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Configuration spaces of complex and real spheres

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Abstract

We study the GIT-quotient of the Cartesian power of projective space modulo the projective orthogonal group. A classical isomorphism of this group with the inversive group of birational transformations of the projective space of one dimension less allows us to interpret these spaces as configuration spaces of complex or real spheres.

*To Rob Lazarsfeld on the occasion
of his 60th birthday*

1 Introduction

In this paper we study the moduli space of configurations of points in complex projective space with respect to the group of projective transformations leaving invariant a nondegenerate quadric. More precisely, if $\mathbb{P}^n = \mathbb{P}(V)$ denotes the projective space of lines in a linear complex space V equipped with a non-degenerate symmetric form $\langle v, w \rangle$, we study the GIT-quotient

$$\begin{aligned} \mathbf{O}_n^m &:= \mathbb{P}(V)^m // \mathrm{PO}(V) = \mathrm{Proj} \left(\bigoplus_{d=0}^{\infty} H^0(\mathbb{P}(V)^m, \mathcal{O}_{\mathbb{P}(V)}(d)^{\boxtimes m})^{\mathrm{O}(V)} \right) \\ &\cong \mathrm{Proj} \left(\bigoplus_{d=0}^{\infty} (S^d(V^*)^{\boxtimes m})^{\mathrm{O}(V)} \right). \end{aligned}$$

If $m \geq n + 1 = \dim(V)$ then generic point configurations have 0-dimensional isotropy subgroups in $\mathrm{O}(V)$, and since $\dim \mathrm{O}(n + 1) = \frac{1}{2}n(n + 1)$ we expect that $\dim \mathbf{O}_n^m = mn - \frac{1}{2}n(n + 1)$ when $m \geq n + 1$.

Let

$$R(n; m) = \bigoplus_{d=0}^{\infty} (S^d(V^*)^{\otimes m})^{O(V)}.$$

It is a finitely generated graded algebra with graded part $R(n; m)_d$ of degree d equal to $(S^d(V^*)^{\otimes m})^{O(V)}$. After polarization, $R(n; m)_d$ becomes isomorphic to the linear space $\mathbb{C}[V^m]_{d, \dots, d}^{O(V)}$ of $O(V)$ -invariant polynomials on V^m which are homogeneous of degree d in each vector variable. The first fundamental theorem (FFT) of invariant theory for the orthogonal group [19, Chapter 2, Section 9] asserts that $\mathbb{C}[V^m]^{O(V)}$ is generated by the bracket functions $[ij] : (v_1, \dots, v_m) \mapsto \langle v_i, v_j \rangle$. Using this theorem, our first result is the following:

Theorem 1.1 *Let Sym_m be the space of symmetric matrices of size m with the torus*

$$T^{m-1} = \{(z_1, \dots, z_m) \in (\mathbb{C}^*)^m : z_1 \cdots z_m = 1\}$$

acting by scaling each i th row and i th column by z_i . Let \mathbb{S}_m be the toric

variety $\mathbb{P}(\mathrm{Sym}_m)/T^{m-1}$. Then \mathbf{O}_n^m is isomorphic to a closed subvariety of \mathbb{S}_m defined by the rank condition $r \leq n + 1$.

For example, when $m \leq n + 1$, we obtain that \mathbf{O}_n^m is a toric variety of dimension $\frac{1}{2}n(n + 1)$.

The varieties \mathbf{O}_1^m are special since the connected component of the identity of $\mathrm{O}(2)$ is isomorphic to $\mathrm{SO}(2) \cong \mathbb{C}^*$. This implies that \mathbf{O}_1^m admits a double cover isomorphic to a toric variety $(\mathbb{P}^1)^m//\mathrm{SO}(2)$. We compare this variety with the toric variety $X(A_{m-1})$ associated with the root system of type A_{m-1} (see [2, 15]). The variety $X(A_{m-1})$ admits a natural involution defined by the standard Cremona transformation of \mathbb{P}^{m-1} and the quotient by this involution is a generalized Cayley 4-nodal cubic surface Cay_{m-1} (equal to the Cayley cubic surface if $m = 3$). We prove that \mathbf{O}_1^m is isomorphic to Cay_{m-1} for odd m and equal to some blow-down of Cay_{m-1} when m is even.

The main geometric motivation for our work is the study of configuration spaces of complex and real spheres. It has been known since F. Klein and S. Lie that the inversive group¹ defining the geometry of spheres in dimension n is isomorphic to the projective orthogonal group $\mathrm{PO}(n+1)$ (see, e.g., [8, Section 25]). Thus any problem about configurations of m spheres in \mathbb{P}^n is equivalent to the same problem about configurations of m points in \mathbb{P}^{n+1} with respect to $\mathrm{PO}(n+1)$. The last two sections of this paper give some applications to the geometry of spheres.

2 The first fundamental theorem of invariant theory

Let V be an $(n+1)$ -dimensional vector quadratic space, i.e., a vector space together with a nondegenerate symmetric bilinear form whose values we denote by $\langle v, w \rangle$. Let $G = \mathrm{O}(V)$ be the orthogonal group of V and

$\mathrm{PO}(V) = \mathrm{O}(V)/\{\pm I\}$. Consider the diagonal action of G on V^m . The first fundamental theorem of invariant theory for the orthogonal group (see [14, Chapter 11, 2.1; 19, Chapter 2, Section 9]) asserts that any G -invariant polynomial function on V^m is a polynomial in the bracket functions

$$[ij] : V^m \rightarrow \mathbb{C}, (v_1, \dots, v_m) \mapsto \langle v_i, v_j \rangle, \quad 1 \leq i, j \leq m.$$

The algebra of G -invariant polynomial functions $\mathbb{C}[V^m]^G$ has a natural multi-grading by \mathbb{N}^m with homogeneous part $\mathbb{C}[V^m]_{(d_1, \dots, d_m)}^G$ equal to the linear space of polynomials which are homogeneous of degree d_i in each i th vector variable. This grading corresponds to the natural action of the torus \mathbb{C}^{*m} by scaling the vectors in each factor. The \mathbb{N} -graded ring $R(n; m)$ in which we are interested is the subring $\bigoplus_{d=0}^{\infty} \mathbb{C}[V^m]_{(d, \dots, d)}^G$. We have

$$R(n; m) \cong \mathbb{C}[V^m]^{\mathrm{O}(V) \times T},$$

where

$$T = \{(z_1, \dots, z_m) \in \mathbb{C}^{*m} : z_1 \cdots z_m = 1\}.$$

Let Sym_m denote the linear space of complex symmetric $m \times m$ matrices. If we view V^m as the space of linear functions $L(\mathbb{C}^m, V)$, then we can define a quadratic map

$$\Phi : V^m \rightarrow \text{Sym}_m, \quad (v_1, \dots, v_m) \mapsto (\langle v_i, v_j \rangle)$$

by composing

$$\mathbb{C}^m \xrightarrow{\phi} V \xrightarrow{b} V^* \xrightarrow{\iota\phi} (\mathbb{C}^m)^*,$$

where the middle map is defined by the symmetric bilinear form b associated with q . It is easy to see that, considering the domain and the range of Φ as affine spaces over \mathbb{C} , the image of ϕ is the closed subvariety $\text{Sym}_m(n+1) \subset \text{Sym}_m$ of symmetric matrices of rank $\leq n+1$. Passing to the rings of regular functions, we get a homomorphism of rings

$$\Phi : \mathbb{C}[\text{Sym}_m] \rightarrow \mathbb{C}[V^m]. \quad (2.1)$$

The map Φ is obviously T -equivariant if we make (z_1, \dots, z_m) act by multiplying the entry x_{ij} of a

symmetric matrix by $z_i z_j$. By passing to invariants, we obtain a homomorphism of graded rings

$$\Phi_T : \mathbb{C}[\text{Sym}_m]^T \rightarrow \mathbb{C}[V^m]^T. \quad (2.2)$$

The FTT can be restated by saying that the image of this homomorphism is equal to the ring $R(n; m)$.

We identify $\mathbb{C}[\text{Sym}_m]$ with the polynomial ring in entries X_{ij} of a general symmetric matrix $X = (X_{ij})$ of size $m \times m$. Note that the action of $(z_1, \dots, z_m) \in (\mathbb{C}^*)^m$ on a symmetric matrix (X_{ij}) is by multiplying each entry X_{ij} by $z_i z_j$. The graded part $\mathbb{C}[\text{Sym}_m]_d$ of $\mathbb{C}[\text{Sym}_m]$ consists of functions which under this action are multiplied by $(z_1 \cdots z_m)^d$. They are obviously contained in $\mathbb{C}[\text{Sym}_m]^T$ and define the grading of the ring $\mathbb{C}[\text{Sym}_m]^T$. The homomorphism Φ^* is a homomorphism of graded rings from $\mathbb{C}[\text{Sym}_m]^T$ to $R(n; m)$.

