ABSTRACT ALGEBRAIC GEOMETRY

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The present article consists mainly of papers reviewed in Referativnyi Zhurnal "Matematika" (Soviet Mathematical Abstracts Journal) during 1960-1971 and is concentrated around questions connected with abstract algebraic geometry. By the latter we mean first of all the foundations as well as those sections which arose in the mid-Fifties under the influence of Serre and Grothendieck. The ideas and methods of the latter turned out to have an immense influence on almost all sections of algebraic geometry (and on many other branches of mathematics) and gave a start to "Modern Algebraic Geometry." A fundamental role in the latter is played by precisely those sections to which the present article is devoted. The year 1958, when Grothendieck announced at the Edinburgh congress his program of investigations in algebraic geometry, serves as the starting date for this article. In it we have not included those papers on commutative algebra and analytical geometry which were written under the direct influence of Grothendieck's paper. We have also left aside such important sections of algebraic geometry as K-theory, the theory of algebraic cycles, the resolution of singularities, the theory of modules, and theory of intersections; the theory of group schemata* and of formal groups has been touched on only incidentally, and they deserve a separate survey.

§1. Foundations of Algebraic Geometry

The establishment of the fundaments of classical algebraic geometry began comparatively long ago. Beginning with Hilbert and his successors (Noether, Krull, van der Waerden) algebraic geometry was based on the theory of polynomial ideals. The results in the papers from this school were summarized in Brogner's book [216]. After the appearance in 1946 of Weil's book [507] valuation theory and field theory (the language of Weil's "generic points") became the commonly-accepted fundaments of algebraic geometry. Weil also introduced new objects of study in algebraic geometry, namely, abstract algebraic varieties. The powerful methods of commutative and, in particular, local algebra were introduced into abstract algebraic geometry (signifying at that time the study of abstract algebraic varieties over an arbitrary field of constants) by Zariski and his school (Samuel, Cohen, etc.). An account of these methods can be found in Samuel [459].

Serre's paper [468] on coherent algebraic sheaves served as the source of a subsequent process of reorganization of the fundaments of algebraic geometry ("Serre's language"). In it for the first time there was introduced into algebraic geometry the ideas and methods of homological algebra and also was extended the notion of algebraic variety (Serre's "algebraic space"). Another point of view on algebraic varieties ("Chevalley's schemata") was developed by Chevalley [154] and Nagata [392, 297].

Finally, in 1958 Grothendieck, by developing and generalizing Serre's ideas, introduced into algebraic geometry the language of functors and of the theory of categories and also essentially extended the notion of an algebraic variety by laying the beginnings of the theory of schemata. Starting with the publication of the first chapters of Grothendieck and Dieudonne's treatise [240-248], the language of the theory of schemata solidly became the custom of algebraic geometers and is now most widespread and commonly accepted. The orderliness, completeness, and geometricity of this theory permitted algebraic geometers not to dwell any longer on the foundations and to return to solving the concrete problems of algebraic geometry, bequeathed by previous generations, as well as to develop the connection of this science with other areas of mathematics.

*Translator's note: I have preferred to use "schema" (plural "schemata") even though there is a tendency nowadays to use "scheme" (plural "schemes").

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1. Development of the Notion of Algebraic Variety. As we see from the preceding, the development of the foundations of algebraic geometry paralleled the development of the notion of an algebraic variety. Beginning with the notion of Weil's abstract variety and ending with the notion of Artin's and Moishezon's notion of an algebraic space, this development was stimulated chiefly by various constructions in algebraic geometry (the Jacobi variety, the Poincare schema, the space of modules, minimal modules, etc.; see §5). The classical definition of an algebraic variety was restricted to affine or projective k-sets over an algebraically closed field k, i.e., closed in the Zariski topology by subsets of an affine or projective space over k. The idea of carrying over the construction of differentiable varieties (with the aid of pasting together) onto algebraic varieties was due to Weil. In [507] he defined abstract algebraic varieties as a system of affine algebraic varieties (V_α) in each of which there are picked out open subsets $W_{\alpha\beta}\subset V_\alpha$ consistently isomorphic with the open subsets $W_{\beta\alpha}\subset V_\beta$. Weil succeeded in extending onto these varieties all the fundamental concepts of classical algebraic geometry. In 1950 Leray [336] introduced the notion of a sheaf on a topological space. Cartan's 1950/51 seminar in Paris was devoted to the development of the theory of sheaves. This notion permitted the definition of differentiable and analytic varieties from one point of view, including them within the general notion of a ringed topological space. In 1955 Serre [468] discovered that a similar definition was applicable also in algebraic geometry. A ringed space locally isomorphic to an affine variety with a sheaf of germs of regular functions on it came to be called an algebraic variety (an algebraic space in Serre's terminology). The additional structure of a ringed space on an algebraic variety permits not only the simplification of various constructions with abstract varieties but also introducing in their study the powerful methods of homological algebra, connected with the theory of sheaves. At the Edinburgh congress in 1958 Grothendieck sketched the perspective for the further generalization of the notion of algebraic variety connected with the theory of schemata [14]. The first definitions of schemata were presented in his report at the Bourbaki seminar in 1959 [219]. The idea of affine schemata was stated independently also by Cartier (unpublished) and by Kähler [299].

Let X be an affine variety over a field k with a coordinate ring k[X]. Its points (in the classical sense) are found in one-to-one correspondence with the homomorphisms f: k[X] → $\bar{k}$, where $\bar{k}$ is the algebraic closure of field k. The kernel of such a homomorphism is a maximal ideal j_f of ring k[X]. The correspondence $f \to j_f$ defines a bijection of the set of points X($\bar{k}$) of variety X with coordinates in $\bar{k}$ (with identification of points conjugate over k) and with the set Spec m (k[X]) of maximal ideals of ring k[X]. Furthermore, the Zariski topology on X($\bar{k}$) corresponds to the spectral topology on Spec m (k[X]) in which closed sets are sets of maximal ideals containing a fixed ideal $I\subset k[X]$. The lattice of a ringed space on X($\bar{k}$) corresponds to an analogous lattice on Spec m (k[X]) in which the fiber at the point $\mathfrak{m}\in\operatorname{Spec} m(k[X])$ of the lattice sheaf is a localization of ring k[X] relative to a multiplicative closed set $S=k[X]\setminus\mathfrak{m}$. Conversely, each k-algebra of finite type without nilpotent elements is isomorphic to the coordinate ring of some affine variety in the sense of Serre, and the correspondence A → Spec m (A) is a bijective correspondence between a k-algebra of the type being considered and affine algebraic varieties (to within isomorphism). Grothendieck generalized this correspondence in two essential respects. First of all he noted that it should define a functor with values in the category of ringed spaces. For every homomorphism of k-algebras φ: A → B, the only reasonable method of defining the morphism Spec m (B) → Spec m (A) is that to a maximal ideal $\mathfrak{m}\subset B$ there must correspond its preimage $\varphi^{-1}(\mathfrak{m})$ in A. However, this ideal no longer has to be maximal, although it always remains prime. Grothendieck suggested that space Spec m (A) be replaced by the space Spec (A) of all prime ideals with analogous spectral topology and with the lattice of a ringed space. This generalization yields the functoriality of the correspondence A → Spec (A) and is analogous to Weil's idea of considering the points of variety X with coordinates in an arbitrary extension K/k of the ground field k. The next remark of Grothendieck was that we need to discard every condition on k-algebra A and to consider that A is an arbitrary commutative ring with unit (sometimes Noetherian). Here field k is replaced by an arbitrary subring $B\subset A$. This generalization is of a meaningful nature since it allows us to explain certain classical phenomena in Italian algebraic geometry "by the presence of nilpotent elements in ring A." A ringed space Spec (A) is called the affine schema corresponding to ring A and is a natural generalization of an affine algebraic variety. The standard construction of pasting together ringed spaces now permits us to give the definition of a schema as a ringed space locally isomorphic to an affine schema. From a new point of view algebraic varieties are reduced schemata of finite type over a field.

The characteristic features of this new theory in algebraic geometry are the following:

1. A commutative algebra becomes a part of algebraic geometry. Namely, this is a theory of local objects of algebraic geometry, i.e., of affine schemata. The advantage of such a viewpoint is two-fold. Firstly, it permits us to carry over all the concepts of commutative algebra into geometric language and by

the same token puts a powerful tool in the hands of the algebraist, namely, geometric intuition. Secondly, by examining affine schemata within the framework of the general theory of schemata, it attracts to their study the powerful methods of algebraic geometry.

2. The ground field k is replaced by an arbitrary ground schema S, i.e., to examine schemata X for which there is given a morphism f: X → S (lattice morphism of an S-schema X). Arbitrary S-morphisms, namely, the mappings φ: $X_1 \to X_2$ for which $f_2 \circ \varphi = f_1$, where f_i: $X_i \to S$ (i = 1, 2) are lattice morphisms, become morphisms of S-schemata. Each schema X is a Z-schema, i.e., a schema over the affine schema S = Spec (Z). The classical notion of the extension of the field of constants, due to Zariski and Weil, is replaced by notion of a change of base. For any S-schema S' and S-schema X we can consider the "change of base," namely, the S'-schema $X_{(S')} = X \underset{S}{\times} S' \to S'$, where $X \underset{S}{\times} S'$ denotes a direct product in the category of S-schemata. Special cases of this operation are such concepts as the reduction of a variety by a prime module, the fiber of a morphism, etc. Classical versions of the definitions of these notions were only slightly geometric and rather awkward.

3. The introduction of nilpotent elements. The presence of nilpotent elements in coordinate rings of arbitrary schemata proved to be natural enough and of frequent occurrence in algebraic geometry. For example, the fibers of a morphism of the usual nonsingular algebraic varieties are schemata with nilpotent elements (for example, Kodaira's "multiple fibers" in the theory of algebraic surfaces). The presence of nilpotent elements in a "schema of modules" or in "Poincare's schema" allowed us to explain the previously not well understood phenomena of classical Italian algebraic geometry as well as certain pathologies of algebraic varieties over a field of positive characteristic (see Mumford [376, 378, 382]). The theory of schemata with nilpotent elements plays an important role in the study of the "infinitesimal" properties of algebraic varieties and served as the foundation of Grothendieck's formal geometry (see §6).

We remark that other attempts, which did not become prevalent, were made to generalize the notion of an algebraic variety, in which a central role was played by the concept of a local ring of the field of algebraic functions (Chevalley's schema [154, 392, 397, 266, 159, 509]). The idea of considering algebraic varieties over arbitrary Dedekind rings is due also to Nagata [392].

A number of generalizations of the concept of a schema arose in certain concrete problems of algebraic geometry. A natural generalization of schemata consists in the immersion of the category (Sch/S) of S-schemata into some larger category. For example, the examination of the category of projective objects of the category (Sch/S) leads to the notion of a formal schema (Grothendieck [219, 240]) and to Serre's pro-algebraic groups. The dual concept of an inductive system turned out to be useful in the algebraic theory of uniformization of Pyatetskii-Shapiro and Shafarevich [45]. An analysis of the category (Sch/S) of contravariant functors on (Sch/S) (or its subcategory consisting of sheaves on a certain Grothendieck topology on S; see §5 further) permitted us to identify S-schemata with representable functors. This point of view proved to be particularly useful for the theory of group S-schemata [169-171]. Grothendieck [236] also suggested that arbitrary ringed spaces (or even the Grothendieck topology) be considered as ground schemata. The theory of such schemata ("relative schemata") was developed in Hakim's dissertation [255]. This notion proved useful in analytical geometry.

An essential generalization of the concept of a schema are algebraic spaces introduced independently by Artin [84] and by Moishezon (he called them "minischemata" [38-40]). In Weil's language an analog of such a concept was introduced in 1965 by Matsusaka (Q-varieties; see [360]) in connection with the theory of modules of polarized algebraic varieties. The majority of definitions and results of the theory of schemata was carried over to algebraic spaces by Knutson [314]. See §5 for details on this theory.

2. Theory of Schemata. From 1960 on, Grothendieck (in collaboration with Dieudonne) started to publish the monumental treatise on algebraic geometry in which it was proposed to establish the foundations of algebraic geometry within the framework of schemata theory. Although no further chapters of this treatise have been published, the majority of the classical results on the foundations have found in it their natural generalization and clarification in the theory of schemata. We remark that the degree of generality in this theory has proved to be so great that the publications of Grothendieck and Dieudonne remain almost unique in schemata theory proper. Papers [98-100, 107, 143, 146, 180, 205, 251-254, 274, 275, 320, 328, 329, 333, 335, 386, 428, 484, 509, 510] were devoted to certain unessential generalizations of Grothendieck's results. A detailed analysis of Grothendieck's results in the theory of schemata would take up too much space. We restrict ourselves only to a survey of the individual highlights of this theory.

Chapter I of the treatise [240, 248] was devoted to the language of schemata; Chapters II and IV [241, 244–247] were devoted to the study and definition of various classes of morphisms of schemata. The cohomology theory of coherent sheaves on schemata (see §2) was developed in Chapter III [242, 243]. The terminology introduced by Grothendieck in relation with the properties of morphisms of schemata includes more than a hundred terms. All properties of Grothendieck morphisms separate into local and global ones, and in their own turn, those are subdivided into three categories: finiteness conditions (morphism of local finite type, locally finitely presented morphism of finite type, finite morphism, quasifinite morphism, etc.), topological conditions (Noetherianness, quasicompactness, open and universally submersive morphisms, etc.), and properties which can be called good (smoothness, flatness, formal nonramifiability, etc.). Among the most important results in the theory of morphisms of schemata are theorems of the following types: criteria of fulfillment of a given property, construction conditions (the set of points at which a given property is fulfilled, constructively), preservation properties after a change of base, descent of a given property, etc. As a rule the proofs of these theorems reduce to the affine case, where they represent sometimes profound results from commutative algebra. Here the technique of passing to the projective limit, developed by Grothendieck, permits us to pass from arbitrary rings to Noether rings and even to algebras of finite type over Z. To the global properties of morphisms refer the properties of projectivity, of the properness of a morphism, investigated in detail in Chapter II.

Grothendieck's results gave a powerful impetus to the development of commutative algebra, by introducing new methods, ideas, and problems. We list some of them:

1. The concept of flatness of a module (introduced by Serre in 1955 [478]) received further development, was given a geometric interpretation, and its role in algebraic geometry and in commutative algebra was stressed.

