

Integral Models and Torsors of Inseparable Forms of \mathbb{G}_a

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ABSTRACT. After recalling some basic facts about F -wound commutative unipotent algebraic groups over an imperfect field F , we study their regular integral models over Dedekind schemes of positive characteristic and compute the group of isomorphism classes of torsors of one-dimensional groups.

0. Introduction

In this paper we study integral models of unipotent commutative algebraic groups over the field K of rational function of a Dedekind scheme S over an algebraically closed field \mathbb{k} of positive characteristic. The interest in unipotent groups over imperfect fields is related to the study of pseudo-reductive algebraic groups [6] and quasi-elliptic fibrations on smooth projective surfaces [4; 7]. The general theory of such groups was started by Tits [34]. Its earlier appearance in algebraic geometry goes back to the work of Rosenlicht [28]. A one-dimensional unipotent group over K not isomorphic to the additive group $\mathbb{G}_{a,K}$ admits a regular (but not smooth) compactification and, via the theory of relative minimal models of two-dimensional schemes, can be realized as the generic fiber of a flat morphism $f : X \rightarrow S$ of regular schemes. In the special case when the genus of a regular compactification is equal to one and S is a proper smooth curve over \mathbb{k} , this leads to quasi-elliptic algebraic surfaces that play an important role in the classification of algebraic surfaces over fields of positive characteristic. Their appearance in the theory of Enriques and K3 surfaces in small characteristic is the main motivation for writing this paper.

The most notorious example of a commutative unipotent algebraic group over a field F is a vector group, a finite-dimensional vector space over F equipped with a structure of an algebraic group over F isomorphic to a direct sum of the additive group \mathbb{G}_a over F . Over any base scheme S , one can consider its form in Zariski topology, a vector group scheme $\mathbb{V}(\mathcal{E})$ over a base scheme S associated to a locally free sheaf \mathcal{E} of \mathcal{O}_S -modules. Its value on any scheme $a : T \rightarrow S$ is the group of global sections of the dual of the sheaf $a^*\mathcal{E}$. One can also consider forms of $\mathbb{V}(\mathcal{E})$ in étale or flat Grothendieck topology. It is known that there are no nontrivial forms of a vector group in étale topology; however, there are forms in flat topology if F is not a perfect field. A nontrivial form of $\mathbb{G}_{a,F}$ is an example of an F -wound unipotent group as defined by Tits [34]: a unipotent algebraic group which does

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not contain any subgroups isomorphic to $\mathbb{G}_{a,F}$. A commutative connected unipotent group U_K of dimension r killed by multiplication by the characteristic p of F can be embedded in a vector group $\mathbb{G}_{a,F}^{r+1}$ as the zero locus of a p -polynomial Φ in $r+1$ variables, a sum of monomials in one variable of degree power of p . After adjoining to the field some p^k -power roots of the coefficients, one obtains a group isomorphic to a vector group, so it becomes a purely inseparable form of a vector group. In the case when U_K is one-dimensional, one can choose Φ to be a Russell polynomial, a p -polynomial of the form $\Phi = u^{p^n} + v + a_1 v^p + \dots + a_m v^{p^m}$. The minimal extension F'/F such that the group becomes isomorphic to $\mathbb{G}_{a,F'}$ is equal to $F(a_1^{1/p^n}, \dots, a_m^{1/p^n})$. By giving appropriate weights to the variables u, v one can homogenize the affine curve given by equation $\Phi(u, v) = 0$ in \mathbb{A}_F^2 to obtain a weighted homogenous compactification \bar{U} of degree $p^{\max\{m,n\}}$ in the weighted homogenous projective plane $\mathbb{P}(1, 1, p^{\min\{m,n\}})$. If the polynomial $u^{p^n} - a_m v^{p^m}$ is irreducible in F , the compactification \bar{U} is a regular projective curve of arithmetic genus $\frac{1}{2}(p^{\min\{m,n\}} - 1)(p^{\max\{m,n\}} - 2)$. If this is not the case, one takes a normalization of this curve to arrive at a regular compactification of smaller arithmetic genus.