Let

$$\det X = \sum_{\sigma \in \mathfrak{S}_m} \epsilon(\sigma) X_{\sigma(1)1} \cdots X_{\sigma(N)N}$$

be the determinant of X . The monomials $d_\sigma = X_{\sigma(1)1} \cdots X_{\sigma(N)N}$ will be called the *determinantal* terms. Note that the number $k(m)$ of different determinantal terms is less than $m!$. It was known since the 19th century [16, p. 46] that the generating function for the numbers $k(m)$ is equal to

$$1 + \sum_{m=1}^{\infty} \frac{1}{m!} k(m) t^m = \frac{e^{\frac{1}{2}t + \frac{1}{4}t^2}}{\sqrt{1-t}}.$$

For example,
 $k(3) = 5, k(4) = 17, k(5) = 73, k(6) = 338.$

Each permutation σ decomposes into disjoint oriented cycles. Consider the directed graph on m vertices which consists of the oriented cycles in σ ; i.e., we take a directed edge $i \rightarrow \sigma(i)$ for each vertex i . Suppose there is a cycle τ in σ of length ≥ 3 . Write $\sigma = \tau \nu = \nu \tau$, and define $\sigma' = \tau^{-1} \nu$. Since our matrix is symmetric, the determinantal term $d_{\sigma'}$ corresponding to σ' has the same

value as d_σ , and furthermore σ' has the same sign as σ (so there is no canceling), and so we may drop the orientation on each cycle. We may therefore envision the determinantal terms as 2-regular undirected graphs on m vertices (where 2-cycles and loops are admitted). Thus for each 2-regular graph having k cycles of length ≥ 3 , there correspond 2^k determinantal terms.

Proposition 2.1 *The ring $\oplus_{d=0}^{\infty} \mathbb{C}[\text{Sym}_m]_{2d}^T$ is generated by the determinantal terms.*

Proof A monomial $X_{i_1 j_1} \cdots X_{i_k j_k}$ belongs to $\mathbb{C}[\text{Sym}_m]_d^T$ if and only if

$$z_{i_1} \cdots z_{i_k} z_{j_1} \cdots z_{j_k} = (z_1 \cdots z_m)^d$$

for any $z_1, \dots, z_m \in \mathbb{C}^*$. This happens if and only if each $i \in \{1, \dots, m\}$ occurs exactly d times among $i_1, \dots, i_k, j_1, \dots, j_k$. Consider the graph with set of vertices equal to $\{1, \dots, m\}$ and an edge from i to j if X_{ij} enters into the monomial. The above

property is equivalent to the graph being a regular graph of valency d . The multiplication of monomials corresponds to the operation of adding graphs (in the sense that we add the sets of the edges). It remains to use the fact that any regular graph of valency $2d$ is equal to the union of regular graphs of valency 2 (this is sometimes called a “2-factorization” or Petersen’s factorization theorem) [13, Section 9]. \square

Corollary 2.2 *A set $([v_1], \dots, [v_m])$ is semi-stable for the action of $O(V)$ on $(\mathbb{P}^n)^m$ if and only if there exists $\sigma \in \mathfrak{S}_m$ such that $\langle v_{\sigma(1)}, v_1 \rangle \cdots \langle v_{\sigma(m)}, v_m \rangle$ is not equal to zero.*

We can make it more explicit.

Proposition 2.3 *A point set $([v_1], \dots, [v_m])$ is unstable if and only if there exists $I \subseteq J \subseteq \{1, \dots, m\}$ such that $|I| + |J| = m + 1$ and $\langle v_i, v_j \rangle = 0$ for all $i \in I$ and $j \in J$.*

Proof Since $(\mathbb{C}[\mathrm{Sym}_m]^{(2)})^T$ is generated by determinantal terms, we obtain that a matrix $A = (a_{ij})$ has all determinantal terms equal to zero if and only if it represents an unstable point in $\mathbb{P}(\mathrm{Sym}_m)$ with respect to the torus action. Now the assertion becomes a simple consequence of the Hilbert–Mumford numerical criterion of stability.

It is obvious that if such subsets I and J exist then all determinantal terms vanish. So we are left with proving the existence of the subsets I and J if we have an unstable matrix.

Let $r : t \mapsto (t^{r_1}, \dots, t^{r_m})$ be a nontrivial 1-parameter subgroup of the torus T . Permuting the points, we may assume that $r_1 \leq r_2 \leq \dots \leq r_m$. We also have $r_1 + \dots + r_m = 0$. We claim that there exist i, j such that $i + j = m + 1$ and $r_i + r_j \leq 0$. If not, then each of $r_1 + r_m, r_2 + r_{m-1}, \dots, r_{\lfloor \frac{m+1}{2} \rfloor} + r_{\lceil \frac{m+1}{2} \rceil}$ is strictly positive, which contradicts $\sum_i r_i = 0$.

Since our symmetric matrix $A = (a_{ij})$ is unstable, by the Hilbert–Mumford criterion there must exist r such that $\min\{r_i + r_j : a_{ij} \neq 0\} > 0$. Permute the points if necessary so that $r_1 \leq r_2 \leq \cdots \leq r_n$. Let i_0, j_0 be such that $r_{i_0} + r_{j_0} \leq 0$ and $i_0 + j_0 = m + 1$. We may assume that $i_0 \leq j_0$ since the above condition is symmetric in i_0, j_0 . Now, since the entries of r are increasing, we have that $r_i + r_j \leq 0$ for all $i \leq i_0$ and $j \leq j_0$. Hence $a_{ij} = 0$ for all $i \leq i_0$ and $j \leq j_0$. Now let $I = \{1, \dots, i_0\}$ and $J = \{1, \dots, j_0\}$. \square

Similarly, we can prove the following:

Proposition 2.4 *A point set $([v_1], \dots, [v_m])$ is semi-stable but not stable if and only if*

$$m = \max\{|I| + |J| : I \subseteq J \subseteq \{1, \dots, m\} \text{ and } \langle v_i, v_j \rangle = 0 \text{ for all } i \in I, j \in J\}.$$

Proof Suppose that $A = (a_{ij} = \langle v_i, v_j \rangle)$. Let

$$m'(A) = \max\{|I| + |J| : I \subseteq J \subseteq \{1, \dots, m\} \text{ and } \langle v_i, v_j \rangle = 0 \text{ for all } i \in I, j \in J\}.$$

Suppose that A is semi-stable but

not stable. Since A is not unstable, we know by the prior proposition that $m'(A) \leq m$. So we are left with showing that $m'(A) = m$.

Since A is not stable, there is a 1-parameter subgroup $r : t \mapsto (t^{r_1}, \dots, t^{r_m})$ such that $r_i + r_j \geq 0$ whenever $a_{ij} \neq 0$. We shall reorder the points so that $r_1 \leq r_2 \leq \dots \leq r_n$. Recall also that $\sum_i r_i = 0$. Since some $r_i \neq 0$, we know that $r_1 < 0 < r_n$. We claim there is some i, j such that $i + j = m$ and $r_i + r_j < 0$. Otherwise, each of $r_1 + r_{m-1}, r_2 + r_{m-2}, \dots, r_{\lfloor \frac{m}{2} \rfloor} + r_{\lceil \frac{m}{2} \rceil}$ would be non-negative. This implies that $r_1 + \dots + r_{m-1} \geq 0$. But since $r_n > 0$, we have that $r_1 + \dots + r_m > 0$, a contradiction. Hence, the claim is true. Now, take $i_0 \leq j_0$ such that $i_0 + j_0 = m$ and $r_{i_0} + r_{j_0} < 0$. Now, we must have that $a_{ij} = 0$ for all $i \leq i_0$ and $j \leq j_0$. Let $I = \{1, \dots, i_0\}$ and $J = \{1, \dots, j_0\}$. Then $I \subseteq J \subseteq \{1, \dots, m\}$, $|I| + |J| = m$, and $a_{ij} = 0$ for all $i \in I, j \in J$. Thus $m'(A) = m$.

Conversely, suppose that $m'(A) = m$

. Then A is not unstable (if A were unstable, Proposition 2.3 implies that $m'(A) \geq m + 1$). Let $I \subseteq J \subseteq \{1, \dots, m\}$, such that $|I| + |J| = m$, and $a_{ij} = 0$ for all $i \in I, j \in J$. Reorder points if necessary so that $I = \{1, \dots, i_0\}$ and $J = \{1, \dots, j_0\}$. Let $r : t \mapsto (t^{r_1}, \dots, t^{r_m})$ be defined as follows. Let $r_i = -1$ for $i \leq i_0$, let $r_i = 1$ for $i > m - i_0$, and let $r_i = 0$ otherwise. The sum $\sum_i r_i$ is zero and not all r_i are zero, so this defines a 1-parameter subgroup of the torus T . Also, if $r_i + r_j < 0$ and $i \leq j$, then $i \leq i_0$ and $j \leq j_0$, which implies that $a_{ij} = 0$. Thus $r_i + r_j \geq 0$ whenever $a_{ij} \neq 0$. Hence A is not stable. \square

The second fundamental theorem (SFT) of invariant theory for the group $O(V)$ describes the kernel of the homomorphism (2.1) (see [14, p. 407; 19, Chapter 2, Section 17]).

Consider the ideal $\mathcal{I}(n, m)$ in $\mathbb{C}[V^n]^{O(V)}$ generated by the *Gram functions*

$$\gamma_{I,J} : (v_1, \dots, v_m) \mapsto \det \begin{pmatrix} \langle v_{i_1}, v_{j_1} \rangle & \dots & \langle v_{i_{n+2}}, v_{j_{n+2}} \rangle \\ \vdots & \ddots & \vdots \\ \langle v_{i_{n+2}}, v_{j_1} \rangle & \dots & \langle v_{i_{n+2}}, v_{j_{n+2}} \rangle \end{pmatrix}$$

where

$I = (1 < i_1 \cdots < i_{n+2}), J = (1 < j_1 \cdots < j_{n+2})$ are subsets of $[1, m]$. We set $\gamma_I = \gamma_{I,I}$.