2. The creation of the technique of passing to the projective limit, mentioned above.

3. The connection of the notion of the depth (or homologic dimension) of a module, introduced and developed by Serre [49], with cohomology theory and, in particular, with local cohomology theory [227, 234].

4. The creation of the theory of excellent rings, generalizing and systematizing the results of Zariski and Nagata on local Noetherian rings [401].

5. The theory of Henselian rings, first established by Nagata, was essentially developed and was made the foundation of the theory of étale cohomologies, of smooth morphisms, etc. [247, 453, 467].

6. The theory of descent (see §5 further on) provided an influx of new ideas and problems into commutative algebra (for example, see [455]).

7. The application of global methods of algebraic geometry and of cohomology theory permitted the solving of certain problems of factorial rings [15–17, 227, 344].

Concrete problems of algebraic geometry made it necessary to study schemata of a more special type. For example, the theory of singularities of algebraic varieties was connected with the study of local schemata, i.e., of open sets of schemata of the form Spec (A), where A is a local ring. A natural generalization of the concept of an algebraic group into the language of schemata (group schema, see [169–171]) proved to be suitable for the study of the reduction of an Abelian variety. In this language Raynaud [444] gave a very simple and natural definition of the Neron model of an Abelian variety. This concept, introduced in Weil's language by Neron [414], plays an important role in the arithmetic of Abelian varieties. Lichtenbaum [340] and Shafarevich [463] developed the theory of two-dimensional regular schemata over a Dedekind ring and, in particular, the theory of "arithmetic surfaces" (also see [112, 128]). Other aspects of the application of schemata theory in the arithmetic of algebraic varieties are the theory of finite group schemata [421, 424, 425, 497], the theory of p-divisible groups [51], the theory of Abelian schemata (Mumford [380, 385]), Greenberg schemata [209, 210].

The technique of blowing down sheaves of ideals to schemata, due to Grothendieck [241] (also see [321]), proved to be a powerful technical tool in the resolution of the singularities of algebraic varieties over a field of zero characteristic (Hironaka [269]). The concept of an ample sheaf, introduced by Grothendieck in [241], played an important role in the theory of projective immersions of abstract algebraic varieties (see §4). See [20, 32, 173, 174, 194, 348, 476] for surveys and textbooks on schemata theory.

3. General Facts on Algebraic Varieties. Parallel with the development of schemata theory investigations were continued on the general properties of algebraic varieties in the spirit of classical algebraic

geometry. Kuan's results [317-319] relate to theorems of Bertini type: if a variety possesses a certain property, then its common hyperflat section also possesses this property. An analog of Bertini's theorem for local domains was obtained by Chow [157]. Cartier [148] introduced the notion of a local principal divisor (Cartier's divisor) and gave a criterion for the rationality of a divisor. Chevalley's 1958/59 seminar [466] was devoted to the theory of divisors. The last section of Chapter 4 of Grothendieck's "Elements" [447] was devoted to the general properties of Cartier's divisors and their connection with Weil's divisors. Papers [134, 178, 267, 305, 408, 420, 461, 462, 466] also are devoted to the theory of divisors. Interesting results have been obtained on endomorphisms of algebraic varieties. Ax [96, 97] proved that any injective endomorphism is surjective. Another proof of this fact was given by Borel [133]. The lattice of an algebraic group was introduced in Ramanugam's paper [437] on the automorphism group of a complete algebraic variety. Later Matsumura and Oort [358] reproved this result in a stronger form (see §5). Many papers have been devoted to the differential properties of algebraic varieties. Each variety X over a field k defines a sheaf $\Omega^1_{X/k}$ of germs of one-dimensional regular differential forms on X. The fiber of this sheaf at a point $x \in X$ is the $\mathcal{O}_{X,x}$-module $\Omega^1_{\mathcal{O}_{X,x}/k}$ of the Kähler differentials of the ring $\mathcal{O}_{X,x}$ over field k. The general properties of the sheaves $\Omega^1_{X/k}$ (and, more generally, even of sheaves $\Omega^1_{X/S}$ for an arbitrary S-schema X) were studied by Grothendieck in [247] (also see [467]). Many properties of algebraic varieties were expressed in terms of the properties of sheaf $\Omega^1_{X/k}$. Thus, for example, the smoothness of X over a perfect field k is equivalent to the local freedom of sheaf $\Omega^1_{X/k}$ (Grothendieck [247, 467], Nakai [405], Kunz [322]). The case of an imperfect field k was investigated by Asaeda [90]. Grothendieck [247] obtained analogous criteria for the smoothness of a morphism within the framework of schemata theory. Other properties of varieties (for example, normality), expressed in terms of the properties of $\Omega^1_{X/k}$, were investigated in [185, 304, 325, 342, 498, 503]. The book [111] is devoted to the analogs of these results in analytical geometry. In [342] Lipman conjectured that if char (k) = 0, then variety X is nonsingular if and only if the dual sheaf $\Theta^1_X = \mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_{X/k}, \mathcal{O}_X)$ of tangent vectors is locally free. Papers [316, 342, 343] are devoted to this conjecture. Arima [75, 76], Kodama [315], and Kunz [323] addressed themselves to differential forms of the second kind on an algebraic variety over a field of positive characteristic. The general properties of finite coverings of algebraic varieties are taken up by Abhyankar [69] and Popp [435]. One of the central results of this theory is the Zariski-Nagata purity theorem for the ramification set. This theorem states that the set of ramification points of a finite covering f: V → W is a divisor if V is normal and W is nonsingular. In the case when the ground field k has a zero characteristic, this result was proved by Zariski in 1958 [514]. Nagata [396] proved it simultaneously (for arbitrary k). Later on, this theorem (in a different version) was proved over again by various authors [18, 93, 139, 195, 227, 325, 326]. Various analogs and generalizations of the Zariski-Nagata purity theorem were obtained by Grothendieck (for the Brauer group [231] and for a birational morphism [247]), Dolgachev [18, 179] (for the property of smoothness of a family of curves), and Artin (SGA4). For textbooks on algebraic geometry (without schemata theory) see [55, 101, 154, 195, 297, 331, 459, 460].

§2. Cohomologies of Algebraic Varieties and of Schemata

As is well known, the foundation for the topological and transcendental methods of studying complex algebraic varieties is the presence, on the one hand, of the theory of cohomologies with complex coefficients and, on the other hand, of the theory of cohomologies with coefficients in a coherent analytic sheaf. The profoundness of the results, generalizing the classical results of Picard, Poincare, Lefschetz, obtained by these methods at the beginning of the Fifties (Hodge, Hirzebruch, Kodaira, Chern), raised the natural question of the presence of analogs of cohomology theories for abstract algebraic varieties. The necessity for such a theory was stressed, on the other hand, by Weil in connection with his significant conjectures in diophantine geometry. In 1955 Serre constructed a theory of coherent algebraic sheaves. In 1962 there appeared the theory of étale (or covering) cohomologies of Grothendieck — the first of the theories of the Weil cohomologies destined to replace cohomologies with complex coefficients. Later on, other theories of Weil cohomologies were proposed by a number of authors (Lubkin, Monsky, Washnitzer, Grothendieck, Verthelot).

1. Cohomologies of Coherent Algebraic Sheaves. In the fundamental paper [46, 468] Serre included algebraic varieties in the general category of locally ringed spaces and for the latter defined the concept of a coherent sheaf of modules over a lattice sheaf of rings. He also developed the cohomology theory of such sheaves. In the case when X is an algebraic variety over the complex number field C, any coherent algebraic sheaf F on X induces on the corresponding analytic space X^{an} a coherent analytic sheaf F^{an}. Serre's classical results [478] (well known under the designation "GAGA") show that for projective varieties

there holds the canonic isomorphism of cohomology spaces $H^i(X, F) \simeq H^i(X^{an}, F^{an})$. Moreover, Serre has proved that for any coherent algebraic sheaf $\mathcal{F}$ on X^{an} there holds the isomorphism $F^{an} \simeq \mathcal{F}$, where F is some coherent algebraic sheaf on X. Later on, these results were generalized by Grothendieck [217, 442] to the case of complete varieties (not necessarily projective) and by Hartshorne [261, 264] to the case of quasiprojective varieties. Grothendieck transferred Serre's theorems to a formal geometry (see §6). An essential development of Serre's work were Grothendieck's results on the cohomologies of coherent sheaves on arbitrary schemata [242, 243]. The following finiteness theorem is a central result. For any proper morphism $f: X \to S$ and coherent sheaf F on X, the sheaves $R^q f_*(F)$ associated with the presheaf $U \to H^q(f^{-1}(U), F/f^{-1}(U))$ are coherent.

In the case when S is the spectrum of field k and X is a projective variety over k, this result (establishing in this case the finite dimensionality of the k-spaces $H^q(X, F)$) was proved by Serre. Serre's affineness criterion [469], contained in the characterization of affine varieties by the property of $H^1(X, I) = 0$ for any sheaf of ideals I, was carried over to the case of affine schemata by Grothendieck [242] and by Nastold [412] (also see [194]). The techniques of Grothendieck's formal geometry allowed us to give a cohomological generalization of the connection theorem and of Zariski's theory of holomorphic functions (see §6). The general theory of duality for coherent algebraic sheaves is another direction in the generalization of Serre's results. Grothendieck gave an outline of this theory at the Edinburgh congress [14] and in a report at the Bourbaki seminar [218]. Serre's theorem [513] states that for any nonsingular projective variety over a field k of dimension n and for a locally free sheaf L on X the spaces $H^i(X, L)$ and $H^{n-i}(X, \check{L} \otimes \omega_X)$ are dual to each other. Here ω_X is the sheaf of germs of regular n-forms on X, and $\check{L}$ is a sheaf dual to L.

In report [218] Grothendieck gave certain generalizations of this theorem and then at the Edinburgh congress [14] formulated and discussed the general duality theorem for arbitrary complete varieties. Hartshorne's seminar [259] was devoted to this theorem (and its generalization for an arbitrary proper S-schema). The general formalism of duality, established in the language of arbitrary Weil categories, developed in [259, 501], was then used in other duality theories [118, 417, 499]. Another approach to the proof of the general duality theorem is due to Deligne (see [259], Appendix). The connection between these approaches was shown by Verdier [502]. The Altman-Kleiman seminar [71] was devoted to the specialization of Grothendieck's results for the case of projective varieties. A very beautiful approach to the theory of duality on a curve was indicated by Tate [52]. Also see Nastold [411, 413]. The language of sheaf theory proved to be very convenient for the formulations and proofs of many results of classical algebraic geometry, namely, the questions connected, first of all, with the Riemann-Roch theory and the theory of linear systems (see [31, 513]).

Sampson and Washnitzer [457, 458] obtained important results on the behavior of $H^i(X, F)$ relative to monoidal transforms and the Künneth formula. The latter was significantly generalized by Grothendieck [242]. The behavior of cohomologies relative to proper mappings and algebraic correspondences was studied by Snapper [492, 495] and Matsumura [357]. The first of them proved a very important theorem stating that for any coherent sheaf F on a nonsingular variety X and Euler divisors $D_1, \ldots, D_k$ the characteristic $\chi(X, F \otimes \mathcal{O}_X(n_1 D_1 + \ldots + n_k D_k))$ is a polynomial in $n_1, \ldots, n_k$ [493]. This result was used by him to define the intersection number for divisors [494] and was then generalized to quasiprojective varieties by Cartier [149]. Borelli [134] introduced the concept of a divisoral variety a special case of which is a nonsingular or quadiprojective variety and extended Snapper's theorem to this variety. A generalization of Snapper's theorem is due also to Moishezon [37].

While the cohomologies of projective and affine varieties have been studied in sufficient detail, there have remained many questions on the cohomologies of arbitrary quasiprojective varieties. The first result in this direction was Grothendieck's theorem [234] stating that $H^n(X, F) = 0$ for any coherent sheaf F on an n-dimensional variety X all of whose irreducible components are improper. An elementary proof of this theorem (not using local cohomology theory) was given by Kleiman [310]. Fundamental results on the cohomological dimension of quasiprojective varieties were obtained by Hartshorne [260, 261]. In particular, he has proved that $cd(\mathbf{P}^3 \setminus C) = 1$ if C is a curve, and this put an end to the attempts to give a cohomological proof to Kneser's old problem [313]. Together with Goodman he gave a characterization of varieties the cohomology groups on which are finite-dimensional.

The main technical tool in the preceding theory is the theory of local cohomologies of coherent sheaves, due to Grothendieck [227, 234]. By generalizing the classical definition of cohomologies with compact supports, he defined groups (respectively, sheaves) of local cohomologies $H^i_Z(X, F)$ (respectively, $\mathcal{H}^i_Z(F)$) of sheaf F relative to a closed subschema $Z \subset X$. Here are the most important results of this theory:

a) A cohomological characterization is given of the important concept of the depth, $\operatorname{depth}_Z(F)$, of a sheaf F relative to subschema Z. This concept is the global variant of the notion of the depth (or homological dimension) of an A-module M, due to Serre [49]. There holds the theorem $\operatorname{depth}_Z(F) \geqslant k \Leftrightarrow \mathscr{H}^i_Z(F)=0$, $i<k$.

b) The exact sequence

$$\ldots \to H^i_Z(X,F) \to H^i(X,F) \to H^i(X \setminus Z,F) \to H^{i+1}_Z(X,F) \to \ldots$$

allows the connection of groups $H^i(X, F)$ and $H^i(X \setminus Z, F)$.

c) A theorem on local duality is proved permitting the computation of the group $\operatorname{Ext}^i_A(M, A)$ for an A-module M in terms of the local cohomologies $H^i_{\mathfrak{m}}(M)=H^i_{\{\mathfrak{m}\}}(\operatorname{Spec}(A), \tilde{M})$ (under certain restrictions on ring A).

d) The theorems on the finiteness of sheaves $\mathscr{H}^i_Z(F)$, due to Grothendieck [227] and then generalized by Hartshorne [262] and Peskine [430].