We will be mostly interested in the case when the field $F = K$. In this case one can study integral models \mathbf{U} of a one-dimensional K -wound unipotent group U_K over schemes S obtained by localizations or completions of a smooth algebraic curve C over \mathbb{k} . The theory of relative minimal models allows us to realize \mathbf{U} as an open subscheme in a regular projective scheme $X \rightarrow S$ whose generic fiber X_K is a regular compactification of a K -wound unipotent group U_K over K of some genus g . If $g > 0$, the group scheme \mathbf{U} is the Néron model of U_K , and it is also isomorphic to a closed group subscheme of the Néron model \mathbf{J} of the Jacobian variety $\text{Jac}(X_K)$ of X_K [5, 9.5, Theorem 4]. We say that a K -wound unipotent group U_K is of genus g if the arithmetic genus of X_K is equal to g . In the case $g = 1$, U_K coincides with its Jacobian variety and an integral model $X \rightarrow S$ is an example of a Jacobian quasi-elliptic fibration on a surface. The surfaces admitting such fibration were first studied by Bombieri and Mumford in [4].

The theory of Néron models of elliptic curves and, more generally, abelian varieties has an old history and was extensively studied in algebraic geometry and number theory. It is an important tool in the study of principal homogeneous spaces (torsors) of abelian varieties. The isomorphism classes of torsors of an abelian variety A over K form a commutative group, the *Weil–Châtelet group* $\text{WC}(A, K)$ isomorphic to the group of Galois cohomology $H^1(K, A)$ identified with the group of étale cohomology over $\text{Spec}(K)$ of the abelian sheaf represented by A . The usual local-to-global study of this group splits into three parts: the study of the group of torsors over a local field K_v obtained from K by a formal completion with respect to a discrete valuation v of K , the study of the subgroup of $\text{WC}(A, K)$ of isomorphism classes of torsors which are trivial for all valuations v of K , the *Tate–Shafarevich group* $\text{III}(A, K)$, and finally the study of the group of obstructions for reconstructing the isomorphism class of a torsor from its localizations. The group $\text{III}(A, K)$ admits a realization as the group of

étale cohomology $H_{\text{ét}}^1(C, \mathbf{A})$, where \mathbf{A} is the Néron model of A over C considered as an abelian sheaf in étale topology. The group of obstructions is the group of étale cohomology $H_{\text{ét}}^2(C, \mathbf{A})$. The existence of Néron models of commutative K -wound unipotent groups allows one to use the same approach to the study of their torsors over K . In the case of one-dimensional groups over fields K of characteristic $p = 3$, this study was first initiated by Lang [17] and has been extended to characteristic 2 in [7]. In this paper we treat the case of arbitrary genus g .

Here is the content of the paper.

In Section 1 we remind some known general facts about commutative unipotent groups over imperfect fields. Here we introduce their weighted homogenous compactifications and compute their canonical sheaves.

In Section 2, we specialize to the case of one-dimensional groups and introduce the notion of a unipotent algebraic group U of genus g . We discuss a known classification of Russell equations defining groups of genus 0 and 1. As we have already mentioned earlier, the Jacobian variety of a regular compactification of a genus $g > 0$ curve is a commutative F -wound unipotent group of dimension g . Although U is a p -torsion group, the group $\text{Jac}(X)$ is a p^s -torsion, where s is the smallest positive integer such that U_K splits over K^{1/p^s} or becomes of genus 0 [1]. An interesting question for which we do not have an answer is whether U coincides with the p -torsion part of $\text{Jac}(X)$.

In Section 3 we study minimal models of a unipotent group of genus g over a Dedekind scheme over a field of characteristic $p > 0$. We give an explicit description of such a model over a global base as a closed subscheme of a group scheme locally isomorphic to a vector group scheme over the base.

In Section 4, we study integral models of regular compactifications of rational K -wound one-dimensional unipotent groups over the field K of rational functions of a smooth projective curve C . They are isomorphic to minimal ruled surfaces $X \rightarrow C$ with an inseparable bisection as the boundary of an integral model of the group. For example, if $C \cong \mathbb{P}^1$, then $X \cong \mathbb{F}_1$ and the boundary is the pre-image of the strange conic in the plane with the pole at the center of the blow-up.