The pre-image of this ideal in $\mathbb{C}[\text{Sym}^2(V^*)] \cong \mathbb{C}[\text{Sym}_m]$ is the determinant ideal $\mathcal{D}_m(n+1)$ of matrices of rank $\leq n+1$. The SFT asserts that it is the kernel of the homomorphism (2.1). Then

$$\text{Ker}(\Phi_T) = \mathcal{D}_m(n+1) \cap \mathbb{C}[\text{Sym}_m]^T$$

and it is finitely generated by polynomials of the form $\mathbf{m}\Delta_{I,J}$, where \mathbf{m} is a monomial in X_{ij} of degree $(k, k, \dots, k) - \deg(\Delta_{I,J})$ for some $k \geq 2$.

Our naive hope was that $\text{Ker}(\Phi_T)$ is generated only by polynomials of the form $\mathbf{m}\Delta_{I,J}$, for \mathbf{m} having degree $(2, 2, \dots, 2) - \deg(\Delta_{I,J})$. This is not true even if we restrict it to the open subset of semi-stable points in Sym_m with respect to the torus action. The symmetric matrix

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

has rank 4, but for any (i, j) the product $a_{ij}A_{ij}$ (where A_{ij} is the complementary minor) is equal to zero. Thus our naive relations make it appear that A has rank 3. Also, $a_{31}a_{42}a_{13}a_{24}a_{55} \neq 0$, so the matrix represents a semi-stable point. It can be shown that no counterexample exists with $m < 5$.

The following T -invariant polynomial vanishes on rank 3 matrices and is nonzero when evaluated on the matrix A above:

$$a_{13}^2 a_{14}^2 a_{25}^2 \Delta_{\{2,3,4,5\},\{2,3,4,5\}}.$$

Hence we need to consider higher-degree relations. We can at least give a bound on the degree of such relations, again appealing to Petersen's factorization theorem.

Proposition 2.5 *The ideal $\text{Ker}(\Phi_T)$ is*

generated by polynomials of the form $\mathbf{m}\Delta_{I,J}$, for \mathbf{m} having degree at most $(2(n+2), 2(n+2), \dots, 2(n+2)) - \deg(\Delta_{I,J})$.

Proof It is clear that $\text{Ker}(\Phi_T)$ is generated by relations of the form $\mathbf{m}\Delta_{I,J}$ where \mathbf{m} is a monomial of degree $(2k, 2k, \dots, 2k) - \deg(\Delta_{I,J})$, for arbitrary k . Suppose that $k > n + 2$. The monomial \mathbf{m} corresponds to the multigraph Γ' with edges ij for each X_{ij} dividing \mathbf{m} , counting multiplicity. Choose any term from $\Delta_{I,J}$; similarly this term corresponds to a multigraph Γ'' . The graph Γ'' has exactly $n + 2$ edges.

The union $\Gamma = \Gamma' \sqcup \Gamma''$ is a $2k$ -regular graph. By Petersen's factorization theorem, we know that Γ completely factors into k disjoint 2-factors. Since $k > n + 2$, at least one of these 2-factors is disjoint from Γ'' . Hence, this 2-factor must be a factor of Γ' . This means that the monomial \mathbf{m} is divisible by the T -invariant monomial \mathbf{m}_0 corresponding to the 2-factor of Γ' :

$$\mathfrak{m} = \mathfrak{m}_0 \cdot \mathfrak{m}'.$$

Hence, the relation $\mathfrak{m}\Delta_{I,J}$ is equal to $\mathfrak{m}_0 \cdot (\mathfrak{m}'\Delta_{I,J})$, where $\mathfrak{m}'\Delta_{I,J} \in \text{Ker}(\Phi_T)$ has smaller degree. \square

Conjecture 2.6 A recent conjecture of Andrew Snowden (informal communication) implies that there is a bound $d_0(n)$ such that $R(n; m)$ is generated in degree $\leq d_0(n)$ for all m . Further, after choosing a minimal set of generators (each of degree $\leq d_0(n)$), his conjecture also implies that there is a bound $d_1(n)$ such that the ideal of relations is generated in degree $\leq d_1(n)$ for all m . His conjecture applies to all GIT quotients of the form $X^m // G$, where G is linearly reductive and X is a G -polarized projective variety.

One of our goals was to prove (or perhaps disprove) his conjecture for this case of $\mathbb{P}(V)^m // \text{PO}(V)$. We were not able to do so. However, we have shown that the second Veronese subring is generated in lowest degree,

providing small evidence of the first part of his conjecture. Furthermore, Proposition 2.5 is a small step toward proving an m -independent degree bound on the generating set of the ideal (again for the second Veronese subring only).

3 A toric variety

The variety $\mathbb{S}_m = \text{Sym}_m // T = \text{Proj } \mathbb{C}[\text{Sym}_m]^T$ is a toric variety of dimension $m(m-1)/2$. We identify the character lattice of \mathbb{C}^{*m} with \mathbb{Z}^m . We have $\text{Sym}_m = \oplus \mathbb{C}X_{ij}$, where X_{ij} is an eigenvector with the character $\mathbf{e}_i + \mathbf{e}_j$. The lattice M of characters of the torus $(\mathbb{C}^*)^{m(m-1)/2}$ acting on \mathbb{S}_m is equal to the kernel of the homomorphism $\mathbb{Z}^{m(m+1)/2} \rightarrow \mathbb{Z}^m, \mathbf{e}_{ij} \mapsto \mathbf{e}_i + \mathbf{e}_j$. It is defined by the matrix A with (ij) -spot in a k th row equal:

$$a_{k,ij} = \begin{cases} 0 & \text{if } k \neq i, j; \\ 1 & \text{if } k = i \neq j, \text{ or } k = j \neq i; \\ 2 & \text{if } k = i = j. \end{cases}$$

Let

$$S = \{x \in \mathbb{Z}_{\geq 0}^{m(m+1)/2} : Ax = 2d(\mathbf{e}_1 + \dots + \mathbf{e}_m), \text{ for some } d \geq 0\}$$

be the graded semigroup. Then

$$\mathbb{C}[\mathrm{Sym}_m]^T = \mathbb{C}[S].$$

In other words, the toric variety \mathbb{S}_m is equal to the toric space \mathbb{P}_Δ , where Δ_m is the convex polytope in $\{x \in \mathbb{R}^{m(m+1)/2} : Ax = (2, \dots, 2)\}$ spanned by the vectors $v_\sigma, \sigma \in \mathfrak{S}_m$, such that a_{ij} is equal to the number of edges from i to j in the regular graph corresponding to the determinantal term d_σ . For example, if $m = 3$, $\sigma = (12)$ defines the v_σ with $a_{12} = 2, a_{33} = 1$ and $a_{ij} = 0$ otherwise. Thus the number of lattice points in the polytope Δ is equal to the number $k(m)$ of determinantal terms in a general symmetric matrix.

Proposition 3.1

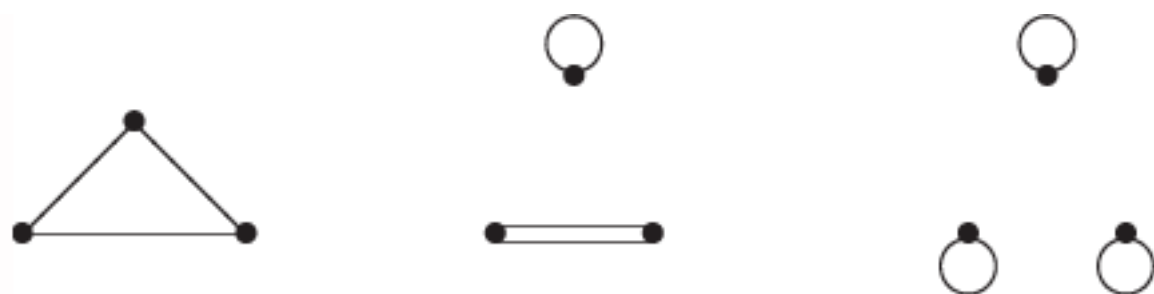
$$\#(d\Delta_m) \cap M = \#\{\text{regular graphs with valency } 2d\}.$$

Proof This follows easily from Proposition 2.1. \square

4 Examples

Example 4.1 Let $n = 2$ and $m = 3$. We are interested in the moduli space of 3-points in \mathbb{P}^2 modulo the group of projective transformations leaving invariant a nonsingular conic. The group $\mathrm{PO}(3) \cong \mathrm{PSL}_2$ is a 3-dimensional group. So, we expect a 3-dimensional variety of configurations.