We mention the most important applications of this theory. The concept $\operatorname{depth}_Z(F)$ is closely connected with the question of the coherent extension of a sheaf $F|X-Z$. In particular, results in item a) permit us to prove again a part of the assertions of M. Baldassarri and his students [101-106, 351-355] on the characterization of torsion-free sheaves as well as one of Hartshorne's results [256]. The results in item d) were applied to the theorems on the finiteness of cohomologies for quasiprojective varieties [261, 428]. Sheaves of local cohomologies play an important role in duality theory [259]. With their aid we can construct the canonic injective resolvent of a sheaf (the Cousin complex), which plays one of the central roles in this theory. Sharp [485] has studied this complex in the case of affine schemata. The theorems of finiteness of local cohomologies were applied for the generalization of the comparison and existence theorems (see §6) to the case of not necessarily proper morphisms [227]. In [262] Hartshorne proved the affine duality theorem stated as a conjecture by Grothendieck in [227].

Important applications of local cohomology theory relative to "Lefschetz-type" theorems on the comparison of the Picard group (or the fundamental group) of schema X and of its subschema Y of codimension 1 were obtained in [227]. This comparison is carried out in three stages. At first $\operatorname{Pic}(\hat{X})$ and $\operatorname{Pic}(U)$ are compared, where U is an open set containing Y while $\hat{X}$ is the formal completion of X along Y (see §6). Next, $\operatorname{Pic}(U)$ and $\operatorname{Pic}(X)$ are compared, and finally $\operatorname{Pic}(\hat{X})$ and $\operatorname{Pic}(\hat{Y})$. Local cohomology theory has a bearing only on the first two stages, the last stage being studied by the methods of formal geometry. The investigation of the second stage is connected with the condition of factoriality of the rings of points $x \in X/U$. Here Grothendieck obtained important results, one of which is the affirmative answer to the following conjecture of Samuel: a local Noetherian ring A, being a complete intersection and factorial in codimension ≤ 3, is a factorial ring (see Giraud's report [199] for a survey of these results). In the investigation of the analogous stage for a fundamental group Grothendieck generalized the Zariski-Nagata purity theorem (see §1) to the case of local Noetherian rings of dimension ≥ 3, being complete intersections.

Grothendieck obtained both local as well as global versions of Lefschetz's theorems on hyperflat sections. A special case of these results is, for example, the following theorem. For any complete intersection X in P^n_k of dimension ≥ 3 (respectively, ≥ 2), $\operatorname{Pic}(X) \simeq Z$ and is induced by a hyperflat section (respectively, $\pi_1(X) = 0$). These results generalize the classical facts true for the case when X is nonsingular and complex (cf. [41, 197, 198]). A generalization of Grothendieck's theorem was given by Raynaud [438]; also see Hartshorne [261].

2. Weil Cohomologies of Schemata. In 1949 Weil pointed out the need for some theory of the cohomologies of abstract algebraic varieties, analogous to the theory of complex cohomologies, which would possess all the formal properties (Poincare duality, Künneth formula) for the derivation of an analog of Lefschetz's formula on fixed points. More precisely, such a theory should be described by the following assignments and axioms:

There exists a contravariant functor $X \to H^*(X)$ from the category of smooth projective k-varieties into the category of finite-dimensional anticommutative graded algebras over a field K of zero characteristic. Here the following properties should be fulfilled:

A. Poincare duality. Namely: a) groups $H^i(X) = 0$ for $i > 2n$ ($n = \dim X$), b) the orientation isomorphism $H^{2n}(X) \simeq K$ exists, c) the canonical coupling $H^i(X) \times H^{2n-i}(X) \to H^{2n}(X) \simeq K$ is nondegenerate.

B. Künneth formula. Let $p_i: X_1 \times X_2 \to X_i$ (i = 1, 2) be projections. The canonic mapping $a \otimes b \to p_1^*(a) \cdot p_2^*(b)$ defines the homomorphism $H^*(X_1) \underset{K}{\otimes} H^*(X_2) \simeq H^*(X_1 \times X_2)$.

C. Gysin homomorphism. There exists the homomorphism of groups γ_X: $C^p(X) \to H^{2p}(X)$, where $C^p(X)$ is the group of algebraic cycles of codimension p of variety X. Such a homomorphism should be functorial with respect to X, multiplicative ($\gamma_{X \times Y}(Z \times W) = \gamma_X(Z) \otimes \gamma_Y(W)$), and central (if P is a point, then $C^*(P) \simeq Z \to H^*(P) \simeq K$ is a canonic imbedding).

The presence of such a theory permits us in a formal manner to derive Lefschetz's formula for the fixed points of the correspondence and, in particular, if k is the algebraic closure of a finite field, to obtain proofs of the first two conjectures of Weil on the ζ-function (rationality and functional equation) (see [311]). In case k = C the functor $X \to H^*(X^{an}, C)$ yields the theory of Weil cohomologies.

Serre's attempts to construct such a theory, starting from the theory of coherent sheaves, was not crowned with success. To be precise, in [471] Serre proposed to examine the cohomology $H^i(X, W(\mathcal{O}_X))$ with coefficients in the sheaf of rings of Witt vectors over local rings X. Although these cohomologies proved useful for many questions connected with algebraic varieties over a field of positive characteristics (for example, see [381]), they turned out to be unfit for Weil's theory. Even for Abelian varieties the cohomologies $H^i(X, W(\mathcal{O}_X))$, considered as modules over the Witt ring W(k), are not finitely generated (Serre [479]). We have not succeeded even in obtaining the "true" Betti numbers of variety X with the aid of coherent sheaves [286, 471].

A completely different approach to the definition of Weil cohomologies was first suggested by Grothendieck [141]. The first publication on this theory (Grothendieck's covering or étale cohomologies) appeared only in 1962 in the form of reports at Artin's seminar at Harvard [78]. Surveys [1, 30, 79, 83, 200] are devoted to sketching the milestones in this theory. The Artin-Grothendieck seminar in 1963/64 (SGA 4) as well as Verdier's paper [499] was devoted to the proofs of the fundamental theorems on étale cohomologies (with the aid of these theorems it was established that a variant of the theory of étale cohomologies, i.e., of l-adic cohomologies, is the theory of Weil cohomologies). These results as well as the applications to ζ-functions on algebraic varieties were the subject of Grothendieck's seminar of 1965 (SGA 5). Tate [50], Kleiman [311], Grothendieck [228], and Deligne [164] (cf. the survey [44]) have discussed the applications of this theory to the arithmetic questions of algebraic varieties. Other methods of determining Weil cohomologies were suggested by Lubkin in 1967 [345] and by Grothendieck [237] (see later on about them).

3. Grothendieck Cohomologies of Schemata. The construction of these cohomologies is based on the general notion of Grothendieck's topology. The latter signifies the giving of a certain category T with fiber bundles and of a collection Cov(T) of families of morphisms $\{U_i \to U\}_{U \in T,\ i \in I}$, called the coverings of object U. Here certain natural axioms should be fulfilled, turning into the usual axioms for a topological space if as T we take the category of open sets of some topological space (morphisms turn into embeddings). The general formalism of Grothendieck's topology and of its variant, leading to the notion of a topologized category (site), was developed in (SGA 4) (also see Giraud [200] and [468, 469]). At the present time this theory plays an independent role in homological algebra, category theory, and logic. If X is a schema, if T is some subcategory of the category of X-schemata (Sch/X), and if the families $\{U_i \xrightarrow{f_i} U\}_{i \in I}$ of X-morphisms into T such that $U = \bigcup_{i \in I} f_i(U_i)$ are coverings, then we obtain the notion of Grothendieck's topology of schema X, associated with the category T. For example, if as T we take the category of étale morphisms, we then obtain the étale topology for schema X, which we denote X_{et}. The idea for such a topology was suggested by Grothendieck in Serre [470]. If T is the category of strictly flat quasicompact morphisms, then we obtain the $f$pqc-topology of schema X, denoted X_{fpqc}. Other Grothendieck topologies of schemata – the quasifinite X_{qf}, the strictly flat finitely presented X_{fppf}, etc. – are obtained analogously. In particular, if T is the category of Zariski-open subsets of X, then we obtain the usual Zariski topology X_{Zar} of schema X. Later on, Grothendieck generalized the notion of a sheaf with values in a category C to a topological space by defining a sheaf on the Grothendieck topology T as a certain contravariant functor F: T → C satisfying a certain axiom (the sheaf axiom). For any Abelian sheaf (C is the category of Abelian groups) the cohomology groups $H^i(T, F)$ are defined by the usual methods (in the non-Abelian case, cohomology theory was developed by Giraud [202, 204]). In particular, for schema X there have been defined various Grothendieck cohomologies: the étale cohomology $H^i(X_{et}, F)$, the $f$pqc-cohomology $H^i(X_{fpqc}, F)$, etc. All of the usual formalism of sheaf theory takes place within the framework of the theory of sheaves on Grothendieck

topologies [78, 169] (SGA 4). When X = Spec (k), where k is a field (i.e., X is a "point"), the giving of a sheaf F on X_{et} is equivalent to the giving of the Gal $(\bar{k}/k)$-module $F(\bar{k}) = \varinjlim_{k} F(K)$, where K/k are all possible extensions of the Galois field k. The groups of cohomologies $H^i(X_{et}, F)$ are precisely the Galois cohomologies $H^i(\mathrm{Gal}(\bar{k}/k), F(\bar{k}))$. From this point of view the theory of Galois cohomologies, set forth, for example, in Serre's book [48], can be considered as a point cohomology theory. Commutative group X-schemata [421] furnish examples of Abelian sheaves on X_{et} and X_{fpqc}. For example, the multiplicative group schema G_m defines the sheaf $G_{m,X}\ (U \to G_{m,X}(U) = \Gamma(U, \mathcal{O}_U^*))$; the constant group A defines the constant sheaf A_X $(U \to A^{\pi_0(U)})$. A generalization of "Hilbert's theorem 90," proved by Grothendieck, states that $H^1(X_{et}, G_{m,X}) \simeq \mathrm{Pic}(X)$. The group $H^2(X_{et}, G_{m,X})$ turned out to be closely related to the Brauer group Br (X) of schema X. The latter concept, generalizing the classical concept of the Brauer group of a field, was introduced by Auslander and Goldman [95] for the case of commutative rings and was extended to the case of arbitrary schemata by Grothendieck. The foundations of this theory and certain important computations are contained in three fundamental papers [299-301] of Grothendieck. Other computations of the Brauer group are to be found in [23, 366, 464]. The first computations of étale cohomologies for algebraic curves and surfaces were made by Artin. In this case the groups $H^i(X_{et}, (Z/n)_X)$ ((n, char (k_n)) = 1) are completely analogous to the classical groups $H^i(X^{an}, Z/n)$ if X is a complex curve or surface. Later on Artin proved a comparison theorem $H^i(X_{et}, (Z/n)_X) \simeq H^i(X^{an}, Z/n)$ for algebraic varieties over field C (SGA 4) (also see [200]). The computation in [443] of cohomologies $H^i(X_{et}, \mathfrak{A})$, where $\mathfrak{A}$ is the sheaf associated with the Neron model of an Abelian variety A over the field of rational functions of curve X, allowed us to interpret and generalize the results of Ogg [416] and of Shafarevich [53] on principal homogeneous spaces over A. When X is a two-dimensional regular schema over a homogeneous regular schema B, group Br (X) proves to be closely related with the Shafarevich group $H^1(B_{et}, \mathfrak{A})$, where $\mathfrak{A}$ is the Neron model of the Jacobi variety of the common fiber X. In the geometric case when X is an elliptic surface, this connection was discovered by Shafarevich [54]. In the case of algebraic surfaces X the Betti numbers, calculated as $\dim_{\mathbf{F}_l}(H^i, H_{et}, (Z/l)_X)$ (l is a prime not equal to char k), coincide with those determined previously by Igusa [287]. Furthermore, in this case the group Br (X) (its divisible part) can be interpreted as an analog of the group of transcendental cycles on X. We also note the decisive role of the Brauer group in the Artin and Mumford resolution of Lurotte's conjecture [165]. The fundamental properties of étale cohomologies were proved by Artin and Grothendieck (SGA 4). Artin proved a theorem on homological dimension (cd X $\leq$ dim X for algebraic varieties) and a finiteness theorem: the group $H^i(X, F)$ is finite for any periodic sheaf F either on a proper schema X or on an excellent schema X of characteristic zero. To Grothendieck are due a "theorem on change of base," a Künneth formula, and a theory of cohomologies with compact supports. Verdier [499, 500] announced general duality theorems and Lefschetz formulas analogous to the Atiyah-Singer-Bott formulas for elliptic complexes. Lefschetz formulas for étale cohomologies were considered by Raynaud [439, 441]. For any algebraic variety X over k and for a prime $l \neq$ char k the étale cohomologies define l-adic cohomologies of X. By definition, $H^i(X, Z_l) = \varprojlim_{k} H^i(X_{et}, (Z/l^k_X))$. The Q_l-spaces $H^i(X, Q_l) = H^i(X, Z_l) \otimes_{Z_l} Q_l$ play the role of Weil cohomologies for X. The interesting question of the independence of the Betti numbers $b_i(X; l) = \dim_{Q_l} H^i(X, Q_l)$ from l has been answered only in special cases (for example, dim X $\leq$ 2). See [50, 228] as well as the survey [44] for the application of l-adic cohomologies to arithmetic.

Using the concept of the Grothendieck topology, we can give a construction of the classifying space for an algebraic or a discrete group G [236]. A detailed exposition of the corresponding theory is contained in Giraud's book [204]. Grothendieck also gave an algebraic definition of Chern classes of the representation of a discrete group G over an arbitrary field k [236]. The generalization of these ideas to arbitrary Lie groups and their representations was given by Geronimus [9-11].

Shatz [486-490] and Mazur [363, 364] devote themselves to the study of the general properties of other Grothendieck cohomologies as well as to certain computations. Mumford [379] applied the Grothendieck topology to module problems. Mazur's papers (cf. [27]) contain a computation of flat cohomologies ($fpqc$ or $fppf$) for number schemata as well as applications to the arithmetic of algebraic varieties. The topologies X_{fpqc} and X_{fppf} have proved to be particularly useful in the theory of group schemata [168-171]. For example, we mention the interpretation of groups of principal homogeneous spaces of a group S-schema G as the group $H^1(S_{fppf}, G)$ [34, 369, 490]. We remark that by virtue of Grothendieck's theorem [231], if G is a smooth schema, then $H^1(S_{fppf}, G) = H^1(S_{et}, G)$. A part of parallel results on Grothendieck cohomologies in another language was obtained in [151-153, 177].