In Section 5 we begin the study of torsors of unipotent groups of genus g by considering the local case where the ground field is the field of formal power series over an algebraically closed field \mathbb{k} of characteristic p . The Weil–Châtelet group in this case is the quotient group $K/\Phi(K \oplus K)$, where Φ is the p -polynomial defining the group. In spite of this simple realization the group is very difficult to compute, and we succeeded to do this only in the case $g = 0$, where it is trivial, and in the case $g = 1$, where it was computed in the case $p = 3$ by W. Lang. An integral model $X \rightarrow \text{Spec}(R)$ of this group over the ring of integers $R = \mathbb{k}[[t]]$ is a genus one fibration with the closed fiber of multiplicity p . The important invariant of this fibration is the length $l(T)$ of the torsion part T of the R -module $H^1(X, \mathcal{O}_X)$. A torsor is tame (resp. wild) when $T = 0$ (resp. $T \neq 0$). An open problem which we have been unable to solve is to compute this invariant in terms of the Russell equation of the model and compare it with the computation of this invariant in terms of the infinitesimal neighborhoods of the multiple fiber given by Raynaud in his unpublished work (see [7, 4.2]).

In Section 6 we treat the case of global fields, we compute the Tate–Shafarevich group $H_{\text{ét}}^1(C, \mathbf{U})$ and the obstruction group $H_{\text{ét}}^2(C, \mathbf{U})$ of the Néron model \mathbf{U} of \mathbf{U}_K over a complete smooth curve C with the field of rational functions K . We also discuss here a mysterious relationship between the rank of the elementary abelian group of global sections $\mathbf{U}(C)$ and the p -rank of some hyperelliptic curves in characteristic $p > 2$.

1. Commutative Unipotent Algebraic Groups over Nonperfect Fields

Throughout this paper F denotes a field of characteristic $p > 0$. In this section we review some basic facts about commutative unipotent algebraic groups over F with emphasis on the case when F is imperfect. There are several sources to refer to, for example, [6, Appendix B], [15; 24; 34]. Recall that a linear algebraic group \mathbf{U} over a field F is called *unipotent* if it is isomorphic to a subgroup of the algebraic group of upper triangular matrices over F with 1 at the diagonal. We will be interested only in commutative unipotent algebraic groups. There is a classification of commutative connected unipotent algebraic groups over a perfect field F [8, Chapter 5, §1]. No classification is known in the case when F is imperfect.

Each commutative unipotent algebraic group over F is a p^n -torsion group. It admits a composition series with p -torsion quotients. A p -torsion group is isomorphic to a codimension one subgroup of a vector group $\mathbb{G}_{a,F}^{r+1}$ defined as the zero locus of a *p -polynomial*

$$\Phi(x_1, \dots, x_{r+1}) = \sum_{i=1}^{r+1} \left(\sum_{j=0}^{k_i} c_j^{(i)} x_i^{p^j} \right), \quad c_{k_i}^{(i)} \neq 0, \tag{1.1}$$

(see [6, Proposition B1.1.13]). The polynomial $\Phi^{\text{pp}} := \sum_{i=1}^{r+1} c_{k_i}^{(i)} x_i^{p^{k_i}}$ is called the *principal part* of Φ . We assume that at least one of the coefficients $c_0^{(1)}, \dots, c_0^{(r+1)}$ is not zero. If this condition is satisfied, then we say that Φ is a *separable p -polynomial*. This condition guarantees that the closed subscheme $V(\Phi)$ of \mathbb{A}_F^{r+1} is smooth, and hence \mathbf{U} is a smooth affine group scheme.

If, for example, $c_0^{(i)} \neq 0$, then the restriction to $\mathbf{U} = V(\Phi)$ of the projection $\mathbb{G}_{a,F}^{r+1}$ to $\mathbb{G}_{a,F}^r$ given by $(x_1, \dots, x_{r+1}) \rightarrow (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{r+1})$ is an étale isogeny of degree p^{k_i} .

PROPOSITION 1.1. *Let G be a connected subgroup of $\mathbb{G}_{a,F}^{r+1}$, $r \geq 1$, given by a separable p -polynomial Φ . There exists a radical extension F' of F such that $G_{F'} \cong \mathbb{G}_{a,F'}^r$,*