We have five determinantal terms given by the following graphs:



Let t_0, t_1, t_2, t_3, t_4 be generators of the ring $R(2; 3)$ corresponding, respectively, to the triangle, to the three graphs of the second type, and the one graph of the third type. We have the cubic relation

$$t_1 t_2 t_3 - t_0^2 t_4 = 0.$$

Thus our variety is a cubic threefold in \mathbb{P}^4 . Its singular locus consists of three lines

$$t_0 = t_1 = t_2 = 0, \quad t_0 = t_1 = t_3 = 0, \quad t_0 = t_2 = t_3 = 0.$$

Let H_i be the hyperplane section of the cubic by the coordinate hyperplane $t_i = 0$. Then

- H_0 : point sets with two points conjugate with respect to the fundamental conic. H_0 is the union of three planes $\Lambda_i : t_0 = t_i = 0, i = 1, 2, 3$.
- H_4 : one of the points lies on the fundamental conic. It is the union of three planes $\Pi_i : t_4 = t_i = 0, i = 1, 2, 3$.
- H_i is the union of two planes Λ_i and $\Pi_i, i = 1, 2, 3$.
- $\Lambda_i \cap \Lambda_j$ is a singular line on \mathbb{S}_3 , the locus of point sets where one point is the intersection point of the polar lines of two other points.
- $\Pi_i \cap \Pi_j$: two points are on the fundamental conic.
- $\Pi_i \cap \Lambda_i$: two points are conjugate, the third point is on the conic.
- $\Pi_i \cap \Lambda_j, i \neq j$: one point is on the conic, and another point lies on the tangent to the conic at this point.
- $\Lambda_1 \cap \Lambda_2 \cap \Lambda_3$ is the point

representing the orbit of ordered self-conjugate triangles.

- $\Pi_1 \cap \Pi_2 \cap \Pi_3$ is the point representing the orbit of ordered sets of points on the fundamental conic.

The singular point $\Lambda_1 \cap \Lambda_2 \cap \Lambda_3 = [0, 0, 0, 0, 1]$ represents the orbit of ordered self-polar triangles. Recall that unordered self-conjugate triangles are parameterized by the homogeneous space $\text{PO}(3)/\mathfrak{S}_4$. It admits a smooth compactification isomorphic to the Fano threefold of degree 5 and index 2 [11, Theorem (2.1) and Lemma (3.3)] (see also [1, 2.1.3]).

Example 4.2 Let us look at the variety \mathbf{o}_1^3 . It is isomorphic to the subvariety of \mathbf{o}_2^3 representing collinear triples of points. The equation of the determinant of the Gram matrix of three points is

$$2t_0 - t_1 - t_2 - t_3 + t_4 = 0. \quad (4.1)$$

It is a hyperplane section of \mathbb{S}_3 isomorphic to a cubic surface S in \mathbb{P}^3 with equation

$$t_1 t_2 t_3 + 2t_0^3 - t_0^2 t_1 - t_0^2 t_2 - t_0^2 t_3 = 0. \quad (4.2)$$

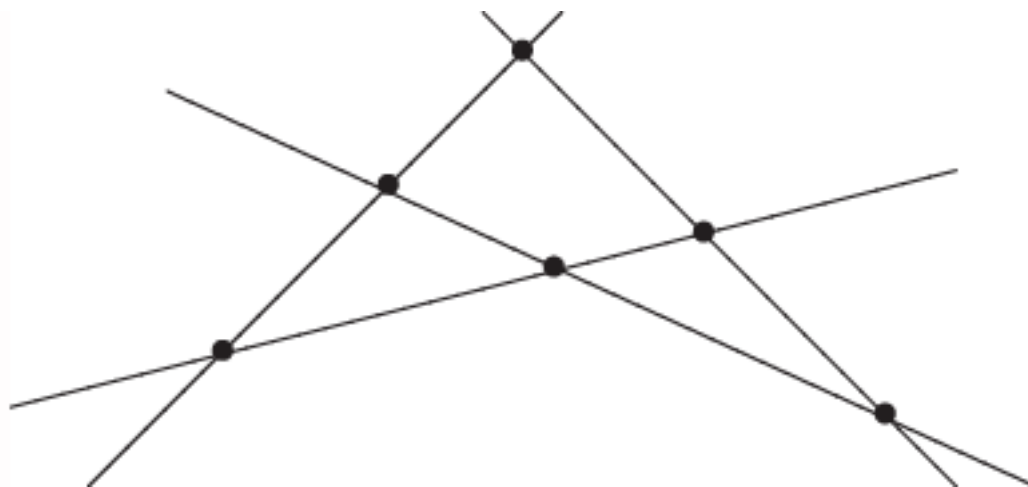
The surface is projectively isomorphic to the 4-nodal Cayley cubic surface given by the equation

$$x_0 x_1 x_2 + x_0 x_2 x_3 + x_0 x_1 x_3 + x_1 x_2 x_3 = 0.$$

Its singular points are $[t_0, t_1, t_2, t_3] = [1, 1, 1, 1], [0, 0, 0, 1], [0, 0, 1, 0], [0, 1, 0, 0]$. Since the surface is irreducible, and all collinear sets of points satisfy (4.1), we obtain that the surface represents the locus of collinear point sets. It is also isomorphic to the variety \mathbf{O}_1^3 of 3-points on \mathbb{P}^1 . The additional singular point $[1, 1, 1, 1]$ not inherited from the singular locus of \mathbf{O}_2^3 is the orbit of three collinear points $[v_1], [v_2], [v_3]$ such that the determinantal terms of the Gram matrix $G(v_1, v_2, v_3)$ are all equal. This is equivalent to all principal minors being equal to zero and the squares

of the discriminant terms $d_{(123)}$ and $d_{(321)}$ being equal. This gives two possible points $[t_0, t_1, t_2, t_3] = [\pm 1, 1, 1, 1]$. We check that the point $[-1, 1, 1, 1]$ does not satisfy (4.2). Thus the point $[1, 1, 1, 1]$ is determined by the condition that the principal minors of the Gram matrix $G(v_1, v_2, v_3)$ are equal to zero. This implies that $[v_1] = [v_2] = [v_3]$. It follows from the stability criterion that this point is not one of the two isotropic points.

It is immediate that $R(1; 2)^{(2)}$ is freely generated by two determinantal terms and hence $\mathbf{O}_1^2 \cong \mathbb{P}^1$. The three projections \mathbf{O}_1^3 to \mathbf{O}_1^2 are a regular map. If we realize \mathcal{S}_3 as the image of the anticanonical system of the blow-up of six vertices of a complete quadrilateral in the plane, then the three maps are defined by the linear system of conics through three subsets of four vertices, no three lying on one side of the quadrilateral. We can show that these are the only regular maps from \mathcal{S}_3 to \mathbb{P}^1 .



Finally, observe that we can use the conic to identify the plane with its dual plane. In this interpretation a triple of points becomes a triple of lines, the polar lines of the points with respect to the conic. Intersecting each line with the conic, we obtain three ordered pairs of points on a conic.

Note that a set of six distinct points on a nonsingular conic can be viewed as the set of Weierstrass points of a hyperelliptic curve C of genus 2. An order on this set defines a symplectic basis of the \mathbb{F}_2 -symplectic space $\text{Jac}(C)[2]$ of 2-torsion points of its Jacobian variety $\text{Jac}(C)$. The GIT-quotient of the subvariety of $(\mathbb{P}^2)^6$ of ordered points on a conic by the group SL_3 is isomorphic to the Igusa quartic in \mathbb{P}^4 (see [3, Chapter 1,

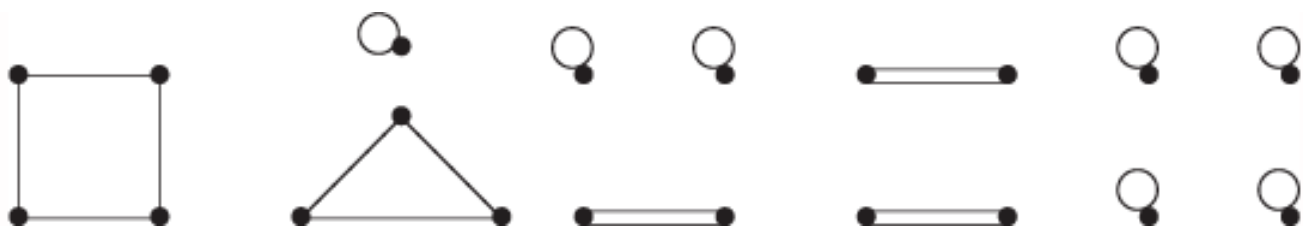
Example 3]). A partition of the set of Weierstrass points in three pairs defines a maximal isotropic subspace in $\text{Jac}(C)[2]$. An order of the three pairs chooses a basis in this space. The moduli space of principally polarized abelian surfaces A equipped with a symplectic basis in $A[2]$ is isomorphic to the quotient of the Siegel space $\mathcal{Z}_2 = \{X \in \text{Sym}_4 : \text{Im}(X) > 0\}$ by the group $\Gamma(2) = \{M \in \text{Sp}(4, \mathbb{Z}) : M \equiv I_4 \pmod{2}\}$. The moduli space of principally polarized abelian surfaces, together with a choice of a basis in a maximal isotropic subspace of 2-torsion points, is isomorphic to the quotient of \mathcal{Z}_2 by the group $\Gamma_1(2) = \left\{M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : A - I_2 \equiv C \equiv 0 \pmod{2}\right\}$. Thus, we obtain that our variety \mathbf{O}_2^3 is naturally birationally isomorphic to the quotient $\mathcal{Z}_2/\Gamma_1(2)$ and this variety is isomorphic to the quotient of $\mathcal{Z}_2/\Gamma(2)$ by the group $G = \mathfrak{S}_2 \times \mathfrak{S}_2 \times \mathfrak{S}_2$. The Satake compactification of $\mathcal{Z}_2/\Gamma(2)$ is isomorphic to the Igusa quartic. In [12], Mukai shows that the Satake compactification of $\mathcal{Z}_2/\Gamma_1(2)$ is isomorphic to the double cover of \mathbb{P}^3

branched along the union of 4-coordinate hyperplanes. It is easy to see that it is birationally isomorphic to the cubic hypersurface defining \mathbf{O}_2^3 . A remarkable result of Mukai is that the Satake compactifications of $\mathcal{Z}_2/\Gamma(2)$ and $\mathcal{Z}_2/\Gamma_1(2)$ are isomorphic.