4. Other Cohomology Theories of Schemata. a) De Rham cohomologies of algebraic varieties. The beginnings of this theory were set forth in Grothendieck's fundamental paper [232]. For any S-schema X Grothendieck defines the de Rham cohomology $H^i_{DR}(X/S)$ as the hypercohomology $H^n(X, \Omega^{\cdot}_{X/S})$ of a complex of sheaves of relative differentials of schema X. When S = Spec (C) and X is a smooth algebraic variety over S, Grothendieck, developing the ideas of Atiyah and Hodge [91], proved the comparison theorem $H^n_{DR}(X/C) \simeq H^n(X^{an}, C)$. This theorem permits us to compute the cohomologies of a complement to a hypersurface in a projective space with the aid of the cohomology classes induced by rational differential forms with poles on this hypersurface. This result was used by Atiyah, Bott, and Gårding in [6].

More generally, Grothendieck defined the sheaves of relative de Rham cohomologies as $\mathscr{H}^i_{DR}(X/S) = R^i f_*(\Omega^{\cdot}_{X/S})$, where on the right stand the values of the i-th hyperderivative functor of the direct image functor of sheaf f_* on the de Rham complex $\Omega^{\cdot}_{X/S}$. If f: X → S is a proper smooth morphism of complex algebraic varieties, then from Grothendieck's comparison theorem follows the existence of a canonic isomorphism of analytic coherent sheaves $\mathscr{H}^i_{DR}(X/S)^{an} \simeq R^i f^{an}_*(C) \otimes_C \mathscr{O}_{S^{an}}$. Katz and Oda [25] gave an algebraic definition of a canonic integrable connection (the Gauss-Manin connection) on the sheaves $\mathscr{H}^i_{DR}(X/S)$. In the preceding situation this definition reduces to the connection on a vector bundle definable by a locally free sheaf $\mathscr{H}^i_{DR}(X/S)^{an}$ whose sheaf of horizontal sections coincides with $R^i f^{an}_*(C)$. When X/S is an algebraic curve over a functional field, the Katz-Oda construction reduces to the one proposed earlier by Manin in [28]. In a slightly less general case an algebraic definition of the Gauss-Manin connection was given by Grothendieck [237]. Katz gave an algebraic proof of the regularity theorem for the Gauss-Manin connection [302]. Analytical proofs were given by Griffith [215] and Deligne [163]. The concept of a regular connection generalizes the classical notion of a differential equation with a regular singular point and is the subject of Deligne's book [163] (also see [302, 350]). A generalization of Grothendieck's comparison theorem also is given in it.

Katz [300, 301] gave an interpretation of Dwork's results [182] with the aid of de Rham cohomologies and of the Gauss-Manin connection on them. Oda [415] studied the connections between $H^i_{DR}(X/k)$, where X is a complete variety over a perfect field k of characteristic p > 0, and the Dieudonne modules corresponding to the Picard schema $\mathrm{Pic}_{X/k}$.

Since the spaces $H^i_{DR}(X/k)$ havd been defined, generally speaking, over a field of zero characteristic, the de Rham cohomologies are not the Weil cohomology theory. Nevertheless, Hartshorne [261] has proved Poincare duality for $H^n_{DR}(X/k)$, while in the case of a zero characteristic he has given an algebraic definition of the Gysin homomorphism and of the Künneth formula. Herrera and Liebermann [265] proved that for a smooth subschema $Y \subset X$ of an algebraic variety over field C there holds the isomorphism $H^*(Y, C) \simeq \varprojlim_k H^*_{DR}(Y^{(k)}/C)$, where $Y^{(k)}$ is the k-th infinitesimal neighborhood of subschema Y. This result was obtained independently also by Deligne. The latter has also investigated in detail the connection between de Rham cohomologies and Hodge cohomologies for varieties over a field of zero characteristic [162].

b) Crystal cohomologies. The de Rham cohomologies found their natural generalization in the framework of crystal cohomologies. The basic constructions of this theory and also the program for future investigations in this area formed the subject of Grothendieck's 1966 lectures [237]. The fundamental results of this theory, establishing that crystal cohomologies are Weil cohomologies, were announced later on by Berthelot [113-121].

For any S-schema X Grothendieck proposed to examine topologized category $(X/S)_{Crys}$ whose objects are all nilpotent S-imbeddings $U \subset T$, where U is a Zariski-open subset of X, while the ideal of U in T is provided with a lattice of separated degrees. Morphisms of the pair (U, T) are defined in the natural way, and the Grothendieck topology on $(X/S)_{Crys}$ is induced by the Zariski topology on X. The sheaf on $(X/S)_{Crys}$ is given by the system of sheaves $F_{(U,T)}$ on T. The lattice sheaf $\mathscr{O}_{X/S}$ is defined by the system $F_{(U,T)} = \mathscr{O}_T$. If X is smooth over S, then each coherent sheaf $\mathscr{F}$ on schema X, provided with an integrable connection, defines a sheaf of modules on $(X/S)_{Crys}$. Berthelot [115] proved the isomorphism $H^*((X/S)_{Crys}, \mathscr{F}_{Crys}) \simeq H^*(X_{Zar}, \mathscr{F} \otimes \Omega^{\cdot}_{X/S})$, where on the right stand the hypercohomologies of the complex $\mathscr{F} \otimes \Omega^{\cdot}_{X/S}$ of coherent sheaves on X, defined by the connection on $\mathscr{F}$. When the S-schema is of zero characteristic, this result was proved by Grothendieck [237]. In particular, if $\mathscr{F} = \mathscr{O}_X$, this result shows that $H^*_{DR}(X/S) \simeq H^*((X/S)_{Crys}, \mathscr{O}_{X/S})$ and, in particular, the de Rham cohomologies can be computed without differential forms (!). In order to obtain the Weil cohomologies from crystal cohomologies, for any schema X_0 over a perfect field k we need to set

$H^*_{Crys}(X_0/k) = \lim_{\leftarrow n} H^*((X_0/S_n)_{Crys}, \mathcal{O}_{X_n/S_n})$, where S_n = Spec $(W/p^{n+1}W)$ is the spectrum of the quotient ring of the Witt ring W(k). If X_0 is proper, smooth on k, and is lifted up to the very same schema X over S = Spec (W(k)), then there holds the isomorphism $H^*_{Crys}(X_0/k) \simeq H^*_{DR}(X/S)$. This result shows the invariance of the de Rham cohomologies $H^*_{DR}(X/S)$ relative to the different liftings of schema X_0 up to schema X. This result was obtained by other methods by Lubkin [345] and Monsky-Washnitzer (cf. [237]). The cohomology groups $H^*_{Crys}(X_0/k) \otimes_{W(k)} K$ (K is the field of particular rings W(k)) satisfy the conditions of Weil cohomologies [121]. In [116] Berthelot constructed a theory of local crystal cohomologies and in [122, 123], together with Illusie, defined the Chern classes of a sheaf with value in crystal cohomologies. A comparison theorem of crystal cohomologies with classical ones over field C is due to Deligne (unpublished). Grothendieck's work on representations of profinite groups [239] has application also to crystal cohomologies. The application of crystal cohomologies to Dieudonne modules and to p-divisible groups is contained in Grothendieck's report at the Nice congress and also in Messing's dissertation [365].

We remark that a theory of crystal cohomologies is a p-adic theory of cohomologies in the sense of Grothendieck [237]. That is, in contrast to the theory of l-adic cohomologies, it gives information on the p-torsion in the cohomologies (p = char k). Other Grothendieck cohomologies (the *fpqc*- or the *fppf*-cohomologies) give good p-adic cohomologies only in small dimensions (since by virtue of Artin's theorem, $H^i(X_{fppf}, Z/p^k) = 0$, $i > \dim X + 1$).

c) The p-adic cohomologies of Monsky-Washnitzer and of Lubkin. Other possible approaches to the theory of p-adic cohomologies were suggested by Monsky-Washnitzer [372-375] and by Lubkin [346, 347]. In a brief note [374] in 1964 the first authors proposed to define cohomologies in the following way. First of all they defined a certain class of smooth algebras over a Witt ring W(k) (the Monsky-Washnitzer "w.c.f.g."-algebras) and proved that for any smooth k-algebra A_0 there exists a functorial lifting up to an algebra A with $A \otimes_{W(k)} k \simeq A_0$. The assignment $A_0 \to H^*_{DR}(A/W(k))$ defines a cohomological functor from the category of smooth k-algebras into the category of W(k)-modules. By localizing this functor relative to the Zariski topology, Monsky and Washnitzer obtain the cohomology sheaves $\mathcal{H}^*_{MW}(X)$ for any smooth schema X over a perfect field k. Global sections of these sheaves furnish "good" invariants in small dimensions. For example, for a curve X of genus g, $H^0(X, \mathcal{H}^i_{MW}(X))$ is a free W(k)-module of rank 2g. Working only with k-algebras of finite type, Monsky succeeded in using his own theory to prove a fixed-point formula (using completely continuous operators) and the rationality of the ζ-function of smooth schema [372, 373]. The question of the globalization of the Monsky-Washnitzer idea for obtaining a theory of Weil cohomologies remains open until now (see [237] for a discussion of this).

In [346] Lubkin, on the basis of the Monsky-Washnitzer ideas, gave a definition of p-adic Weil cohomologies for the subcategory of varieties liftable to characteristic zero. We remark that by virtue of Serre's example [475], such varieties do not exhaust the class of all smooth projective varieties. Also, as Grothendieck's crystal cohomologies, these cohomologies can be computed on the basic only of the usual Zariski topology of schema X. If X is a lifting of a nonsingular projective variety X over a field k up to a flat proper $\mathcal{O}$-schema, where the residue field of ring $\mathcal{O}$ is k, while the fraction field K has zero characteristic, then Lubkin assumed $H^*(X) = H^*_{DR}(X \otimes_{\mathcal{O}} K/K)$ and proved that this is not dependent on the lifting of X to X. In this paper the author developed a general technique of hypercohomologies $H^i(X, U, F^*)$ of finite complexes of sheaves on a topological space X with respect to a module of an open set U.

d) Combinatorial Lubkin cohomologies. In [345] Lubkin, using an étale topology of a schema X, associated with each such schema a certain prosimplicial complex $S(X) = \lim_{\leftarrow U} S(X, U)$ and, after this, defined $H^*(X, M) = \lim_{\rightarrow U} H^*(S(X, U), M)$ for any Abelian group M. The author's main theorem asserts that these cohomologies coincide with the Grothendieck étale cohomologies $H^*(X_{et}, (M)_X)$. The author announced theorems analogous to the fundamental Artin-Grothendieck theorems and, from these theorems, derived in a standard manner the first two Weil conjectures as well as proved that the classes of algebraic cycles are finitely generated with respect to numerical equivalence. The latter fact follows formally from any theory of Weil cohomologies [311].

e) Grothendieck's theory of motifs. The presence of so many different cohomology theories of schemata posed the question of the creation of some "universal cohomology theory."

The idea of such a theory was suggested by Grothendieck in 1967 (unpublished). The first publication on this theory, in which the main constructions of Grothendieck were set forth, belongs to Manin [33]. Later on, this theory as well as Manin's results, which we take up below, was the subject of Demazure's report [167]. Grothendieck's construction is extremely sharp and, roughly speaking, consists of the following. The category V/k of algebraic varieties over a field k is immersed at first into an additive category CV/k (by an introduction of supplementary morphisms, i.e., of all possible correspondences relative to some theory of intersection C) and next into a pseudo-Abelian category $\widetilde{CV/k}$ (by an association formally of the kernels and images of all possible projectors into CV/k). Every object (motif) of category $\widetilde{CV/k}$ is represented as a direct sum of "n-dimensional pieces of varieties," i.e., of objects of the form (X, p), where X is an n-dimensional variety and p is a projector from $C^n(X \times X)$. The motif $h(X) = (X, id)$ is called the motif of variety X, and its expansion into the direct sum $h(X) = \bigoplus_{i=0}^{2n} h^i(X)$ yields the definition of the "groups of motif cohomologies $h^i(X)$" of variety X. Any functor of cohomologies $X \to H^*(X)$ passes through functor h. Grothendieck called $H^*(X)$ a realization of motif h(X). Unfortunately, the proof that the functor $X \to h(X)$ itself is a theory of Weil cohomologies is based on the as yet unproved assumptions on algebraic cycles (the standard conjectures; see [238, 311]).

Nevertheless, Manin succeeded in showing in [33] that, without using the hypothetical part of theory of motifs, we can progress rather far in their computations and obtain important applications. This paper clarifies the behavior of motifs under monoidal transformations and finite coverings and also defines the "motif intermediate Jacobian" of a three-dimensional unirational variety. Using the functoriality of the latter, Manin proved the Weil conjectures for such varieties. The articles [56, 57] of Shermenev serve as continuations of Manin's work, in which Shermenev computes in explicit form the motif of a cubic hypersurface, of an Abelian variety, and of a Weil hypersurface (the latter has not been published).

In conclusion we remark that a cohomology theory of schemata, established chiefly in Grothendieck's fundamental papers, has turned into a fascinating area of algebraic geometry with a lot of unsolved problems and with rich applications.

§3. Fundamental Group and Homotopic Invariants of Schemata

As the foundation of the algebraic definition of the fundamental group of a schema we take the classical fact that this group should classify nonramified coverings. By the latter in schemata theory we mean an arbitrary finite étale morphism. When X is a normal irreducible algebraic variety, the group $\pi_1(X)$ was defined by Abhyankar [63] as a Galois group (a profinite group) of a maximal nonramified extension of the rational function field k(X) of variety X. By virtue of Riemann's classical existence theorem, when the ground field k is the complex number field, $\pi_1(X)$ is a profinite completion of the usual fundamental group (for example, see [442]).

In 1960 Grothendieck [467] gave a definition of $\pi_1(X)$ for any schema X, which reduces to the above definition if X is normal and irreducible. For this he defined the concept of a Galois covering of schema X, proved the prorepresentability of the functor $X' \to \mathrm{Hom}_X(\xi, X')$ from the category of such coverings into the category of sets (ξ is a geometric point of schema X), and then set $\pi_1(X, \xi) = \varprojlim_i \mathrm{Aut}(P_i)$, where $(P_i)_{i \in I}$ is the prorepresentative object of the functor being considered. For a connected schema X the group $\pi_1(X, \xi)$ does not depend (to within isomorphism) on ξ. Later Grothendieck [170] modified the definition of a fundamental group (groupe fondamental élargi) so as to take into consideration the infinite coverings of schema X. When schema X is one-branched (for example, is normal), these groups coincide. In the general case the extended group $\widetilde{\pi}_1(X)$ is not profinite (for example, for a rational curve X with a regular double point, $\widetilde{\pi}_1(X) \simeq Z$). Murre's lectures [391] were devoted to Grothendieck's definition.

