1)^{n-1} \dim_k H^0\,(X, O_X/\vartheta_{X/Y}), \tag{5.3}$$

where $\vartheta_{X/Y}$ is the Jacobi sheaf of the morphism f (or different in the sense of [4]). In case $\dim X = 2$, formula (5.3) was proved in [4].

The following conjecture is connected with the question of "localization" of formula (5.2).

Conjecture 2. *Let* $\widetilde{X}_y = X \otimes_Y O_{Y,y}$, *and let* $i: \widetilde{X}_y \hookrightarrow X$ *be the canonical imbedding. If* X *is smooth, and the fibers of* f *are reduced, then*

$$c_y(f; l) = \sum_{i=0} (-1)^{i+n-1} \dim_k \operatorname{Ext}^i (O_{\widetilde{X}_y}, i^*(K_.(f))).$$

It holds in case $\dim X = 2$ and $p = 0$ (see [7]). An unclear proof of this fact can be found also in Jung ([8], Chapter VI).

Conjecture 3. *The invariants* $\alpha_y^i(f; l) = \delta_y(R^i f_* \mu_{l,X})$ *do not depend on the prime* $l \neq p$. *In particular, the invariants* $\alpha_y(f; l)$ *and* $c_y(f; l)$ *do not depend on* l.

Assume further that the general fiber of f is smooth; then by virtue of Corollary 1.6 it suffices to prove this conjecture only for the values $1 \leq i \leq n$, where $n = \dim X_\eta$. Moreover, as in the proof of Proposition 5.1 (case b)), we may assume that $i > 1$. In particular, if $\dim X = 3$, it remains to verify the invariance of $\alpha_y^2(f; l)$.

In the general case it is easy to verify that the preceding conjecture follows from a conjecture of Serre and Tate on l-adic representations (see [21], Appendix).

Let $\widetilde{\eta}_y$ be the general point of the scheme $\operatorname{Spec}(O^b_{Y,y})$ and $\overline{X}_{\widetilde{\eta}_y}$ be the geometric general fiber of the corresponding morphism $\widetilde{f}_y: X \otimes_Y O^b_{Y,y} \to \operatorname{Spec}(O^b_{Y,y})$. The Galois group G_y of the field $K_y = k(\widetilde{\eta}_y)$ acts on the rational l-adic cohomology $H^i_l(y) = H^i(X_{\widetilde{\eta}_y}, \mathbf{Q}_l)$. Let $\epsilon^i_y(X_\eta; l)$ be the codimension of the invariants $H^i_l(y)^{G_y}$, and let

$$\varepsilon_y(X_\eta; l) = \sum_{i=1}^{2n-1} (-1)^i \varepsilon_y^i(X_\eta; l).$$

Following Serre and Tate, we could define the exponent of the conductor (respectively the conductor) at a point y of the generic fiber X_η of the morphism f by setting

$$\bar{c}_y(X_\eta; l) = \varepsilon_y(X_\eta; l) + \alpha_y(f; l)$$

$$\left(\text{respectively,} \quad \overline{C}(X_\eta; l) = \sum_{y \in \overline{Y}} \bar{c}_y(X_\eta; l)\, y \right).$$

Conjecture 4. *Assume that the Y-scheme* X *is Y-Néron in the sense of Raynaud* [14]. *Then*

$$\bar{c}_y(X_\eta; l) = -c_y(f; l) + b_{2n}(X_y) - 1.$$

We note that this conjecture holds in case X is a smooth algebraic surface.

Furthermore, we assume that all the singular points of fibers of the morphism f are isolated and the scheme X is smooth. Let x be a closed point of X, and $y = f(x)$. Consider the canonical morphism $f_x: \operatorname{Spec}(O^b_{X,x}) \to \operatorname{Spec}(O^b_{Y,y})$ induced by the natural imbedding $O^b_{Y,y} \hookrightarrow O^b_{X,x}$. Let $X(x)$ denote the K_y-scheme $\operatorname{Spec}(O^b_{X,x} \otimes_{O^b_{Y,y}} K_y)$, and let $\overline{X}(x) = X(x) \otimes_{K_y} \overline{K}_y$ be the geometric general fiber of the morphism f_x. The

Galois group $G_y = \mathrm{Gal}(\overline{K}_y/K_y)$ acts in a natural way on $\overline{X}(x)$. Thus we have defined a representation

$$\rho_x^i: G_y \to \mathrm{Aut}_{\mathbf{F}_l}(H^i(\overline{X}(x), \mu_{l,\overline{X}(x)})),$$

which we shall call the *local monodromy* at the point x. This representation is an algebraic analogue of the local Picard-Lefschetz monodromy studied in [25], [26].