Remark 4.3 Assume $m = n + 1$. Fix a volume form on V and use it to identify the linear spaces V^* and $\wedge^n V$. This identification is equivariant with respect to the action of $\mathbf{O}(V)$ on V and $\mathbf{O}(V^*)$, where the orthogonal group of V^* is with respect to the dual quadratic form on V^* . Passing to the configuration spaces, we obtain a natural birational involution $F : \mathbf{O}_n^{n+1} \dashrightarrow \mathbf{O}_n^{n+1}$. If G is the Gram matrix of vectors v_1, \dots, v_{n+1} , then the Gram matrix G^* of the vectors $w_i = v_1 \wedge \dots \wedge v_{i-1} \wedge v_{i+1} \wedge \dots \wedge v_{n+1} \in V^*$ is equal to the adjugate matrix of G (see [1, Lemma 10.3.2]). In the case $n = 2$, the birational involution corresponds to the involution defined by conjugate triangles (see [1, 2.1.4]). Using the modular

interpretation of \mathbf{O}_2^3 from the previous example, the involution F corresponds to the Fricke (or Richelot) involution of $\mathcal{Z}_2/\Gamma_1(2)$ (see [12, Theorem 2]).

Example 4.4 Now let us consider the variety \mathcal{O}_1^4 of 4-points in \mathbb{P}^1 modulo $\mathrm{PO}(2) \cong \mathbb{C}^* \times \mathbb{Z}/2\mathbb{Z}$. It is another threefold. First we get the 5-dimensional toric variety of symmetric matrices of size 4. The coordinate ring is generated by 17 $(3+4+6+3+1)$ determinantal terms:



Let

$x_1, x_2, x_3, y_1, y_2, y_3, y_4, z_1, \dots, z_6, u_1, u_2, u_3, v$ be the variables. We have additional equations expressing the condition that the rank of matrices is less than or equal to 2. We can show that the equations are all linear:

$$a_{ij} \det A_{ij} = 0.$$

Their number is equal to 10 but there are three linear dependencies found

by expanding the determinant expression along columns.

We may also consider the spaces $\mathbf{SO}_n^m = \mathbb{P}(V)^m // \mathbf{O}^+(V)$, where $\mathbf{O}^+(V) = \mathbf{O}(V) \cap \mathbf{SL}(V)$ is the special orthogonal group. Note that $\mathbf{PO}(V) \cong \mathbf{PO}^+(V)$ if $\dim V$ is odd. Thus we will be interested only in the case when $\dim V$ is even. In this case $\mathbf{PO}^+(V)$ is a subgroup of index 2 in $\mathbf{PO}(V)$, so the variety \mathbf{SO}_n^m is a double cover of \mathbf{O}_n^m . We have

$$\mathbf{SO}_n^m = \text{Proj } R^+(n; m),$$

where $R^+(n; m) = \bigoplus_{d=0}^{\infty} (S^d(V)^* \otimes^m)^{\mathbf{O}^+(V)}$. There are more invariants now. The additional invariants in $\mathbb{C}[V^m]^{\mathbf{O}^+(V)}$ are the Plücker brackets

$$p_{i_1, \dots, i_{n+1}} : (v_1, \dots, v_m) \mapsto v_{i_1} \wedge \dots \wedge v_{i_{n+1}},$$

where we have fixed a volume form on V . There are additional basic relations (see [19, Chapter 2, Section 17])

$$p_{i_1, \dots, i_{n+1}} p_{j_1, \dots, j_{n+1}} - \det([i_\alpha, j_\beta]_{1 \leq \alpha, \beta \leq n+1}) = 0, \quad (4.3)$$

$$\sum_{j=1}^{n+1} (-1)^j \sum p_{i_1, \dots, \hat{i}_j, \dots, i_{n+1}} [i_j, i_{n+2}] = 0. \quad (4.4)$$

The graded part $R^+(n; m)_d$ is spanned by the monomials $p_{I_1} \cdots p_{I_k} [i_1 j_1] \cdots [i_s j_s]$, where each index $j \in [1, m]$ appears exactly d times. Using the first relation in (4.3), we may assume that at most one Plücker coordinate p_I appears. Also, we see that the product of any two elements in $R^+(n; m)_d$ belongs to $R(N; m)_{2d}$.

5 Points in \mathbb{P}^1 and generalized Cayley cubics

The group $SO(2)$ is isomorphic to the 1-dimensional complex torus \mathbb{C}^* . We choose projective coordinates in \mathbb{P}^1 to identify a quadric in \mathbb{P}^1 with the set $Q = \{0, \infty\}$ so that $SO(2)$ acts by $\lambda : [t_0, t_1] \mapsto [\lambda t_0, \lambda^{-1} t_1]$. The points on Q are the fixed points of $SO(2)$. The group $O(2)$ is generated by $SO(2)$ and the transformation $[t_0, t_1] \mapsto [t_1, t_0]$.

Recall that there is a Chow

quotient $(\mathbb{P}^1)^m // \mathbb{C}^*$ defined by the quotient fan of the toric variety $(\mathbb{P}^1)^m$ (see [6]).

Lemma 5.1 *Consider $(\mathbb{P}^1)^m$ as a toric variety, the Cartesian product of the toric varieties \mathbb{P}^1 . Then the Chow quotient $(\mathbb{P}^1)^m // \mathrm{SO}(2)$ is isomorphic to the toric variety $X(A_{m-1})$ associated with the root system of type A_{m-1} defined by the fan in the dual lattice of the root lattice of type A_{m-1} formed by the Weyl chambers.*

Proof The toric variety $(\mathbb{P}^1)^m$ is defined by the complete fan Σ in the lattice \mathbb{Z}^m with 1-skeleton formed by the rays $\mathbb{R}_{\geq 0}e_i$ and $\mathbb{R}_{\leq 0}e_i$, $i = 1, \dots, m$. The action of $\mathrm{SO}(2)$ on the torus $(\mathbb{C}^*)^m \subset (\mathbb{P}^1)^m$ is defined by the surjection of the lattices $\mathbb{Z}^m \rightarrow \mathbb{Z}$ given by the map $e_i \rightarrow e_1, i = 1, \dots, m$. Thus the lattice M of characters of the torus $(\mathbb{C}^*)^{m-1}$ acting on $(\mathbb{C}^*)^m / \mathbb{C}^*$ can be identified with the sublattice of \mathbb{Z}^m spanned by the vectors $\alpha_1 = e_1 - e_2, \dots, \alpha_{m-1} = e_{m-1} - e_m$. This is the root lattice of type A_{m-1} .

The dual lattice N is the lattice $\mathbb{Z}^m / \mathbb{Z}e$, where $e = e_1 + \dots + e_m$. The quotient fan is defined as follows. For any coset $y \in N_{\mathbb{R}} = \mathbb{R}^m / \mathbb{R}e$, one considers the set

$$\mathcal{N}(y) = \{\sigma \in \Sigma : y + \mathbb{R}e \cap \sigma \neq \emptyset\}.$$

A coset $y + \mathbb{R}e \in N_{\mathbb{R}}$ is called admissible if $\mathcal{N}(y) \neq \emptyset$. Two admissible cosets $y + \mathbb{R}e$ and $y' + \mathbb{R}e$ are called equivalent if $\mathcal{N}(y) = \mathcal{N}(y')$. The closure of each equivalence class of admissible cosets is a rational polyhedral convex cone in $N_{\mathbb{R}}$ and the set of such cones defines a fan Σ' in $N_{\mathbb{R}}$ which is the quotient fan.

In our case, Σ consists of open faces σ_I of the 2^m m -dimensional cones

$$\sigma_{I,J} = \{(x_1, \dots, x_m) \in \mathbb{R}^m : (-1)^{\delta_I} x_i \geq 0, (-1)^{\delta_J} x_i \leq 0\},$$

where I, J are subsets of $[1, m]$ such that $I \cup J = [1, m]$ and δ_K is a delta-function of a subset K of $[1, m]$. The cones of maximal dimension correspond to pairs of complementary subsets I, J . The k -dimensional cones

correspond to the pairs I, J with $\#I \cap J = m - k$.