Another definition of the fundamental group (coinciding with Grothendieck's extended fundamental group) was given by Lubkin [345].

The series of papers [63-68] by Abhyankar was devoted to the first algebraic computations of the fundamental group. In the main they were devoted to the computation of the group $\pi_1(V \setminus W)$, where $W \subset V$ is a divisor on a nonsingular algebraic variety V of dimension ≥ 2. In case char k > 0 there has actually been computed the subgroup $\pi_1^t(V \setminus W) \subset \pi_1(V \setminus W)$ classifying the coverings which are prolonged up to tamely ramified coverings of V. Abhyandar's results generalize the classical computations of the Zariski group

$\pi_1(\mathbf{P}^2 \setminus C)$, where C is a curve. Serre's article [473] and Popp's book [435] survey Abhyankar's work. Edmunds [182] also concerns himself with this same group of questions. In his fundamental paper [219] Grothendieck computed the group $\pi_1^t(X \setminus (P_1, \ldots, P_n))$ for a nonsingular point curve X. This result played a central role in the papers of Shafarevich [53] and Ogg [416]. Grothendieck's computations were based on the lifting of the curve to characteristic zero and on the application of the topological results in [440]. An analogous computation for a complete curve X (the first of which computation was proved without the use of the methods of formal geometry) was carried out also by Popp [431, 432]. The assumption of tame ramification was removed by Fulton [191]. A purely algebraic computation of $\pi_1(X \setminus \{P_1, \ldots, P_n\})$ is as yet not known. In [435] Popp showed that everything reduces to the three-point problem, i.e., to the case when $n = 3$ (also compare with [176]).

The method for computing the fundamental group with the aid of lifting to characteristic zero is based on the following theorem of Grothendieck. If f: $X \to \operatorname{Spec}(A)$ is a proper morphism with a closed connected fiber X_0, where A is a full Noetherian local ring, then $\pi_1(X) \cong \pi_1(X_0)$. This same theorem, with A replaced by an arbitrary Henselian Noetherian local ring, was proved by Artin [85]. From it follows the theorem on the change of base in Grothendieck's étale cohomologies. Artin also compared the groups $\pi_1(U)$ and $\pi_1(\hat{U})$, where U is an open set in Spec (A) and $\hat{U}$ is its preimage in Spec $(\hat{A})$ [81, 85].

See §2 on the Lefschetz theorem for the fundamental group. A variant of the Lefschetz theorem for the case of a quasiprojective variety V was proved by Popp [434]. In the classical case, when V is an open set in a projective space, this result was proved (with gaps) by Zariski in 1937. De Bruin's paper [144] also was devoted to Zariski's theorem. In [433] (also see [435]) Popp studied the behavior of the fundamental group of the complement of a curve on a smooth surface as the curve varies within a family.

Questions connected with the local fundamental group, i.e., with the calculations of $\pi_1(U)$, where U is an open set of the spectrum of a local Henselian ring, turned out to be very interesting. An impetus to these questions was provided by Mumford [26], who computed the fundamental group of the boundary ∂V of some "ε-neighborhood" of a normal singular point P on a complex algebraic surface. In particular, Mumford proved a criterion for simplicity (conjectured by Abhyankar): point P is nonsingular if and only if $\pi_1(\partial V) = 0$. An algebraic analog of set ∂V (as was noted apparently for the first time by Grothendieck; cf. [227], exp. XII) is the schema $X' = \operatorname{Spec} A \setminus \{\mathfrak{m}\}$, where A is the Henselization (or completion) of the local ring of a point P of an arbitrary normal algebraic surface and $\mathfrak{m}$ is a maximal ideal of ring A. Using Artin's theorem [82] on the lifting of a two-dimensional singularity to characteristic zero and his results from [81], we can apply Mumford's topological computations to compute the group $\pi_1(X')^{(p)}$, namely, the factor group of $\pi_1(X')$ by the normal divisor generated by a Sylow p-subgroup ($p = \operatorname{char} k$). Grothendieck and Murre [249] proved the topological finite-generability of the group $\pi_1(X')^{(p)}$. We note that Mumford's criterion of simplicity does not carry over directly to the case of a positive characteristic p. Indeed, by virtue of Nagata's example [398] (also see [82]) there exist an irregular ring R and a radical morphism f: $\operatorname{Spec}(R') \to \operatorname{Spec}(R)$, where R' is a regular ring. Hence it follows that $\pi_1(\operatorname{Spec}(R) \setminus \{\mathfrak{m}\}) \cong \pi_1(\operatorname{Spec}(R') \setminus \{\mathfrak{m}\}) = 0$ (the spaces Spec (R) and Spec (R') are homeomorphic). It is as yet not known whether the condition that X' be homeomorphic to the point spectrum of a regular ring is a necessary one for the vanishing of $\pi_1(X')$.

We note Serre's result [472] on the simple-connectedness of a unirational variety and Grothendieck's theorem on the birational invariance of $\pi_1(X)$ [467]. Using the theory of descent, Grothendieck established many important properties of π_1; in particular, he constructed the start of the exact sequence of homotopy groups (the first six terms) for a flat proper morphism.

The computation of the fundamental group of an Abelian variety (actually, contained in Lang and Serre [332]) can be found also in [467]. Serre [474] computed π_1 for an arbitrary algebraic group.

We remark that the Galois theory of rings (for example, see [151]) in commutative algebra was developed in parallel with the "geometric" study of the fundamental group of schemata. Many results of this theory are easily interpreted and reduced to the corresponding results on the algebraic fundamental group or on the étale cohomologies of affine schemata. Takeuchi devoted his paper [496] to the connection of these theories.

The definition of higher homotopic invariants of schemata was proposed independently by Lubkin [345] and by Artin-Mazur [83, 88]. A detailed account of the latter theory is contained in [89]. Both these theories are very close to each other and are based on the Verdier-Lubkin construction of a functor from the category of locally Noetherian schemata into the category of pro-objects of the homotopy category of simplicial sets. Taking the composition of this functor with the functor of "geometric realization," Artin and

Mazur obtained a certain pro-CW-complex X_{et} canonically comparable to schema X and called the homotopy type of schema X. The Verdier-Cartier construction of hypercoverings (SGA4, exp. V, Appendix) permits us to generalize the preceding construction and to determine the homotopy type of an arbitrary Grothendieck topology. The generalization of homotopy theory to pro-objects allows us to determine the homotopy groups $\pi_1(X_{et})$. For example, $\pi_1(X_{et})$ is an "extended" Grothendieck fundamental group. Artin and Mazur introduced important notions of a profinite completion $\hat{K}$ of a cell complex K and, by generalizing Riemann's existence theorem, proved that $C/X_{et} \simeq \hat{X}_{cl}$ for a normal schema X over a field. Here X_{cl} is the usual CW-complex defined by the complex space X^{an}. From this theorem it follows, in particular, that $\pi_1(X_{et}) \simeq \hat{\pi}_1(X_{cl})$, and for simply connected X, $\pi_i(X_{et}) \simeq \hat{\pi}_i(X_{cl})$, $i > 1$, where a profinite completion of the group occurs everywhere on the right. Among the other results in [89] we note, for example, the following fact. If s_1, s_2: $k \subset C$ are distinct immersions of an algebraically closed k into a field C, then $\hat{X}^1_{cl} \simeq \hat{X}^2_{cl}$ holds for the corresponding complex varieties X^1, X^2 obtained from the k-variety X by a change of base. By virtue of the well-known example of Serre [477], here it is impossible, in general, to remove the $\wedge$.

§4. Ample Sheaves. Projective Imbeddings of Abstract Varieties

Ever since Weil introduced the notion of an abstract algebraic variety, the natural question was posed of their projective immersibility. As Weil himself proved, the Jacobian variety of a curve (its construction was the original purpose for the introduction of an abstract variety) is always projective. Later on, Chow proved that any homogeneous space is projective [156]. The first example of an incomplete abstract variety not immersible into a projective space was presented by Nagata in 1956 [393]. Next he constructed an example of a complete nonprojective algebraic surface (with singular points) [394]. Finally, in 1958 he constructed and example of a nonsingular nonquasiprojective surface (which, moreover, is even rational) [395]. Later on, examples of nonprojective varieties were constructed by Hironaka [268].

In 1961 Moishezon [35] announced a criterion of projectivity of nonsingular abstract varieties. This criterion was next generalized to singular varieties [36], and a detailed account of previously obtained results appeared in 1964 [37].

As is well known, each birational mapping φ: $X \to P^n$ is given by some linear system, i.e., by an invertible sheaf $\mathscr{L}$ on X and by a set of its sections $s_1, \ldots, s_k$. The sheaf $\mathscr{L}$ is said to be very ample if the complete linear system $|\mathscr{L}|$ defines the closed imbedding $\varphi: X \subset P^n$ ($n = \dim_k H^0(X, \mathscr{L}) - 1$). The sheaf $\mathscr{L}$ is said to be ample if some tensor power $\mathscr{L}^{\otimes m}$ of it is very ample. The projective criterion is in fact an ampleness criterion for the sheaf. The first such criterion, coinciding with the corresponding Moishezon criterion, was given for nonsingular projective surfaces by Nakai in 1960 [404]. In 1963 he succeeded in extending his own result to arbitrary projective schemata [406]. At the present time this criterion (generalized to complete schemata by Kleiman [307]) is called the Nakai-Moishezon criterion and is stated as follows. A sheaf $\mathscr{L}$ on a complete k-schema X of dimension $n > 1$ is ample if and only if $(D_1, \ldots, D_{n-1}, \mathscr{L}) > 0$. for any effective divisors $D_1, \ldots, D_{n-1}$ on X. Here the intersection number is determined with the aid of Snapper's theorem (see §2).

A beautiful ampleness criterion is due to Seshardi (cf. [261]).

A numerical criterion for a very ample sheaf on a nonsingular surface is due to Mumford [381]. Beautiful characterizations of very ample sheaves in the language of the geometry of cones of ample sheaves are contained in Kleiman's paper [309] and in Mumford's book [381].

The following Kleiman's theorem (expressed as a consequence of Chevalley's conjecture) serves as an example of a "nonnumerical" projectivity criterion [309]. A nonsingular variety X is projective if and only if each finite set of closed points of X is contained in an open affine set. An example of a "nonnumerical ampleness criterion" is Grauert's criterion [341].

If an effective divisor D on schema X is ample (i.e., the sheaf $\mathscr{L} = \mathscr{O}_X(D)$ is ample), then the complement $X \setminus D$ is affine. Gizatullin [12] and Goodman [206] independently proved that the converse also is true in the case of surfaces all of whose singular points lie on D. Hence we obtain a new proof (also see [309]) of the old Zariski theorem stating that an algebraic surface X is projective if all its singular points lie in an affine piece (Zariski had further assumed the normality of X). By virtue of Artin's result [77] each normal surface over a finite field is projective. The Gizatullin-Goodman theorem does not carry over directly to the case of varieties of dimension greater than two (for example, the Zariski variety; cf. [206]). A generalization of this theorem is due to Goodman and Hartshorne [206, 261].

Hartshorne [258] defined an ample vector bundle (or locally free sheaf). One of the equivalent definitions of an ample sheaf $\mathcal{E}$ consists of the requirement that the tautological invertible sheaf $\mathcal{O}_P(1)$ on $P=\mathbf{P}_X(\mathcal{E})$ be ample. Various characterizations and properties of an ample sheaf are proved in [258]. For example, if char k = 0, then the tensor power of an ample sheaf is ample. Barton [109] succeeded in proving this result in the case of a positive characteristic.

In [261] Hartshorne discussed various possible generalizations of the notion of an ample divisor on a subvariety $W\subset X$ of larger codimension. These definitions are closely related to the question of the cohomological dimension of the complement $X\setminus W$ and also to the ampleness of the normal bundle to W. Griffiths [215] suggested that analogous and other possible properties of ampleness of a subvariety be taken as a definition in the analytic case.

Block and Gieseker [130] proved Hartshorne's conjectures in the zero-characteristic case: if a sheaf $\mathcal{E}$ is ample, then its Chern classes $c_i(\mathcal{E})$ are numerically positive. The latter signifies that $(c_i(\mathcal{E})\cdot Y)>0$ for any effective cycle Y of codimension n−i. Kleiman [312] (and Barenbaum [7], independently) has proved this result earlier for the case of surfaces over an arbitrary field.

Chow's results on the projectivity of homogeneous spaces were generalized to a considerable degree by Raynaud to the case of arbitrary group schemata [449]. We note, for example, that even an Abelian S-schema for an arbitrary schema S proves to be not necessarily projective (we need to require the normality of S).

By generalizing his own result of [399], Nagata proved a theorem which is very useful for applications: each S-schema of finite type can be imbedded as an open subvariety into a proper S-schema.

§5. Construction Techniques in Algebraic Geometry

1. General Results and Technical Tools. a) Representable functors. In a series of reports [220-226] at the Bourbaki seminar in 1959/62 Grothendieck subjected to detailed analysis, significantly clarified, and generalized certain classical constructions in algebraic geometry (the Poincare variety, the Chow variety, etc.). The fundamental concept introduced for this purpose by Grothendieck was the concept of a representable functor. We consider a certain category $\mathcal{C}$, and we let $\hat{\mathcal{C}}$ be the category of contravariant functors on $\mathcal{C}$ with value in the category of sets. Grothendieck's fundamental idea consists in that the functor $h:\mathcal{C}\to\hat{\mathcal{C}}$, defined as $h(X):Y\to \mathrm{Hom}_{\mathcal{C}}(Y, X)=X(Y)$ for any $X\in\mathcal{C}$, effects the imbedding of category $\mathcal{C}$ onto the complete subcategory $\hat{\mathcal{C}}$. Objects from $\hat{\mathcal{C}}$ of the form h(X) are called representable functors. For example, in application to schema theory, when $\mathcal{C}=(\mathrm{Sch}/S)$, this remark of Grothendieck permits us to identify the S-schema X with the functor "of a point with value in an S-schema" h(X): Y → Hom_S(Y, X) = X(Y). In this context Grothendieck succeeded in showing that all the known constructions in algebraic geometry are examples of representable contravariant functors on the subcategory (Sch/k) consisting of k-varieties. For example, the classical Poincare construction consists of associating with each nonsingular variety X over an algebraically closed field k the factor group $\mathrm{Pic}^0(X)$ of divisors algebraically equivalent to zero by the subgroup of principal divisors. The problem of the existence of the Poincare variety is the following. Does there exist for a given variety X a universal variety $\mathfrak{B}(X)$ parametrizing the group $\mathrm{Pic}^0(X)$? The latter signifies that there exists a "universal class of divisors" Θ on $X\times\mathfrak{B}(X)$ such that for any $D\in\mathrm{Pic}^0(X)$ we can find a unique point $z\in\mathfrak{B}(X)$, for which $D=\mathrm{np}_1(X\times\{z\}\cdot\Theta)$. In the case of an affirmative answer to this question the group $\mathrm{Pic}^0(X)$ is in an essential way provided with the structure of an algebraic variety (the elements of $\mathrm{Pic}^0(X)$ are found in bijective correspondence with the points of variety $\mathfrak{B}(X)$). Grothendieck remarked that under the existence condition the variety $\mathfrak{B}(X)$ represents the following functor on the category of k-varieties: $Y\to\mathrm{Pic}^0(X\underset{k}{\times}Y)$ (here the divisor Θ corresponds to the identity morphism $\mathfrak{B}(X)\to\mathfrak{B}(X)$).