Proposition 5.2. a) $H^i(\overline{X}(x), \mu_{l,\overline{X}(x)}) = 0$, $i > n$.

b) *The spaces* $H^i(\overline{X}(x), \mu_{l,\overline{X}(x)})$ *are finite-dimensional over the field* $\mathbf{F}_l$.

c) *If the morphism* f *is smooth at the point* x, *then* $H^i(\overline{X}(x), \mu_{l,\overline{X}(x)}) = 0$ *for* $i > 0$.

Proof. First we shall prove c). Since the morphism f is smooth at the point x, there exists an open set U containing x, and an étale Y-morphism $U \to Y[T_1, \cdots, T_n]$, where

$$n = \dim X_\eta \text{ и } Y[T_1, \ldots, T_n] = Y \underset{\mathbf{Z}}{\otimes} \mathrm{Spec}(\mathbf{Z}[T_1, \ldots, T_n])$$

(see [5], Chapter IV, 17.11.4). From this it obviously follows that the scheme $X(x)$ is isomorphic to the local henselian scheme $\mathrm{Spec}(K_y\{T_1, \cdots, T_n\})$, where $K_y\{T_1, \cdots, T_n\}$ is the henselization of the localization of the ring of polynomials $K_y[T_1, \cdots, T_n]$ at the point $(0, \cdots, 0)$. Thus the scheme $\overline{X}(x) = \mathrm{Spec}(\overline{K}_y\{T_1, \cdots, T_n\})$ is strictly henselian, and consequently $H^i(\overline{X}(x), \mu_{l,\overline{X}(x)}) = 0$, $i > 0$.

Now we shall prove assertion b). Let $\widetilde{X}_y = X \otimes_Y O^h_{Y,y} \to \mathrm{Spec}(O^h_{Y,y})$ be the canonical base change morphism and $\overline{X}_{\widetilde{\eta}_y} = \widetilde{X}_y \otimes_{O^h_{Y,y}} \overline{K}_y$ its geometric general fiber. Consider the canonical morphism i: $\overline{X}_{\widetilde{\eta}_y} \to \widetilde{X}_y$, which is a composition of the canonical projection morphisms $\overline{X}_{\widetilde{\eta}_y} \to \widetilde{X}_y \otimes_{O^h_{Y,y}} K_y \to \widetilde{X}_y$. Let

$$E_2^{p,q} = H^p(\widetilde{X}_y, R^q i_* \mu_{l,\overline{X}_{\widetilde{\eta}_y}}) \Rightarrow H^{p+q}(\overline{X}_{\widetilde{\eta}_y}, \mu_l) \tag{5.4}$$

be the Leray spectral sequence for the morphism i and the sheaf $\mu_{l,\overline{X}_{\widetilde{\eta}_y}}$. For any closed point $x \in X_y$ we have ([16], exposé VIII)

$$(R^q i_* \mu_{l,\overline{X}_{\widetilde{\eta}_y}})_x = H^q(\overline{X}(x), \mu_{l,\overline{X}(x)}).$$

By c) it follows from this that the sheaves $R^q i_* \mu_{l,\overline{X}_{\widetilde{\eta}_y}}$ for $q > 0$ are concentrated at the singular points of the fiber X_y. Thus in the spectral sequence (5.4) the terms $E_2^{p,q}$ with p, $q > 0$ are equal to zero. Since the terms $E_2^{p,0}$ are finite-dimensional over $\mathbf{F}_1$ and the abutments H^{p+q} are also finite-dimensional, the terms $E^{0,q}$ are finite-dimensional.

Assertion a) follows from the fact that $\overline{X}(x)$ is an affine scheme of dimension n ([16], exposé XIV). This proves the proposition.