Let $y = (y_1, \dots, y_m) \in \mathbb{R}^m$ with $y_i \neq y_j, i \neq j$ and let $s \in \mathfrak{S}_m$ be a unique permutation such that $y_{s(1)} > y_{s(2)} > \dots > y_{s(m)}$. Then $y + \mathbb{R}e$ intersects $\sigma_{I,J}$ if and only if $s(I) = \{1, \dots, k\}$ for some $k \leq m$ or \emptyset and $J = [1, m] \setminus I$. Since \mathfrak{S}_m has only one orbit on the set of pairs of complementary subsets of $[1, m]$, we see that the interiors of maximal cones in the quotient fan are obtained from the image of the subset

$$\{y \in \mathbb{R}^m : y \cdot (e_i - e_{i+1}) \geq 0, i = 1, \dots, m-1\}$$

in $N_{\mathbb{R}}$. This is exactly one of the Weyl chambers in $N_{\mathbb{R}}$. All other cones in the quotient fan are translates of the faces of the closure of this chamber. This proves the assertion. \square

It is known that the toric variety $X(A_{m-1})$ is isomorphic to the blow-up of \mathbb{P}^{m-1} of the faces of the coordinate simplex (see, e.g., [2, Lemma 5.1]). Let

$$\tau_{m-1} : \mathbb{P}^{m-1} \dashrightarrow \mathbb{P}^{m-1}, [t_0, \dots, t_{m-1}] \mapsto [1/t_0, \dots, 1/t_{m-1}]$$

be the standard Cremona transformation of \mathbb{P}^{m-1} . The variety $X(A_{m-1})$ is isomorphic to a minimal resolution of indeterminacy points of the standard involution (see [1, Example 7.2.5]). Equivalently, $X(A_{m-1})$ is isomorphic to the closure of the graph of τ_{m-1} in $\mathbb{P}^{m-1} \times \mathbb{P}^{m-1}$. It is given by the 2×2 -minors of the matrix

$$\begin{pmatrix} F_0(x) & F_1(x) & \dots & F_{m-1}(x) \\ y_0 & y_1 & \dots & y_{m-1} \end{pmatrix},$$

where $F_i(x) = (x_0 \cdots x_{m-1})/x_i$. It follows from this formula that the standard involution τ_{m-1} of $X(A_{m-1})$ is induced by the switching involution ι of the factors of $\mathbb{P}^{m-1} \times \mathbb{P}^{m-1}$. The image of composition of the embedding $X(A_{m-1})$ in $\mathbb{P}^{m-1} \times \mathbb{P}^{m-1}$ and the Segre embedding $\mathbb{P}^{m-1} \times \mathbb{P}^{m-1} \hookrightarrow \mathbb{P}^{m^2-1}$ is equal to the intersection of the Segre variety with the linear subspace of dimension $m - 1$ defined by

$$t_{00} = t_{11} = \dots = t_{m-1,m-1}, \quad (5.1)$$

where we use the coordinates $t_{ij} = x_i y_j$ in \mathbb{P}^{m^2-1} . So $X(A_{m-1})$ is isomorphic to a closed smooth subvariety of $\mathbb{P}^{m(m-1)}$ of degree $\binom{2(m-1)}{m-1}$.

Consider the embedding of $(\mathbb{P}^{m-1} \times \mathbb{P}^{m-1})/\langle \iota \rangle$ in $\mathbb{P}^{\frac{1}{2}(m+2)(m-1)}$ given by the linear system of symmetric divisors of type $(1, 1)$. Its image is equal to the secant variety of the Veronese variety $v_2(\mathbb{P}^{m-1})$ isomorphic to the symmetric square $\text{Sym}^2 \mathbb{P}^{m-1}$ of \mathbb{P}^{m-1} . The image of $X(A_{m-1})/\langle \tau_{m-1} \rangle$ in $\mathbb{P}^{\frac{1}{2}(m+2)(m-1)}$ is equal to the intersection of the secant variety with a linear subspace L of codimension $m - 1$ given by (5.1). It is known that the singular locus of the secant variety is equal to the Veronese variety. The singular locus of the embedded $X(A_{m-1})/\langle \tau_{m-1} \rangle$ is equal to the intersection of L with the Veronese subvariety $v_2(\mathbb{P}^{m-1})$ and consists of 2^{m-1} points. We have $\dim L = \frac{1}{2}m(m - 1)$ and $\deg \text{Sym}^2 \mathbb{P}^{m-1} = \frac{1}{2}\binom{2(m-1)}{m-1}$. So $X(A_{m-1})/\langle \tau_{m-1} \rangle$ embeds into $\mathbb{P}^{\frac{1}{2}m(m-1)}$ as a subvariety of degree $\frac{1}{2}\binom{2(m-1)}{m-1}$ with 2^{m-1} singular points locally isomorphic to

the singular point of the cone over the Veronese variety $v_2(\mathbb{P}^{m-2})$. We call it the *generalized Cayley cubic* and denote it by Cay_{m-1} .

It follows from above that Cay_{m-1} is isomorphic to the subvariety of the projective space of symmetric $m \times m$ matrices with the conditions that the rank is equal to 2 and the diagonal elements are equal.

In the case when $m = 3$, the variety $X(A_2)$ is a del Pezzo surface of degree 6, the blow-up of three non-collinear points in \mathbb{P}^2 , and Cay_2 is isomorphic to the Cayley 4-nodal cubic surface in \mathbb{P}^3 . The variety Cay_3 is a 3-dimensional subvariety of \mathbb{P}^6 of degree 10 with eight singular points locally isomorphic to the cone over the Veronese surface.

It is known that the Chow quotient birationally dominates all the GIT-quotients [7, Theorem (0.4.3)]. So we have a \mathbb{G}_m -equivariant birational morphism

$$\Phi_m : X(A_{m-1}) \rightarrow \mathrm{SO}_1^m$$

which, after dividing by the involution τ_{m-1} , defines a \mathfrak{S}_m -equivariant birational morphism

$$\Phi_m^c : \mathrm{Cay}_{m-1} \rightarrow \mathbf{O}_1^m.$$

For example, take $m = 3$. The variety Cay_2 is the Cayley 4-nodal cubic, the morphism Φ_m is an isomorphism. Take $m = 4$. We know from Proposition 2.4 that the variety so_1^4 has six singular points corresponding to strictly semi-stable points defined by vanishing of two complementary principal matrices of the Gram matrix. They are represented by the point sets of the form (a, a, b, b) , where $a, b \in \{0, \infty\}$. The morphism Φ_4 resolves these points with the exceptional divisors equal to the exceptional divisors of $X(A_3) \rightarrow \mathbb{P}^3$ over the edges of the coordinate tetrahedron. The morphism Φ_4^c resolves three singular points of \mathbf{O}_4^1 and leaves unresolved the eight singular points coming from the fixed points of τ_3 . Altogether, the variety \mathbf{O}_4^1 has 11 singular points:

eight points locally isomorphic to the cones of the Veronese surface and three conical double points. The latter three singular points correspond to strictly semi-stable orbits.

It is known that $X(A_{m-1})$ is isomorphic to the closure of a general maximal torus orbit in PGL_m/B , where B is a Borel subgroup [9, Theorem 1]. Let P be a parabolic subgroup containing B defined by a subset S of the set of simple roots, and W_S be the subgroup of the Weil group \mathfrak{S}_m generated by simple roots in S . Let $\phi_S : \mathrm{PGL}_m/B \rightarrow \mathrm{PGL}_m/P$ be the natural projection. The image of $X(A_{m-1})$ in PGL_m/P is a toric variety $X(A_{m-1})_S$ defined by the fan whose maximal cones are \mathfrak{S}_m -translates of the cone $W_S\sigma$, where σ is a fundamental chamber ([4, Theorem 1]). The morphism $\phi_S : X(A_{m-1}) \rightarrow X(A_{m-1})_S$ is a birational morphism which is easy to describe.

We believe, but could not find a proof, that for odd m , the morphism

Φ_m and Φ_m^c are isomorphisms. If m is even, then the morphism Φ_m is equal to the morphism ϕ_S , where S is the complement of the central vertex of the Dynkin diagram of type A_{m-1} .

6 Rational functions

First we shall prove the rationality of our moduli spaces.

Theorem 6.1 *The varieties \mathbf{O}_n^m are rational varieties.*

Proof The assertion is trivial when $m = n + 1$ because in this case the variety is isomorphic to the toric variety \mathbb{S}_m . If $m < n + 1$, a general point set spans $\mathbb{P}(W)$, where W is a subspace of $\mathbb{P}(V)$ of dimension m . Since $\mathbf{O}(V)$ acts transitively on a dense orbit of the Grassmannian $G(m, V)$ (the subspaces containing an orthogonal basis), we may transform a general set to a subset of a fixed $\mathbb{P}(W)$. This shows that the varieties \mathbf{O}_n^m and \mathbf{O}_{m-1}^m are birationally isomorphic.