The existence problem in algebraic geometry now becomes the problem of the representability of a contravariant functor on the category of schemata.

The language of representable functors turned out to be very convenient for defining many classical objects in algebraic geometry within the framework of schemata theory (the Grassmann schema, vector bundles, the flag schema, etc. [248]).

The first profound theorems on representability belong to Grothendieck. In 1964 Murre gave a criterion for the representability of an Abelian functor (i.e., with value in Abelian groups) and applied it to the construction of the Poincare schema [389]. The criterion was generalized by Matsumura and Oort [358] to arbitrary group functors (not necessarily Abelian). Murre's report [390] was devoted to one criterion of Grothendieck. Paper [491] also relates to general representability criteria.

b) Prorepresentable functors. An important necessary condition for the representability of a functor is its prorepresentability. The latter concept was introduced by Grothendieck in [221]. We consider the category of pro-$\mathscr{C}$-pro-objects of a category $\mathscr{C}$, i.e., arbitrary projective systems $(X_i)_{i\in I}$ of objects of category $\mathscr{C}$ (the morphisms of projective systems are defined in the natural way). Each functor $F:\mathscr{C}\to(\mathrm{Ens})$ is continued by a natural method up to the functor $\bar{F}:\mathrm{pro}\text{-}\mathscr{C}\to(\mathrm{Ens})$. To do this we need to set $\bar{F}((X_i)_{i\in I})=\varprojlim_{i\in I} F(X_i)$.

A functor F is said to be prorepresentable if $\bar{F}$ is representable, in other words, if there exists a pro-object $(P_i)_{i\in I}$ such that $F(X)=\varprojlim_{i\in I}\mathrm{Hom}(X, P_i)$ for any $X\in\mathscr{C}$.

Grothendieck formulated important criteria for the prorepresentability of a functor F on category $\mathscr{C}$ with fiber bundles and with a final object $\emptyset$. The following conditions are necessary conditions for prorepresentability: 1) $F(\emptyset)=\{\text{one point}\}$, 2) (left-exactness) $F(X\underset{Z}{\times}V)=F(X)\underset{Z(F)}{\times}F(Y)$ for any objects X, Y, Z of category $\mathscr{C}$. Grothendieck showed that in many important cases these conditions are also sufficient. Grothendieck's program of research in this area was carried out by Levelt [339].

Particularly simple is the situation when category $\mathscr{C}$ is Noetherian or Artinian (i.e., the set of pro-objects of each object of category $\mathscr{C}$ possesses the ascending or descending chain condition) while functor F is Abelian. In this case Gabriel [193] proved that a contravariant functor on an Abelian Noetherian category defines a prorepresentable functor on the corresponding dual Artinian category. The application of this fact to the theory of local duality is contained in Grothendieck's seminar [227] (also compare [48, 234]).

The most important application of the theory of prorepresentable functors relates to the case when category $\mathscr{C}$ is the category C_Λ of Artinian Λ-algebras, where Λ is a Noetherian local algebra over its own residue field k.

Schlessinger [58] proved that the functor $F: C_\Lambda \to (\mathrm{Ens})$ is prorepresentable if and only if it is left-exact and $\dim_k F(k[\varepsilon]) < \infty$, where $k[\varepsilon]$ is the algebra of dual numbers. Besides this, he gave a method for verifying the left-exactness of functor F. In this case the prorepresenting object is identified with some complete Λ-algebra $\mathscr{O}$; by the same token, $F(A)=\mathrm{Hom}_\Lambda(A,\mathscr{O})$ for any $A\in C_\Lambda$. If, furthermore, the functor F is formally smooth, i.e., the map $F(A')\to F(A)$ is surjective for each surjection $A'\to A$ in C_Λ, then algebra $\mathscr{O}$ is isomorphic with $\Lambda[[t_1,\ldots,t_n]]$, $n=\dim_k F(k[\varepsilon])$.

The application of the preceding situation to contravariant functors on the category of schemata is based on the following reasoning. Let G be a contravariant functor of the category of preschemata over Spec(Λ) and let $e\in G(\mathrm{Spec}(k))$. For any ring $A\in C_\Lambda$ we denote by $F(A)\subset G(\mathrm{Spec}(A))$ the set of $\xi\in G(\mathrm{Spec}(A))$ such that $G(i)\xi = e$, where i is an imbedding of Spec(k). Artin [86] calls functor G prorepresentable in e if the corresponding functor F is prorepresentable by some Λ-algebra W. If G is a representable functor and X is the schema representing it, then e defines a rational point $x\in X$ and functor F is prorepresentable by the ring $\hat{\mathscr{O}}_{X,x}=W$.

Functor G is said to be effectively prorepresentable in e if there exists an element $z\in G(\mathrm{Spec}(W))$ which induces a compatible system of elements $z_n\in G(\mathrm{Spec}(W/\mathfrak{m}^n))$ with $z_0 = e$. Artin's theorem on the algebraization of formal moduli [86] asserts that in this case there exist a k-schema X of finite type, a closed point $x\in X$, a k-isomorphism $k(x)\simeq k$, and an element $z\in G(X)$ inducing $e\in G(\mathrm{Spec}(k))$ and such that the ring $\hat{\mathscr{O}}_{X,x}$ prorepresents the corresponding functor F (it is assumed, furthermore, that k-algebra Λ has a finite type). In many cases effective prorepresentability follows from prorepresentability and Grothendieck's existence theorem. (The application of this theory to formal moduli is given in the next section.)

Levelt's papers [337, 338] are devoted to the connection between the prorepresentability criteria of Schlessinger and of Grothendieck and also to their generalization.

c) Grothendieck's theory of descent [201, 220, 467]. One of the first problems of the theory of descent was the following one, stated clearly by Weil in [506]. We are given an algebraic variety V over a finite normal extension k' of field k. Can we "lower" V to k (i.e., do there exist a k-variety W and a k'-isomorphism $W\underset{k}{\otimes}k'\simeq V$)? Weil gave an affirmative answer to this question for the case when V is quasiprojective and satisfies natural necessary conditions for such a descent: there exist canonical k'-isomorphisms h_σ: $V\to V^\sigma$ satisfying the "pasting-together condition" $h_{\sigma\tau}=(h_\sigma^\tau)\cdot h_\tau$. Here $\sigma,\tau\in\mathrm{Gal}(k'/k)$, while V^σ is the image of V relative to the action of σ on the coefficients of the equations defining V. Cartier [148] examined the analogous problem for an inseparable extension. Grothendieck noted that one and the same common-category situation lies at the base of the Weil and the Cartier problems.

Let $\mathcal{F}$ be a fiber category over category $\mathcal{C}$, i.e., a category $\mathcal{F}_S$ is associated with each object $S \in \mathcal{C}$ and a functor $\varphi^*: \mathcal{F}_S \to \mathcal{F}_{S'}$ is associated with each morphism φ: S' → S into $\mathcal{C}$. For any such morphism and for an object $X \in \mathcal{F}_S$, the "giving of a descent" onto X consists in the existence of an isomorphism $u: p_1^*(X) \simeq p_2^*(X)$ (where $S' \underset{S}{\times} S' \overset{p_1}{\underset{p_2}{\rightrightarrows}} S'$ are morphisms of a projection) satisfying the pasting-together condition $p_{12}^*(u) \cdot p_{23}^*(u) = p_{13}^*(u)$ (where $S' \underset{S}{\times} S' \underset{S}{\times} S' \overset{p_{12}}{\underset{p_{13}}{\underset{\rightarrow}{\overset{\rightarrow}{\rightarrow}}}} S' \underset{S}{\times} S'$ are morphisms of projections). The giving of a descent is said to be effective if there exists an object $Y \in \mathcal{F}_S$ such that $\varphi^*(Y) \simeq X$. Here the functor $\varphi^*: \mathcal{F}_S \to \mathcal{F}_{S'}$ realizes the equivalence of category $\mathcal{F}_S$ and subcategory $\mathcal{F}_{S'}$ consisting of objects with the effective giving of a descent. The problem of the theory of descent consists in finding the conditions on morphism f in order that any condition of descent on the object $X \in \mathcal{F}_{S'}$ be effective.

The application of this common-category problem to algebraic geometry refers to the case when $\mathcal{C}$ is the category of schemata. The morphism f: S' → S of schemata is called a morphism of effective descent relative to the fiber category $\mathcal{F}$ if any giving of a descent on object $X \in \mathcal{F}_{S'}$ is effective. By generalizing the results of Weil and Cartier, Grothendieck proved that an absolutely flat quasicompact morphism is a morphism of effective descent for the categories of quasicoherent sheafs, of affine or quasiaffine S-schemata, of étale S-schemata of finite type, and of some others. The same is true for a finite surjective morphism and for the categories of unramified coverings, of quasiprojective schemata (if, moreover, the morphism is locally free), or of all schemata (if the morphism is radical) [467].

Important criteria for the descent of group schemata were obtained by Raynaud [449]. In the presence of an effective descent we can consider the additional problem: it is required to preserve the properties of the lowered object (for example, the local freedom of a lowered sheaf, the flatness of a lowered schema, etc.). These problems were solved by Grothendieck in [467].

The ideas of the theory of descent (particularly the last problem) proved to have a great influence on commutative algebra. Among the mass of papers devoted to these problems we note [455]. Among the applications of the theory of descent in algebraic geometry we note the following:

1) The representability of functors was proved first for the category (Sch/S), and next, by applying the theory of descent, the representability of a functor on the category (Sch/S) was obtained. Here S' → S is the morphism of effective descent (see [223-225, 295]).

2) Criteria for the rationality of divisors (Cartier [148], Oort [420]) and their generalizations, namely, "Hilbert's theorem 90" in étale cohomologies (SGA4).

3) Proof of the topological invariance of the étale topology (i.e., the categories X_{et} and $(X_{red})_{et}$ are equivalent).

4) Descent properties for various classes of schema morphism [247, 467].

d) Equivalence relations on schemata. The problem of the theory of descent is a special case of the general problem of existence of a factor of a schema by some equivalence relation. The latter problem is one of the most important technical tools for proving the representability of various functors.

In [222] Grothendieck defined an equivalence relation on an object X of an arbitrary category $\mathcal{C}$ with fiber bundles as a certain subfunctor $R \subset h(X) \times h(X)$ such that $R(T) \subset X(T) \times X(T)$ is the graph of the equivalence relation on the set X(T). If $R \overset{p_1}{\underset{p_2}{\rightrightarrows}} X$ is the restriction of projection morphisms to R, then the factor X/R by the equivalence relation R is called the cokernel (p_1, p_2), i.e., the object representing the covariant functor $Z \to \{\varphi \in X(Z) \mid \varphi p_1 = \varphi p_2\}$. The factor X/R = Y is said to be effective if the canonic monomorphism $R \to X \underset{Y}{\times} X$ is an isomorphism.

A typical example of an equivalence relation is the case when there acts on object X a group object G of category $\mathcal{C}$ (i.e., h(G) is a presheaf of groups on $\mathcal{C}$). By definition, the action of G on X is given by the morphism G × X → X, and the image of the morphism G × X → X × X ((g, x) → (gx, x)) defines an equivalence relation on object X, the factor by which is denoted by X/G.

As is well known (for example, see [268]), the factor X/G does not always exist for the category of schemata. Grothendieck proved the existence of factors X/R in the following cases: 1) the projection

p_1: R → X is a finite morphism, and each equivalence class relative to R is contained in an open affine set; 2) X is quasiprojective over S, and p_1 is a proper morphism; 3) X is quasiprojective over S, and R is closed in $X \underset{S}{\times} X$ [169, 222].

Raynaud [451] proved the existence of an open dense set $U \subset X$ for which U/R exists. Furthermore, if R is closed in $X \underset{S}{\times} X$, then U contains points of codimension ≤ 1. Some other criteria for the representability of factor X/R by a flat equivalence relation (i.e., the projections p_1, p_2: R → X are flat) also were obtained by Raynaud [450, 452]. He has also obtained important applications of these results to the representability of the Neron-Severi schema, of the Poincare schema, and of the Neron model [451].

Corollaries of Grothendieck's results in the existence problem of the factor of a group schema by some subgroup are discussed in [168, 169]. See further on for applications to the Poincare schema. In [380] Mumford obtained profound theorems on the existence of the factor X/G of schema X by a reductive group G.

A more general situation (equivalence prerelations) was studied with the aid of the formalism of groupoids in [168, 169, 450].

e) Artin-Moishezon algebraic spaces. The absence in the general case of a factor of a schema by an equivalence relation forces us to seek a natural extension of the category of S-schemata (Sch/S) in which this factor always exists. The obvious solution is to immerse (Sch/S) into the category of locally ringed spaces. However, this category is too broad to obtain any "reasonable" results in it. If S = Spec (C), then it is natural to seek the factor X/R in the category of complex spaces. Hironaka's example in [268] showed that such a factor can exist in this category but does not possess an algebraic structure (or even a Kählerian one). In 1965 Matsusaka developed the general theory of Q-varieties in Weil's language [360]. In this category the factors of algebraic varieties by an algebraic equivalence exist "by definition." Another approach is due to Grothendieck and has been applied particularly successfully in the theory of group schemata. Here the category (Sch/S) is identified with the category of representable sheaves on the topology S_{fpqc}. Fundamental to such an identification is the theory of descent for absolutely flat quasicompact morphisms.