Proof. Without loss of generality, we may assume that $c_0^{(1)} = 1$. Replacing x_1 with $x_1 + c_0^{(2)}x_2 + \dots + c_0^{(r+1)}x_{r+1}$, we may write $\Phi = x_1 + \Phi_1$, where Φ_1 is a p -polynomial with no monomials of degree 1. Adding to F all p^j -roots of the coefficients $c_j^{(i)}$ of Φ_1 , we obtain a finite purely inseparable extension F' of F

such that $\Phi = x_1 + \Phi_2(x_1, \dots, x_{r+1})^{p^s}$, where Φ_2 is a separable p -polynomial over F' . Let $G \rightarrow \mathbb{G}_{a,F'}$ be a homomorphism of algebraic groups over F' given by $(x_1, \dots, x_{r+1}) \rightarrow \Phi_2(x_1, \dots, x_{r+1})$. Since Φ_2 is separable, the homomorphism is surjective. Its kernel is isomorphic to $V(\Phi_2(0, x_2, \dots, x_{r+1})) \subset \mathbb{G}_{a,F'}^r$. If $r = 1$, then it is an étale cover of \mathbb{G}_a and hence is isomorphic to $\mathbb{G}_{a,F'}$. If $r > 1$, then we apply induction on r to obtain that G is an extension of $\mathbb{G}_{a,F'}^{r-1}$ and \mathbb{G}_a (after replacing F' with some finite radical extension). Since G is a connected p -torsion algebraic group, it must be isomorphic to a vector group \mathbb{G}_a^r (see [6, Lemma B1.10]). \square

We say that G is split over an extension F' if $G_{F'} \cong \mathbb{G}_{a,F'}$. Such F' is called a *splitting extension*.

It follows that any smooth connected unipotent p -torsion algebraic group is a purely inseparable form of \mathbb{G}_a^r . There are no nontrivial separable forms [15, § 2]. Note that for $r > 1$ this fact does not follow from the usual Galois cohomology argument because $\text{Aut}(\mathbb{G}_a^r)$ is strictly larger than $\text{GL}_F(r)$.

It is well known that over a perfect field F every connected commutative unipotent group is isomorphic, as an F -scheme, to affine space \mathbb{A}_F^n . In particular, it admits a nonconstant morphism of algebraic varieties $\mathbb{A}_F^1 \rightarrow \mathbb{U}$.

DEFINITION 1.2. A smooth connected unipotent F -group \mathbb{U} over F is called *F-wound* if every F -morphism of F -schemes $\mathbb{A}_F^1 \rightarrow \mathbb{U}$ is constant.

It is known that a smooth unipotent group is F -wound if and only if it does not contain closed subgroups isomorphic to $\mathbb{G}_{a,F}$ [15, Theorem 4.3.1], [6, Corollary B.2.6].

For example, any nontrivial inseparable form of $\mathbb{G}_{a,F}$ is F -wound.

We refer for the proof of the following lemma to [6, Lemma B.1.7].

LEMMA 1.3. *The group \mathbb{U} given by the zero scheme of a p -polynomial Φ in $\mathbb{G}_{a,F}^{r+1}$ is an F -wound group if the principal part Φ^{pp} of Φ has no zeros in $F^{r+1} \setminus \{0\}$.*

REMARK 1.4. For any separable extension F'/F , the group $\mathbb{U}_{F'}$ is an F' -wound group. However, it may be still F' -wound after an inseparable extension F'/F . For example, if \mathbb{U} is given by equation $u^2 + v + av^2 + bv^4 = 0$, $a^{1/2} \notin F$, then it is F -wound and F' -wound under extension $F' = F(b^{1/4})$. Under this extension the group is isomorphic to a wound group given by equation $y^2 + v + av^2 = 0$.

Another remark is that the converse of Lemma 1.3 is wrong. For example, applying an automorphism $(x, y) \mapsto (x + x^p, y)$ to an F -wound group $y^p + x + ax^p = 0$, we obtain that the principal part has zero in $F^2 \setminus \{0\}$.

The following theorem is Corollary B.2.5 from [6].

THEOREM 1.5. *Every smooth connected unipotent p -torsion commutative algebraic group over F is a direct product of $\mathbb{U} = V \times \mathbb{U}'$, where $V \cong \mathbb{G}_{a,F}^s$ and \mathbb{U}' is a smooth connected F -wound unipotent group.*

We know that $G = \mathbb{G}_{a,F}^r$ admits the projective space \mathbb{P}_F^r as a G -equivariant regular projective compactification \overline{G} of G . It is minimal in the sense that G acts trivially on the complement $\overline{G} \setminus G$.

DEFINITION 1.6. Let G be a connected commutative algebraic group over F . A *minimal compactification* of G is a projective F -scheme \overline{G} such that G admits a G -equivariant closed embedding in \overline{G} and G acts trivially on the complement $\overline{G} \setminus G$.

The following theorem can be found in [5, 10.2, Proposition 11].