If $m > n + 1$, we use the projection map $\mathbf{O}_n^m \dashrightarrow \mathbf{O}_n^{m-1}$ onto the first $m - 1$ factors. It is a rational map with general fiber isomorphic to \mathbb{P}^n . Its geometric generic fiber is isomorphic to the projective space over the algebraic closure of the field K of rational functions of \mathbf{O}_n^{m-1} . In other words, the generic fiber is a Severi–Brauer variety over K (see [17, Chapter X, Section 6]). The rational map has a rational section $(x_1, \dots, x_{m-1}) \mapsto (x_1, \dots, x_{m-1}, x_{m-1})$. Thus the generic fiber is a Severi–Brauer variety with a rational point, hence isomorphic to the projective space over K (see [17, Exercise 1]). Thus the field of rational functions on \mathbf{O}_n^m is a purely transcendental extension of K , and by induction on m , we obtain that \mathbf{O}_n^m is rational. \square

We know that the ring $R(n; m)^{(2)}$ is generated by determinantal terms d_σ of the Gram matrix of m points. If we take σ to be a transposition (ab) , then the ratio $d_\sigma / d_{(12\dots m)}$ is equal to

$$R_{ab} = [ab]^2 / [aa][bb]. \quad (6.1)$$

More generally, for any cyclic permutation $\sigma = (a_1 \dots a_k)$ we can do the same to obtain the rational invariant function

$$R_{a_1 \dots a_k} = \frac{[a_1 a_2] \cdots [a_{k-1} a_k][a_{k-1} a_k][a_k a_1]}{[a_1 a_1] \cdots [a_k a_k]}. \quad (6.2)$$

Writing any permutation as a product of cycles, we see that the field of rational functions on \mathbf{O}_n^m is generated by functions $R_{a_1 \dots a_k}$. Note that

$$R_{a_1 \dots a_k}^2 = R_{a_1 a_2} \cdots R_{a_k a_1}.$$

We do not know whether a transcendental basis of the field can be chosen among the functions $R_{a_1 \dots a_k}$ or their ratios.

7 Complex spheres

An $(n - 1)$ -dimensional sphere is given by an equation in \mathbb{R}^n of the form

$$\sum_{i=1}^n (x_i - a_i)^2 = R^2.$$

After homogenizing, we get the equation in $\mathbb{P}^n(\mathbb{R})$:

$$Q : \sum_{i=1}^n (x_i - a_i x_0)^2 - R^2 x_0^2 = 0. \quad (7.1)$$

The hyperplane section $x_0 = 0$ is a sphere in $\mathbb{P}^{n-1}(\mathbb{R})$ with equation

$$Q_0 : \sum_{i=1}^n x_i^2 = 0. \quad (7.2)$$

The quadric has no real points, and for this reason it is called the imaginary sphere. Now we abandon the real space and replace \mathbb{R} with \mathbb{C} . Equation (7.1) defines a *complex sphere*. A coordinate-free definition of a complex sphere is a nonsingular quadric hypersurface Q in \mathbb{P}^n intersecting a fixed hyperplane H_0 along a fixed nonsingular quadric Q_0 in H_0 . In the real case, we additionally assume that $Q_0(\mathbb{R}) = \emptyset$. If we choose coordinates such that Q_0 is given by equation (7.2), then a quadric in $\mathbb{P}^{n+1}(\mathbb{C})$ containing the imaginary sphere has an equation

$$b \left(\sum_{i=1}^n x_i^2 \right) - 2x_0 \left(\sum_{i=0}^n a_i x_i \right) = 0.$$

If $b \neq 0$, we may assume that $b = 1$ and rewrite the equation in the form

$$\sum_{i=1}^n (x_i - a_i x_0)^2 - (2a_0 + \sum_{i=1}^n a_i^2) x_0^2 = 0,$$

so it is a complex sphere. Consider the rational map given by the linear system of quadrics in \mathbb{P}^n containing the fixed quadric Q_0 with equation (7.2). We can choose a basis formed by the quadric Q_0 and the quadrics $V(x_0 x_i), i = 0, \dots, n$. This defines a rational map $\mathbb{P}^n \dashrightarrow \mathbb{P}^{n+1}$ given by the formulas

$$[x_0, \dots, x_n] \mapsto [t_0, \dots, t_{n+1}] = \left[\sum_{i=1}^n x_i^2, x_0 x_1, \dots, x_0 x_n, x_0^2 \right].$$

The image of this map is a nonsingular quadric in \mathbb{P}^{n+1} given by the equation $Q = V(q)$, where

$$q = -t_0 t_{n+1} + \sum_{i=1}^{n+1} t_i^2 = 0. \quad (7.3)$$

We call Q the *fundamental quadric*.

The quadratic form q defines a symmetric bilinear form on V whose value on vectors $v, w \in V$ is denoted by $\langle v, w \rangle$. The pre-image of a hyperplane section $\sum A_i t_i = 0$ is a complex sphere, or its degeneration. For example, the sphere corresponding to a hyperplane which is tangent to the quadric has zero radius, and hence it is defined by a singular quadric.

The idea of replacing a quadratic equation of a sphere by a linear equation goes back to Moebius and Chasles in 1850, but was developed by Klein and Lie 20 years later. The spherical geometry, as it is understood in Klein's Erlangen program, becomes isomorphic to the orthogonal geometry. More precisely, the *inversive group* of birational transformations of \mathbb{P}^n sending spheres to spheres or their degenerations is isomorphic to the projective orthogonal group $PO(n+2)$.

Let us use the quadric \mathcal{Q} to define

a *polarity duality* between points and hyperplanes in \mathbb{P}^{n+1} . If we use the equation of \mathcal{Q} to define a symmetric bilinear form in \mathbb{C}^{n+2} , the polarity is just the orthogonality of lines and hyperplanes with respect to this form. Under the polarity, hyperplanes become points, and hence spheres in \mathbb{P}^n can be identified with points in \mathbb{P}^{n+1} .

Explicitly, a point $\alpha = [\alpha_0, \dots, \alpha_{n+1}] \in \mathbb{P}^{n+1}$ defines the sphere

$$S(\alpha) : \alpha_0 \left(\sum_{i=1}^n x_i^2 \right) - 2 \sum_{i=1}^n \alpha_i x_0 x_i + \alpha_{n+1} x_0^2 = 0. \quad (7.4)$$

By definition, its *center* is the point $c = [\alpha_0, \alpha_1, \dots, \alpha_n]$, its *radius square* R^2 is defined by the formula

$$\alpha_0^2 R^2 = \sum_{i=1}^n \alpha_i^2 - \alpha_0 \alpha_{n+1} = q_0(\alpha_0, \dots, \alpha_{n+1}). \quad (7.5)$$

Computing the discriminant D of the quadratic form in (7.4), we find

$$D = \alpha_0^{n-1}(\alpha_0 \alpha_{n+1} - \sum_{i=1}^n \alpha_i^2) = \alpha_0^{n-1} q_0. \quad (7.6)$$

This proves the following:

Proposition 7.1 *A complex sphere $S(\alpha)$ is singular if and only if its radius-square is equal to zero, or, equivalently, the point $\alpha \in \mathbb{P}^{n+1}$ lies on the fundamental quadric \mathcal{Q} . The center of a singular complex sphere is its unique singular point.*

Remark 7.2 Spheres of radius zero are points on the fundamental quadric. Thus the spaces \mathbf{O}_n^m contain as its closed subsets the moduli space of m points on the fundamental quadric modulo the automorphism group of the quadric. For example, when $n = 2$, this is the moduli space $P_1^m = (\mathbb{P}^1)^m // \mathrm{SL}(2)$ studied intensively in many papers (see, e.g., [3, 5]).

Many geometrically mutual properties of complex spheres are expressed by vanishing of some orthogonal invariant of point sets in

\mathbb{P}^{n+1} . We give here only some simple examples.

We define two complex spheres in \mathbb{P}^n to be *orthogonal* to each other if the corresponding points in \mathbb{P}^{n+1} are *conjugate* in the sense that one point lies on the polar hyperplane to another point.

Proposition 7.3 *Two real spheres in \mathbb{R}^n are orthogonal to each other (i.e., the radius-vectors at their intersection points are orthogonal) if and only if the corresponding complex spheres are orthogonal in the sense of the previous definition.*

Proof Let

$$\sum_{i=1}^n (x_i - a_i)^2 = r^2, \quad \sum_{i=1}^n (x_i - b_i)^2 = r'^2$$

be two orthogonal spheres. Let (c_1, \dots, c_n) be their intersection point. Then we have

$$0 = \sum_{i=1}^n (c_i - a_i)(c_i - b_i) = \sum_{i=1}^n c_i^2 - \sum_{i=1}^n (a_i + b_i)c_i + \sum_{i=1}^n a_i b_i,$$

$$\sum_{i=1}^n (c_i - a_i)^2 = r^2, \quad \sum_{i=1}^n (c_i - b_i)^2 = r'^2.$$

This gives the equality

$$2 \sum_{i=1}^n a_i b_i - \sum_{i=1}^n a_i^2 - \sum_{i=1}^n b_i^2 + r^2 + r'^2 = 0.$$

It gives a necessary and sufficient condition that two spheres intersect orthogonally. It is clear that the condition does not depend on the choice of intersection point. The corresponding complex spheres correspond to points

$[1, a_1, \dots, a_n, \frac{1}{2}(r^2 - \sum a_i^2)]$ and $[1, b_1, \dots, b_n, \frac{1}{2}(r'^2 - \sum b_i^2)]$. The condition that two points $[\alpha_0, \dots, \alpha_{n+1}]$ and $[\beta_0, \dots, \beta_{n+1}]$ are conjugate is

$$\alpha_0 \beta_{n+1} + \alpha_{n+1} \beta_0 + \sum_{i=1}^n \alpha_i \beta_i = 0.$$

So we see that the two conditions agree. \square

For convenience of notation, we denote $x = \mathbb{C}v \in \mathbb{P}^n$ by $[v]$. We use the symmetric form $\langle v, w \rangle$ in V defined by the fundamental quadric.

We have learnt the statements of the following two propositions from [10].