The concept of an algebraic space, proposed by Artin in 1968 ("étale schemata" or "varieties" in the original terminology) and, from other considerations, independently by Moishezon ("minischemata" in his terminology), connected up in a natural way the points of view of Matsusaka and Grothendieck. By definition, an algebraic space over a schema S is the name given to a sheaf on S_{et}, being a factor of some S-schema U of locally finite type by an equivalence relation R whose projections $R \rightrightarrows U$ are étale morphisms. We note that the factor is considered in the category of sheaves on S_{et}.

When S = Spec (C), the algebraic space over S possesses in natural fashion the structure of Moishezon's analytic space, and conversely, each such space is an algebraic space [40, 87].

The immersion of the category of S-schemata into the category of algebraic S-spaces permits us to examine the question on the representability of a functor in two stages: first, to investigate the question of its representability by an algebraic space, and next, to prove the representability of the latter by some schema. We remark that a satisfactory answer to the first question already yields a quite good solution to the problem being considered, since the various aspects and the naturalness of the concept of an algebraic space, investigated by Artin, Moishezon, Knutson [38-40, 86, 87, 314], allow us to consider the latter as a completely reasonable generalization of an algebraic variety. The general properties and techniques of the theory of schemata were transferred to algebraic spaces by Knutson [314].

In [86] Artin proved a representability criterion for a contravariant functor by an algebraic space. One of them is its effective prorepresentability. The proof of the representability criterion is based on the following ubiquitous Artin theorem on approximations [85]. Let R be a field or an excellent discrete valuation ring, A be the Henselization of some R-algebra of finite type with respect to a prime ideal, and $\mathfrak{m}$ be a maximal ideal of A. Let $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_N)$ be a solution of the system of algebraic equations $f_i(Y) = 0$ with coefficients in A, whose coordinates belong to the $\mathfrak{m}$-adic completion of A. Then for any integer $c > 0$ a solution $y = (y_1, \ldots, y_N) \in A^N$ exists such that $y_i \equiv \bar{y}_i \pmod{\mathfrak{m}^c}$.

A special case of Artin's theorem ($\dim A = 1$) was obtained earlier by Greenberg [213, 214] and Ershov [21].

The close connection between Matsusaka's Q-varieties and algebraic spaces is reflected in the following result of Artin [84]. Let $R \to U \times U$ be a flat equivalence relation and X be the factor U/R in the category of sheaves S_{fppf}. Then X represents an algebraic space. Note, however, that the class of algebraic spaces over C is already a class of Q-varieties since the latter, in general, is not representable by an analytic space (Holmann [276]).

By virtue of Artin's result in [84] the giving of a descent for algebraic spaces relative to an absolutely flat morphism is always effective. In the category of algebraic S-spaces there always exist factors of group spaces by an arbitrary flat S-subgroup.

Papers [5, 84, 454] are devoted to a survey of Artin's results. We leave aside the questions connected with the birational geometry of algebraic spaces and with the theory of singularities [39, 40, 87].

2. Picard Schema. One of the first problems to which Grothendieck applied the techniques presented above was the problem of constructing the Picard schema. The latter was introduced by Grothendieck in [224] as a natural generalization within the framework of schema theory of the Picard variety $\mathfrak{B}(X)$ of a nonsingular algebraic variety X. The first algebraic construction of $\mathfrak{B}(X)$ as an Abelian variety parametrizing the factor group X of divisors algebraically equivalent to zero by the principal divisor was given in 1951 by Matsusaka [359] and was next generalized to the case of a normal variety by Chow and Lang.

In Chevalley's 1958 seminar [466] the Picard variety $\mathfrak{B}_c(X)$ corresponding to Cartier divisors was studied. It was shown that for singular varieties $\mathfrak{B}_c(X)$ possesses (in contrast to $\mathfrak{B}(X)$) good functorial properties [483]. Chevalley [155] constructed the Picard variety $\mathfrak{B}_c(X)$ for complete normal varieties X. This construction was generalized to arbitrary projective varieties by Seshadri [481, 482].

To determine the Picard schema for an arbitrary S-schema X Grothendieck suggested the examination of the relative Picard functor $\mathrm{Pic}_{X/S}$ on the category (Sch/S). The value of this functor on the S-schema S is the group $H^0(S', R^1_{f_{pqc}} f'_*(G_{m,X'}))$, where $f': X' = X \underset{S}{\times} S' \to S'$ is the change-of-base morphism while $R^1_{f_{pqc}} f'_*(G_{m,X'})$ is the sheaf on the Grothendieck topology $S'_{f_{pqc}}$ associated with the presheaf $T \to H^1(T_{f_{pqc}}, G_m) = H^1(T_{et}, G_m)$. Let Pic(T) be the group of classes of invertible sheaves on schema T. When functor $\mathrm{Pic}_{X/S}$ is representable on (Sch/S), the S-schema representing it is called the relative Picard schema of the S-schema X and is denoted by $\mathbf{Pic}_{X/S}$. For example, if X is an algebraic k-scheme having a rational k-point, then Grothendieck shows that $\mathbf{Pic}_{X/k}(S') = \mathrm{Pic}(X \underset{k}{\times} S')/\mathrm{Pic}(S')$ for any k-schema S' and, in particular, $\mathrm{Pic}_{X/k}(k) = \mathrm{Pic}(X)$ is identified with the group of k-rational points $\mathbf{Pic}_{X/k}(k)$ of the schema $\mathbf{Pic}_{X/k}$.

The first theorem on the representability of the functor $\mathrm{Pic}_{X/S}$ is due to Grothendieck and relates to the case when f: X → S is a projective morphism with geometrically integral fibers. In this case the schema $\mathrm{Pic}_{X/S}$ is locally finitely presented separable group S-schema. If S = Spec(k), where k is a field, then the connected component of unity $\mathbf{Pic}^0_{X/S}$ of the schema $\mathbf{Pic}_{X/k}$ is an algebraic k-schema, and the corresponding reduced k-schema $(\mathbf{Pic}^0_{X/k})_{red}$ is precisely the Picard variety $\mathfrak{B}_c(X)$ in Chevalley's definition. The presence of nilpotent elements in the local rings of schema $\mathbf{Pic}^0_{X/k}$ yields much additional information on the Picard variety and permits us to explain various pathologies of algebraic varieties over a field of characteristic p > 0. We remark that by virtue of Cartier's theorem (cf. [169, 381, 422]) the schema $\mathbf{Pic}^0_{X/k}$ is always reduced if p = 0. Igusa has cited an example of a nonsingular algebraic surface F with a one-dimensional Picard variety and with $\dim_k H^1(F, \mathcal{O}_F) = 2$ [286]. Noting that the space $H^1(F, \mathcal{O}_F)$ is identified with a tangent space to the unity of the Picard schema $\mathbf{Pic}^0_{X/k}$, Grothendieck explains this fact in [225] by the irreducibility of the corresponding schema. An analogous explanation can be given for the old Igusa result $\dim_k H^1(X, \mathcal{O}_X) \geq q$, where q is the irregularity of variety X, coinciding with the dimension of its Picard variety $\mathfrak{B}(X)$. Mumford [381], using Serre's techniques from [479] of Bockstein operations on cohomologies of sheaves of Witt rings, investigated the reducibility of the Picard schema of a smooth algebraic surface F. In particular, he proved Grothendieck's theorem [225] that if $H^2(F, \mathcal{O}_F) = 0$, then the schema $\mathrm{Pic}_{F/k}$ is reduced. If base S is a nonsingular curve while the fibers of the morphism f: X → S are curves, the connected component of the unity $\mathbf{Pic}^0_{X/S}$ of schema $\mathrm{Pic}_{X/S}$ is an analog of Igusa's "family of Jacobian varieties" [228] (cf. [447]).

In [389] Murre proved the representability of the Picard functor for an arbitrary proper k-schema X. In [418, 419] Oort investigates the connection between the schemata $\mathrm{Pic}_{X/k}$ and $\mathrm{Pic}_{X_{red}/k}$. This connec-

tion was investigated in particular detail for multiple algebraic curves [417]. Murre's theorem was generalized in [491] to the case of a proper schema over a local Artin ring.

At the Stockholm congress in 1962 Mumford announced a theorem on the representability of the Picard functor, generalizing somewhat the theorem of Grothendieck (the fibers of f are not necessarily irreducible; however, all irreducible components are geometrically irreducible). The proof of this theorem was based on the methods developed in [380]. A specialization of this theorem for the case of a nonsingular algebraic projective surface was presented in [381]. Already in the case treated by Mumford the schema $\mathrm{Pic}_{X/S}$ is not necessarily separable over S.

An essential improvement in the question of the representability of the Picard functor was Artin's theorem in [86] stating that for any proper flat morphism $f\colon X \to S$ (finitely presented if S is non-Noetherian) for which $f'_*(\mathcal{O}_{X'}) = \mathcal{O}_{S'}$ for any change of base $f'\colon X' = X \times_S S' \to S'$, the functor $\mathrm{Pic}_{X/S}$ is an algebraic space over S. Note that the condition $f'_*(\mathcal{O}_{X'}) = \mathcal{O}_{S'}$ (cohomological flatness) is always fulfilled if the fibers of morphism f are reduced [242]. Artin's theorem reduces the question of the representability of the Picard functor by a schema to the question of the representability of a group algebraic space by a schema. Artin himself answered this question affirmatively for the case when the base schema is the spectrum of a local Artin ring, by the same token obtaining anew the result of Murre [389] and of Sivaramakhrishna [491]. Raynaud [447] generalized this result to the case when S is a normal locally Noetherian schema of dimension 1 and the group algebraic space G is smooth over S (the latter condition can be excluded by virtue of Avantaraman's result). Furthermore, Raynaud obtained nontrivial sufficient conditions for the cohomological flatness of the morphism $f\colon X \to S$. A detailed investigation was carried out in [447] for the case when the fibers of f are algebraic curves; in particular, the connection between the schema $\mathrm{Pic}^0_{X/S}$ and the Neron model [414, 444] of the Jacobian variety of the common fiber of f was investigated. Raynaud's results were announced in [445, 446].

In [295] Iversen, using Raynaud's results on the descent of group schemata [449], generalized the classical Weil construction of the Jacobian variety of a curve and constructed the Picard schema for a proper flat morphism with geometrically integral one-dimensional fibers.

Important theorems on the finiteness of the Picard schema, due to Grothendieck, are discussed in [225]. Murre's report [390] gives an account of Grothendieck's results on local representability.

In 1961 Mumford [377] defined the Picard group of a normal singular point on a complex algebraic surface and posed the question of introducing the structure of an algebraic group on it. In [227] Grothendieck sharpened this definition (by examining arbitrary local Hensel rings) and gave an affirmative answer to Mumford's question.

3. Hilbert Schema and Other Constructions. The Hilbert schema was introduced by Grothendieck in [223] as a natural generalization within the framework of schemata theory of the classical Chow variety of algebraic cycles. For any coherent sheaf $\mathcal{F}$ on an S-schema X, Grothendieck defined the functor $\mathrm{Quot}_{\mathcal{F}|X|S}\colon (\mathrm{Sch}/S) \to (\mathrm{Ens})$. The value of this functor on an arbitrary S-schema Z is the set $\mathrm{Quot}_{\mathcal{F}|X|S}(Z)$ of classes by isomorphism of the factor sheaves of the sheaf $\mathcal{F}_Z = \mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{O}_Z$ on $X \times_S Z$ which are flat on Z, finitely presented, and have a support which is proper over Z. For the case $\mathcal{F} = \mathcal{O}_X$ the functor $\mathrm{Quot}_{\mathcal{O}_X|X|S}$ is denoted by $\mathrm{Hilb}_{X/S}$ and is called the Hilbert functor associated with the S-schema X. For any polynomial $P \in \mathbb{Q}[x]$ there is defined a subfunctor $\mathrm{Hilb}^P_{X/S}$ whose value on an S-schema Z is a set of closed subschemata $T \subset X \times_S Z$, flat and proper over Z, such that for any point $z \in Z$ the fiber $T_z = T \otimes_Z k(z)$ has P as the Hilbert polynomial. The functor $\mathrm{Hilb}_{X/S}$ is representable if and only if the functors $\mathrm{Hilb}^P_{X/S}$ are representable, and in this case, for the representing object (the Hilbert schema) we have $\mathbf{Hilb}_{X/S} = \coprod_P \mathbf{Hilb}^P_{X/S}$, where $\mathbf{Hilb}^P_{X/S}$ is the representing schema for $\mathrm{Hilb}^P_{X/S}$.

Grothendieck proved in [223] that when $f\colon X \to S$ is a quasiprojective morphism with a Noetherian schema S, the functor $\mathrm{Hilb}^P_{X/S}$ is representable by schema $\mathbf{Hilb}^P_{X/S}$ of finite type and quasiprojective over S. More generally, in this case the functor $\mathrm{Quot}_{\mathcal{F}|X|S}$ is representable by a quasiprojective schema of locally finite type over S. When the morphism $f\colon X \to S$ is not quasiprojective, the schema $\mathbf{Hilb}_{X/S}$ does not exist, in general. However, by virtue of Artin's result [86] the functor $\mathrm{Quot}_{\mathcal{F}|X|S}$ is representable by an algebraic space locally finitely presented over S for any schema X locally finitely presented over S. By the same token the question of the representability of the functor $\mathrm{Quot}_{\mathcal{F}|X|S}$ is fully answered in the category of algebraic spaces.

The construction of the schema $\mathrm{Hilb}^{P}_{X/S}$ for the case when X is a smooth algebraic surface over field k and P is a polynomial of degree 1 is contained in Mumford's book [381]. In this case the functor $\mathrm{Hilb}^{P}_{X/S}$ coincides with the functor $\mathrm{Div}^{1}_{X/k}$, also studied by Grothendieck [223] for an arbitrary S-schema X. Furthermore, Mumford showed the connection between the Hilbert schema and the classical constructions of Chow and van der Waerden of the Chow variety (also see [187, 380]).

Fogarty [188] investigated the connection between the Hilbert schema of an algebraic surface and its symmetric product.