THEOREM 1.7. *Every smooth connected commutative unipotent p -torsion algebraic group U over F admits a minimal compactification $G \hookrightarrow \overline{G}$. The group is F -wound if and only if the complement $\overline{U} \setminus U$ does not have a rational point over F .*

Proof. By Theorem 1.5, we may assume that U is F -wound. We know that U is isomorphic to the zero scheme $V(\Phi)$ of a separable p -polynomial in $\mathbb{G}_{a,F}^{r+1}$. Let Φ be as in equation (1.1) with

$$k_1 \geq \dots \geq k_{r+1}$$

and

$$\overline{\Phi} = \sum_{i=1}^{r+1} \left(\sum_{j=0}^{k_i} c_i^{(j)} t_0^{p^{k_i-j}} t_i^{p^j} \right) \tag{1.2}$$

be a weighted homogenization of Φ . We view $\overline{\Phi} = 0$ as a hypersurface $X = \overline{U}$ of degree p^r in the weighted projective space

$$\mathbb{P}(q_0, q_1, \dots, q_{r+1}) = \mathbb{P}(1, 1, p^{k_1-k_2}, p^{k_1-k_3}, \dots, p^{k_1-k_{r+1}}).$$

We call such a compactification a *weighted homogeneous compactification*.

It is immediate to see that the complement $\overline{U} \setminus U$ is equal to $V(\Phi^{pp})$ in the hyperplane $V(t_0)$. The group U acts on \overline{U} by translations

$$(a_1, \dots, a_{r+1}) : (t_0, t_1, \dots, t_{r+1}) \mapsto (t_0, t_1 + a_1 t_0, \dots, t_{r+1} + a_{r+1} t_{r+1}).$$

The action is identical on the complement, and also, by Lemma 1.3, the complement has no rational points over F if and only if U is an F -wound group.

Note that $X = \overline{U}$ is not smooth. The singular locus of $X \otimes_F \overline{F}$ is the hyperplane section $V(t_0)$. □

PROPOSITION 1.8. *The weighted homogenous compactification X from above is normal over F if and only if the principal part Φ^{pp} of Φ is reduced over F .*

Proof. By Serre’s criterion of normality, we have to check that X is regular at infinity. We know that the hyperplane $t_0 = 0$ cuts out the hypersurface along the hypersurface $V(\Phi^{pp})$.

Let P be the polynomial obtained from $\overline{\Phi}$ by dehomogenization with respect to the variable t_1 , that is, dividing by $t_1^{p^{k_1}}$ and using the new variables $z_1 = t_0/t_1, z_2 = t_2/t_1^{p^{k_1-k_2}}, \dots, z_r = t_{r+1}/t_1^{p^{k_1-k_{r+1}}}$. We can write $P = \tilde{\Phi}^{\text{PP}}(1, z_2, \dots, z_r) + z_1^p B(z_1, \dots, z_r)$, where $\tilde{\Phi}^{\text{PP}}$ is the dehomogenization with respect to t_1 of the weighted homogeneous polynomial Φ^{PP} of degree p^{k_1} in variables t_1, \dots, t_{r+1} . The quotient of the ring $F[z_1, \dots, z_r]/(P)$ by the ideal (z_1) is isomorphic to $F[z_2, \dots, z_r]/(\tilde{\Phi}^{\text{PP}})$. Thus the ideal (z_1) is prime if and only if $\tilde{\Phi}^{\text{PP}}$ is irreducible over F or, equivalently, Φ^{PP} is irreducible over F . So, Φ^{PP} is irreducible over F if and only if the boundary is defined by a principal prime ideal and hence if and only if X is regular. \square

Let $X = \overline{U}$ be given as in the proof of the previous theorem. A well-known formula for the canonical sheaf of a hypersurface in a weighted projective space gives

$$\omega_X \cong \mathcal{O}_X \left(p^{k_1} - 2 - \sum_{i=2}^{r+1} p^{k_1-k_i} \right). \tag{1.3}$$

For example, if $p = k_1 = k_2 = k_3 = r = 2$, then X is a quartic surface in \mathbb{P}^3 . If we choose the equation in order X is normal and all its singular points are rational double points, then its minimal resolution of singularities is a K3 surface.