Proposition 7.4 *Two complex spheres $S([v])$ and $S([w])$ are tangent at some point if and only if*

$$\det \begin{pmatrix} \langle v, v \rangle & \langle v, w \rangle \\ \langle w, v \rangle & \langle w, w \rangle \end{pmatrix} = 0.$$

Proof Let $\lambda\phi_1 + \mu\phi_2$ be a 1-dimensional space of quadratic forms in V and $V(\lambda\phi_1 + \mu\phi_2)$ be the corresponding pencil of quadrics in \mathbb{P}^n . We assume that it contains a nonsingular quadric. Then the equation $\text{discr}(\lambda\phi_1 + \mu\phi_2) = 0$ is a homogeneous form of degree $n+1$ whose zeros define singular quadrics in the pencil. The quadrics $V(\phi_1)$ and $V(\phi_2)$ are tangent at some point p if and only if p is a singular point of some member of the pencil. It is well known that the corresponding root $[\lambda, \mu]$ of the discriminant equation is of higher multiplicity. If $V(\phi_1) = S([v])$ and $V(\phi_2) = S([w])$ are nonsingular complex

spheres, then the pencil $V(\lambda q_1 + \mu q_2)$ corresponds to the line $\overline{x, y}$ in \mathbb{P}^{n+1} spanned by the points x and y . A point $[\lambda v + \mu w]$ on the line defines a singular quadric if and only if

$$D(\lambda v + \mu w) = (\lambda \alpha_{n+1} + \mu \beta_{n+1})^{n-1} q_0(\lambda v + \mu w) = 0.$$

Our condition for quadrics $V(\phi_1)$ and $V(\phi_2)$ to be tangent to each other is that the equation $q_0(\lambda v + \mu w) = 0$ has a double root. We have

$$q_0(\lambda v + \mu w) = \lambda^2 \langle v, v \rangle + 2\lambda\mu \langle v, w \rangle + \mu^2 \langle w, w \rangle.$$

Thus the condition becomes

$$\det \begin{pmatrix} \langle v, v \rangle & \langle v, w \rangle \\ \langle w, v \rangle & \langle w, w \rangle \end{pmatrix} = 0. \quad \square$$

Proposition 7.5 $n + 1$ complex spheres $S([v_i])$ in \mathbb{P}^n have a common point if and only if

$$\det \begin{pmatrix} \langle w_1, w_1 \rangle & \dots & \langle w_1, w_{n+1} \rangle \\ \langle w_2, w_1 \rangle & \dots & \langle w_2, w_{n+1} \rangle \\ \vdots & \vdots & \vdots \\ \langle w_{n+1}, w_1 \rangle & \dots & \langle w_{n+1}, w_{n+1} \rangle \end{pmatrix} = 0,$$

where w_i are the vectors of coordinates of the polar hyperplane of $[v_i]$.

Proof We use the following known identity in the theory of determinants (see, e.g., [1, Lemma 10.3.2]). Let $A = (a_{ij}), B = (b_{ij})$ be two matrices of sizes $k \times m$ and $m \times k$ with $k \leq m$. Let

$|A_I|, |B_I|, I = (i_1, \dots, i_k), 1 \leq i_1 < \dots < i_k \leq m$, be maximal minors of A and B . Then

$$|A \cdot B| = \sum_I |A_I| |B_I|. \quad (7.7)$$

Let $H_i := \sum_{j=0}^{n+1} a_j^{(i)} t_j = 0$ be the polar hyperplanes of the complex spheres. We may assume that they are linearly independent, i.e., the vectors w_i are linearly independent in V . Otherwise the determinant is obviously equal to zero. Thus the hyperplanes intersect at one point. The spheres have a common point if and only if the intersection point of the hyperplanes H_i lies on the fundamental quadric. Let X be the matrix with rows equal to vectors $w_i = (a_0^{(i)}, \dots, a_{n+1}^{(i)})$. The intersection point has projective coordinates $[C_1, -C_2, \dots, (-1)^{n+1} C_{n+2}]$, where C_j is the maximal minor

obtained from X by deleting the j th column. Let G be the symmetric matrix defining the fundamental quadric. We take in the above formula $A = X \cdot G, B = {}^tX$. Then the product $A \cdot B$ is equal to the LHS of the formula in the assertion of the proposition. The RHS is equal to $\pm (C_1 C_{n+2} - \sum_{i=2}^{n+1} C_i^2)$. It is equal to zero if and only if the intersection point lies on the fundamental quadric. \square

We refer to [18] for many other mutual geometrical properties of circles expressed in terms of invariants of the orthogonal group $O(4)$.

8 Real points

We choose V to be a real vector space equipped with a positive definite inner product $\langle -, - \rangle$. A real point in $\mathbb{P}(V)$ is represented by a nonzero vector $v \in V$. Since $\langle v, v \rangle > 0$, we obtain from Propositions 2.3 and 2.4 that all real point sets (x_1, \dots, x_m) are stable points. Another nice feature of real

point sets is the criterion for vanishing of the Gram functions: $\det G(v_1, \dots, v_k) = 0$ if and only if v_1, \dots, v_k are linear dependent vectors in V .

It follows from the FFT and SFT that the varieties \mathbf{O}_n^m are defined over \mathbb{Q} . In particular, we may speak about the set $\mathbf{O}_N^m(\mathbb{R})$ of their real points.

Theorem 8.1 *Let V be a real inner-product space. Consider the open subset U of linear independent point sets (x_1, \dots, x_m) in $\mathbb{P}(V)(\mathbb{R})$. Then the map*

$$U \rightarrow \mathbf{O}_n^m(\mathbb{R})$$

is injective.

Proof To show the injectivity of the map, it suffices to show that

$$(g(x_1), \dots, g(x_m)) = (y_1, \dots, y_m) \tag{8.1}$$

for $g \in \text{PO}(V_{\mathbb{C}})$ implies that $(g(x_1), \dots, g(x_m)) = (y_1, \dots, y_m)$ for some

$g' \in \text{PO}(V)$. Choose an orthonormal basis in V to identify V with the Euclidean real space \mathbb{R}^{n+1} . The transformation g is represented by a complex orthogonal matrix. If (8.1) holds, we can find some representatives v_i and w_i of points x_i, y_i , respectively, and a matrix $A \in O(V_{\mathbb{C}})$ such that $A \cdot v_i = w_i, i = 1, \dots, m$. This is an inhomogeneous system of linear equations in the entries of A . Since the rank of the matrix $[v_1, \dots, v_n]$ with columns v_i is maximal, there is a unique solution for A and it is real. Thus g is represented by a transformation from $O(n+1, \mathbb{R})$. \square

Let us look at the rational invariants R_{a_1, \dots, a_k} . Let $\phi_{ij}, \pi - \phi_{ij}$, denote the angles between basis vectors of the lines $x_i = \mathbb{R}v_i$. Obviously,

$$R_{ij} = \cos^2 \phi_{ij}$$

is well defined and does not depend on the choice of the bases. Also,

$$R_{ij\dots k} = \cos \phi_{ij} \cdots \cos \phi_{ki}$$

are well defined too. Applying the previous theorem, we see that the cyclic products of the cosines determine uniquely the orbit of a linearly independent point set.

Finally, let us discuss configuration spaces of real spheres. For this we have to choose $V = \mathbb{R}^{n+2}$ to be a real space with quadratic form q_0 of signature $(1, n)$ defined in (7.3). A real sphere with nonzero radius is defined by formula (7.4), where the coefficients $(\alpha_0, \dots, \alpha_{n+1})$ belong to the set $q_0^{-1}(\mathbb{R}_{>0})$. It consists of two connected components corresponding to the choice of the sign of α_0 . Choose the component V^+ where $\alpha_0 > 0$. The image $V^+/\mathbb{R}_{>0}$ of V^+ in the projective space $\mathbb{P}(V^+)$ is, by definition, the *hyperbolic space* \mathbb{H}^{n+1} . Each point in \mathbb{H}^{n+1} can be uniquely represented by a unique vector $v = (\alpha_0, \dots, \alpha_{n+1})$ with

$$\langle v, v \rangle = \sum_{i=1}^{n+1} \alpha_i^2 - \alpha_0 \alpha_{n+1} = 1, \quad \alpha_0 > 0.$$

Each $v \in \mathbb{H}^{n+1}$ defines the orthogonal hyperplane

$$H_v = \{x \in \mathbb{H}^{n+1} : \langle x, v \rangle = 0, q(x) = 1\}.$$

The cosine of the angle between the hyperplanes H_v and H_w is defined by

$$\cos \phi = -\langle v, w \rangle.$$

If $|\langle v, w \rangle| > 1$, the hyperplanes are divergent, i.e., they do not intersect in the hyperbolic space. In this case $\cosh(|\langle v, w \rangle|)$ is equal to the distance between the two divergent hyperplanes. If $\langle v, w \rangle = 1$, the hyperplanes are parallel. By Proposition 2.1, the corresponding real spheres $S(v)$ and $S(w)$ are tangent to each other.

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- 1 Also called the inversion group or the Laguerre group. It is a subgroup of the Cremona group of \mathbb{P}^n generated by the projective affine orthogonal group $\text{PAO}(n + 1)$ and the inversion transformation $[x_0, \dots, x_n] \mapsto [x_1^2 + \dots + x_n^2, x_0x_1, \dots, x_0x_n]$.