Definition. Set

$$b_x(f; l) = \sum_{i=1}^{n} (-1)^{i-1} \dim_{\mathbf{F}_l} (H^i (\overline{X}(x), \mu_{l,\overline{X}(x)})),$$

$$\sigma_x(f; l) = \sum_{i=1}^{n} (-1)^{i-1} \delta (K_y, H^i (\overline{X}(x), \mu_{l,\overline{X}(x)})).$$

Proposition 5.3. *Under the assumptions indicated above*

$$EP(X_y) - EP(X_{\overline{\eta}}) = \sum_{x \in X_y(k)} b_x(f; l), \quad \alpha_y(f; l) = \sum_{x \in X_y(k)} \sigma_x(f; l).$$

In particular

$$c_y(f; l) = \sum_{x \in X_y(k)} (b_x(f; l) + \sigma_x(f; l)).$$

Proof. As we saw in the proof of Proposition 5.2,

$$R^q i_* \mu_{l,\overline{X}_{\widetilde{\eta}_y}} = \bigoplus_{x \in X_y(k)} H^q (\overline{X}(x), \mu_{l,\overline{X}(x)})$$

is a constant sheaf concentrated at the singular points of the fiber X_y. By the spectral sequence (5.4), this implies

$$\chi(\overline{X}_{\widetilde{\eta}_y}, \mu_l) = \chi(\widetilde{X}_y, i_* \mu_{l,\overline{X}_{\widetilde{\eta}_y}}) - \sum_{x \in X_y(k)} b_x(f; l).$$

Since the schemes $\overline{X}(x)$ are connected, $i_* \mu_{l,\overline{X}_{\widetilde{\eta}_y}} = \mu_{l,\widetilde{X}_y}$. By virtue of the base change theorem, from this we have

$$\chi(\widetilde{X}_y, i_* \mu_{l,\overline{X}_{\widetilde{\eta}_y}}) = \chi(X_y, \mu_l) \text{ и } \chi(\overline{X}_{\widetilde{\eta}_y}, \mu_{l,\overline{X}_{\widetilde{\eta}_y}}) = \chi(X_{\overline{\eta}}, \mu_{l,X_{\overline{\eta}}}).$$

Applying Lemma 3.1., we will obtain the equality

$$EP(X_y) - EP(X_{\overline{\eta}}) = \sum_{x \in X_y(k)} b_x(f; l).$$

It is easy to see that, thanks to the action of the group G_y on the sheaves $R^q i_* \mu_{l,\overline{X}_{\widetilde{\eta}_y}}$, the spectral sequence (5.4) is a spectral sequence of G_y-modules (cf. [24], p. 290). By the invariance of the additive function $\delta(K_y)$ in the spectral sequence, we will obtain

$$\alpha_y(f; l) = \sum_{i=0}^{2n} (-1)^i \delta (K_y, H^i (\overline{X}_{\widetilde{\eta}_y}, \mu_l)) =$$

$$= \sum_{i=0}^{2n} (-1)^i \delta (K_y, H^i (\tilde{X}_y, \mu_l)) - \sum_{x \in X_y(k)} \sum_{i=1}^{n} (-1)^i \delta (K_y, H^i (\overline{X}(x), \mu_l)).$$

Since G_y acts as the identity on $H^i(\tilde{X}_y, \mu_l)$, we will obtain the desired equality

$$\alpha_y (f; l) = \sum_{x \in X_y(k)} \sigma_x (f; l).$$

This proves the proposition.

Conjecture 5(1). *For any closed point* $x \in X_y(k)$

$$b_x (f; l) + \sigma_x (f; l) = (-1)^{n-1} \dim_k (O_{X,x}/\vartheta_{X/Y,x}),$$

where $\vartheta_{X/Y}$ *is the Jacobi sheaf of the morphism* f.

Remarks. 1. If all the singular points of the fiber X_y are isolated, Conjecture 3 obviously follows from Conjecture 5 (cf. (5.3)).

2. Conjecture 5 is an algebraic analogue of a theorem of Milnor (see [25], and also [26], Appendix). This theorem also suggestes a conjecture that the groups $H^i(\overline{X}(x), \mu_l)$ are equal to zero for $i \neq 0, n$.

3. Analogous to the local monodromy $\rho_x^i: G_y \to \mathrm{Aut}_{F_l}(H^i(\overline{X}(x), \mu_l))$ we can consider the l-adic representation $\tilde{\rho}_x^i: G_y \to \mathrm{Aut}_{Z_l}(H^i(\overline{X}(x), Z_l))$ and state an analogue of the Serre-Tate conjecture for $\tilde{\rho}_x^i$ (see [21], Appendix). From the validity of these conjectures we would get, for example, that the numbers $\delta_y(f; l)$ do not depend on l.

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