In [257] Hartshorne proved the connectedness (and even the linear connectedness) of the Hilbert schema $\mathrm{Hilb}^{P}_{\mathbf{P}^n_{k}/k}$, generalizing the classical assertion on the connectedness of "the variety of modules of space curves." An algebraic proof of this result (in the language of ideals) is due to Cartier [150]. The Hilbert schemata $\mathrm{Hilb}_{\mathbf{P}^n_{k}/k}$ play an important role in Mumford's construction of the variety of modules of space curves [380]. Mumford [378] showed that the Hilbert schema of nonsingular curves of degree 14 and genus 24 in a three-dimensional projective space over the complex number field has nilpotent elements in local rings. The presence of the latter explains certain pathologies in classical algebraic geometry.

In [221] Grothendieck investigated the representability of other important functors in algebraic geometry. His approach to the study of the automorphism group of an algebraic variety proved to be particularly interesting. Grothendieck defined the automorphism schema $\mathrm{Aut}_{X/S}$ of an S-schema X as the representing object for the functor $\mathrm{Aut}_{X/S}: S' \to \mathrm{Aut}_{S'}(X \underset{S}{\times} S')$ from the category (Sch/S) into the category of groups. In particular, if S = Spec (k) is the spectrum of a field, then the group of rational k-points of the group schema $\mathrm{Aut}_{X/k}$ coincides with the k-automorphism group of the k-schema X. Grothendieck proved the existence of the schema $\mathrm{Aut}_{X/S}$ for any projective S-schema X. The structure of an algebraic group on the automorphism set of a projective algebraic variety had been introduced earlier by Matsusaka and Matsumura (unpublished). In [358] Matsumura and Oort generalized Grothendieck's result to an arbitrary proper S-schema X. The structure of an algebraic group on the automorphism set of a complete variety was introduced earlier by Ramanujam [437]. The schema $\mathrm{Aut}_{X/S}$ is a group S-schema locally finitely presented and, in general, unreduced [358].

We remark that even in the classical case when X = Spec (K), S = Spec (k), where K/k is the finite extension of a field, the schema $\mathrm{Aut}_{K/k}$ yields a new point of view on Galois theory. Namely, the algebraic group $\mathrm{Aut}_{K/k}$ plays the role of the "present-day" Galois group, while its group of k-rational points is precisely the classical Galois group. For example, if K/k is purely inseparable, $\mathrm{Aut}_{K/k}(k) = \{e\}$, and $\dim \mathrm{Aut}_{K/k} > 0$ (!). This classical case was examined in detail in [110]. See [505], for example, for the applications of this point of view to the theory of automorphisms of Lie algebras.

Analogously to the functor $\mathrm{Aut}_{X/S}$, Grothendieck defined and investigated the functors $\mathrm{Hom}_S(X, X')$, $\mathrm{Isom}_S(X, X')$, and others.

In [368, 369] Miyanishi investigated the representability of the functor PH (G; X/S) associating with each S-schema S' the set of classes by isomorphism of principal homogeneous spaces on $X_{S'} = X \underset{S}{\times} S'$ relative to the group S-schema G. This functor was introduced by Grothendieck in [14] as a natural generalization of the Picard functor (the localization of the functor $\mathrm{PH}(G_m; X/S)$ relative to the $f$pqc-topology is the Picard functor $\mathrm{Pic}_{X/S}$). When S = Spec (k) and G is a linear commutative algebraic group, this functor is representable, and the k-schema representing it has a close relation with the Picard variety of divisors of type G, constructed and studied by Bertin [124–127]. Miyanishi proved the representability of the functor PH (G; X/S) for the case when the morphism is cohomologically flat, the Picard functor $\mathrm{Pic}_{X/S}$ is representable by a sum of quasiprojective S-schemata, and G is a finite locally free group S-schema.

In the present survey we do not touch upon the very interesting questions connected with the representability of "global schemata of modules" [166, 192, 285, 379, 380].

§6. Formal Geometry

1. Formal Schemata. The definition and the foundations of the theory of such schemata are contained in [219, 240, 248]. Formal schemata play an important role in the infinitesimal study of schemata and also yield a convenient context for the theory of formal groups (cf. [169]). In the affine case they correspond to the formal spectra of topological algebras, i.e., to the set of closed prime ideals equipped with the Zariski

topology and with a sheaf of local rings. Generally formal schemata are obtained by pasting together affine formal schemata. The most important special case (and, it seems, as yet the only one usable in applications) consists in the examination of a commutative ring A with identity and of an I-adic topology on A. In this case the formal spectrum Sp f(A) is a ringed space whose support is a closed subset Spec (A) defined by the ideal I, while the structure sheaf is the restriction of the sheaf $(\hat{A})$, where $\hat{A}$ is the completion of A in the I-adic topology. The most important construction leading to a formal schema is the construction of the formal completion of the schema X along a closed subschema Y. The latter is a formal schema with a topological space Y and a structure sheaf $\mathcal{O}_{\hat{X}} = \varprojlim_n \mathcal{O}_X/I^{n+1}$, where I is the sheaf of ideals of subschema Y. The schema $Y_n = (Y, \mathcal{O}_X/I^{n+1})$ is called the n-th infinitesimal neighborhood of subschema Y, and $\hat{X} = \varinjlim_n Y_n$. The canonical morphism of ringed spaces i: $\hat{X} \to X$ defines, for any coherent sheaf $\mathcal{F}$ on X, its formal completion $\hat{\mathcal{F}} = \varprojlim_n \mathcal{F}/I^{n+1}\mathcal{F} = i^*(\mathcal{F})$.

In Chap. III of his monumental treatise [242] Grothendieck developed the cohomology theory of formal schemata. The fundamental results of this theory are the theorems of comparison and of existence for proper morphisms. Let f: X → S be a proper morphism of Noetherian schemata, $S_0 \subset S$ be a closed subschema of schema S, $X_0 = f^{-1}(S_0)$, $\hat{X}$ (repsectively, $\hat{S}$) be the formal completion of X (respectively, S) along X_0 (respectively, S_0), i: $\hat{X} \to X$ and j: $\hat{S} \to S$ be corresponding canonic morphisms, $\hat{f}$: $\hat{X} \to \hat{S}$ be the restriction of f on $\hat{X}$. Grothendieck's comparison theorem asserts that for any coherent sheaf F on X the canonic homomorphism of sheaves $\widehat{R^q f_*(F)} \to R^q \hat{f}_*(\hat{F})$ is an isomorphism. In particular, if s ∈ S, then the m-adic completion of the fiber $(R^q f_*(F))_s$ at point s is isomorphic with $\varprojlim_n H^q(f^{-1}(s), F \otimes \mathcal{O}_{S,s}/\mathfrak{m}^n)$. For q = 0 this theory should be considered as a natural generalization of Zariski's "theory of holomorphic functors" [513]. In particular, Zariski's connectedness theorem follows easily from it. Grothendieck also investigated in which case is the limit group isomorphic with the group $H^q(f^{-1}(s), \mathcal{F} \otimes \mathcal{O}_{f^{-1}(s)})$. The results obtained by him are very profound and are connected with the question of the cohomological flatness of morphism f. Grothendieck's existence theorem asserts, under the same assumptions as in the comparison theorem, that for any coherent sheaf $\mathcal{F}$ on schema X there exists a coherent sheaf F on X for which $\hat{F} \simeq \mathcal{F}$. The theorems obtained are formal analogs of Serre's comparison and algebraization theories from [478]. A survey of Grothendieck's results are contained in his report at the Bourbaki seminar [219].

Grothendieck [227] gave a generalization of the comparison and existence theory to the case of not necessarily proper morphisms.

Theorems, important for applications, on the algebraizability of a formal schema X over a complete local ring A, whose support is projective over the residue field of A, are contained in [242].

In [261] Hartshorne proved a formal duality theorem $H^i(\hat{X}, \hat{F}) = \mathrm{Hom}_k(H^{n-i}_Y(X, \mathrm{Hom}_{\mathcal{O}_X}(F, \omega_X)), k)$, where $\hat{X}$ is the completion of a smooth proper schema X relative to a closed subschema Y, ω_X is a canonic sheaf on X, and H^{n-i}_X are local cohomologies relative to Y.

Hartshorne [260, 261], Matsumura [273], and Hironaka [270, 273] studied the formal completion of a subschema Y in a smooth proper k-schema X. In particular, they investigated the question as to when the field of "meromorphic" functions K(X) (i.e., global sections of the sheaf of complete rings of partial local rings of schema X) coincides with the field of rational functions K(X). This coincidence holds when Y is a complete intersection [260, 261] or an effective Cartier divisor with an ample normal sheaf [270] or, finally, a subschema of a projective space [261, 273].

In [261] Hartshorne discovered an interesting relation of these questions with those on the cohomological dimension of the complement $X \setminus Y$ and on the Lefschetz-Grothendieck theory. To him also are due the important theorems on the finite dimensionality of the cohomologies $H^i(X, F)$ [258, 260, 261, 264]. Hironaka investigated the imbeddings of a formal schema Z in a smooth k-schema T so that T ≃ Z and that T would possess the properties of universality relative to rational mappings [270]. Here too an example was presented of a nonalgebraizable formal schema; another such example was given by Hartshorne [261].

2. Formal Deformations. The techniques of formal geometry proved to be particularly useful in the theory of deformation of schemata, designed to generalize to the case of schemata the fundamental results of the Kodaira-Spencer theory of deformations of complex varieties. The basic questions of this theory are the following: a) (The lifting problem.) We are given a k-schema X_0 of finite type and a certain morphism u: Spec (k) → S whose image is a point $s_0 \in S$. Does there exist a flat S-schema X of finite type for which $X_0 \simeq X_{S_0}$? b) (The module problem.) Does there exist a universal family $\mathcal{X} \to M$ for all liftings of

schemata X_0? As a matter of fact the answers to the questions posed are in the negative in the general case. Let us assume that the answer to question a) is in the affirmative for some schema X_0. In this case the formal completion $\hat{X}$ along fiber X_0 is a formal schema over $\mathrm{Spec}(\hat{\mathcal{O}}_{S,s_0})$ whose topological space is isomorphic with schema X_0, while for any integer $n \geq 0$ the schema $X \otimes_S \mathcal{O}_{S,s}/\mathfrak{m}^n$ is the n-th infinitesimal neighborhood of schema X_0. Thus, we arrive at a "formal lifting" of schema X_0. Weakened forms of questions a) and b) are the following formal analogs of them: a') We are given a complete local ring A with a residue field k. Does there exist a formal schema X over A whose topological space is isomorphic with X_0? b') Does there exist a universal family, i.e., a complete local ring $\mathcal{O}$ with residue field k and a formal schema $\mathcal{X} \to \mathrm{Spec}(\mathcal{O})$ such that for any formal deformation $X \to \mathrm{Spec}(A)$ there exists the homomorphism $\mathcal{O} \to A$ of rings for which $X \simeq \mathcal{X} \otimes_{\mathcal{O}} A$? We remark that question b') is a formal analog of Kuranishi's construction [327] of a local variety of modules in analytic geometry.

Grothendieck reformulated question b') as a question on the prorepresentability of the following functor on the category of Artinian k-algebras with residue field k, $F: A \to \{\text{set of A-schemata X of finite type with } X \otimes_A k \simeq X_0\}$. The prorepresenting object for this functor is the formal module schema $\mathcal{X} \to \mathrm{Spec}(\mathcal{O})$. Applying Schlessinger's [58] and Grothendieck's [219] criteria (see §5), we obtain an affirmative answer to question b') in the following case: $\dim_k H^1(X_0, \Theta^1_{X_0}) = m < \infty$ and $H^0(X_0, \Theta^1_{X_0}) = 0$ (here $\Theta^1_{X_0}$ is the sheaf of germs of sections of the tangent bundle to X_0). Moreover, if $H^2(X_0, \Theta^1_{X_0}) = 0$ (for example, X_0 is a curve), then the ring $\mathcal{O} \simeq k[[t_1, \ldots, t_m]]$. This result is precisely analogous to Kuranishi's result cited above. In the case when, in addition, schema X_0 is projective, Grothendieck's algebraization theorem mentioned in Para. 1 allows us to prove the effective prorepresentability of functor F, while Artin's theorem yields the algebraization of the formal module schema for X_0 (see §5, Para. 1).

Grothendieck [467] investigated question a') for a smooth schema X_0. The formal lifting of schema X_0 to a formal A-schema X is effected in the form of a sequential construction of infinitesimal neighborhoods X_n over $S_n = \mathrm{Spec}(A/\mathfrak{m}^n)$ such that $X_n \otimes_{S_n} S_{n-1} \simeq X_{n-1}$. In view of the smoothness of X_0 such a construction can always be made locally. Grothendieck showed that an obstacle to the pasting together of such infinitesimal neighborhoods lies in the group $H^2(X_0, \Theta^1_{X_0})$. Moreover, the continuation of schema X_0 up to a formal schema $X \to \mathrm{Spec}(A)$ is unique if $H^1(X_0, \Theta^1_{X_0}) = 0$. We remark that complex-analytic analogs of this theorem were considered by Grauert [13]. In particular, an affirmative answer to question a') follows from Grothendieck's theory if $H^2(X_0, \Theta^1_{X_0}) = 0$ (for example, X_0 is an affine or smooth curve). If, furthermore, the schema is projective over k, then by virtue of Grothendieck's algebraization theorem we obtain an affirmative answer also to question a) for the schema $S = \mathrm{Spec}(A)$.

Particularly important applications of the preceding theory are the questions connected with the lifting of k-schema X_0 to characteristic zero. In this we are interested in question a), where schema S is a schema of characteristic zero (for example, $S = \mathrm{Spec}(W(k))$ is the spectrum of the Witt vector ring over k). By virtue of Serre's example [475] the answer to this question is not always in the affirmative even for smooth projective surfaces. From Grothendieck's theory immediately follows an affirmative answer to this question for smooth projective curves. When X_0 is an Abelian polarized variety, this question was examined by Grothendieck and Mumford (cf. [383, 424]). For finite group schemata it was considered by Mumford and Oort [424, 425].

In 1968, with the aid of the concept of a cotangent complex of a morphism of schemata, Grothendieck succeeded in making a significant generalization of deformation theory of smooth curves to arbitrary relative schemata [235]. The theory of a cotangent complex and its application to deformation theory was considerably developed by Illusie [290, 292], which in its own turn globalized the local theory of Andree [72] and Quillen [436] (also see [341]).

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