2. Inseparable Forms of \mathbb{G}_a

Specializing to the case $r = 1$, we see that any F -wound one-dimensional unipotent group (automatically commutative) is a subgroup of $\mathbb{G}_{a,F}^2$ given by an equation

$$u^{p^m} + a_{m-1}u^{p^{m-1}} + \dots + a_0u + b_nv^{p^n} + b_{n-1}v^{p^{n-1}} + \dots + b_0v = 0,$$

where $n \leq m$ and $b_n^{p^{-n}} \notin F$.

In fact, we have the following theorem of Russell [29].

THEOREM 2.1. *Each smooth connected one-dimensional unipotent group U is isomorphic to a subgroup of $\mathbb{G}_{a,F}^2$ given by a p -polynomial of the form*

$$\Phi(u, v) = u^{p^n} + v + a_1v^p \dots + a_mv^{p^m} = 0 \tag{2.1}$$

[29]. *The smallest purely inseparable extension F'/F that splits U is equal to $F' = F(a_1^{p^{-1/n}}, \dots, a_m^{p^{-1/n}})$.*

The number n is called the *height* of U .

The polynomial $\Phi(u, v)$ is obviously not unique. For example, if $n \leq m$, then we can also add to a_m any p^n th power of an element from K without changing the isomorphism class. Proposition 2.3 from loc.cit. gives a precise relationship between two polynomials defining isomorphic groups.

We refer to (2.1) as a *Russell equation*. The projections $\pi_1 : \mathbf{U} \rightarrow \mathbb{G}_{a,F}$, $(u, v) \mapsto u$, and $\pi_2 : \mathbf{U} \rightarrow \mathbb{G}_{a,F}$, $(u, v) \mapsto v$ make the Cartesian square

$$\begin{array}{ccc} \mathbf{U} & \xrightarrow{\pi_2} & \mathbb{G}_{a,F} , \\ \downarrow \pi_1 & & \downarrow \tau \\ \mathbb{G}_{a,F} & \xrightarrow{\mathbf{F}'} & \mathbb{G}_{a,F} \end{array}$$

where $\mathbf{F} : \mathbb{G}_{a,F} \rightarrow \mathbb{G}_{a,F}$ is the Frobenius homomorphism and τ is given by the p -polynomial $v + v^p + \dots + a_m v^{p^m}$.

- In this section \mathbf{U} always denotes a smooth connected F -wound unipotent one-dimensional algebraic group over F .

PROPOSITION 2.2. *Any compactification X of \mathbf{U} is unibranched.*

Proof. It is rather obvious. Let F' be a splitting field of \mathbf{U} , then $X_{F'}$ is a compactification of $\mathbb{G}_{a,F'}$. Its normalization \tilde{X} is isomorphic to $\mathbb{P}_{F'}^1$ and has one point as the complement. Thus \tilde{X} and $X_{F'}$ are homeomorphic, and since F'/F is purely inseparable, $X_{F'}$ is homeomorphic to X and has only one point at infinity. \square

PROPOSITION 2.3. *A regular compactification X of \mathbf{U} is a minimal compactification and the boundary consists of one point.*

Proof. This immediately follows from the fact that the action of \mathbf{U} on itself extends to an action of \mathbf{U} on X that acts trivially on the boundary [19, Lemma 2.14]. Let F'/F be a purely inseparable splitting extension of F . Then $X_{F'}$ is a compactification of $\mathbb{G}_{a,F'}$ and its normalization is a regular compactification of $\mathbb{G}_{a,F'}$. It must be isomorphic to $\mathbb{P}_{F'}^1$ with the boundary consisting of one point. Thus $X_{F'}$ has one unibranched singular point and $\mathbb{P}_{F'}^1$ is homeomorphic to X' . Since F'/F is purely inseparable, $X_{F'}$ is homeomorphic to X . Thus the boundary of X consists of one point. \square

Although a regular compactification of \mathbf{U} is unique, there are many nonregular compactification.

Specializing equation (1.2) to the case of a one-dimensional group, we get a compactification of \mathbf{U} to be a curve of degree $p^{\max\{m,n\}}$ in the weighted homogeneous plane $\mathbb{P}(1, 1, p^{\max\{m,n\}-\min\{m,n\}})$ given by equation

$$t_2^{p^n} + t_0^{p^m} t_1 + \dots + a_{m-1} t_0^p t_1^{p^{m-1}} + a_m t_1^{p^m} = 0 \tag{2.2}$$

if $n \leq m$ and

$$t_2^{p^n} + t_0^{p^{m-1}} t_1 + \dots + a_{m-1} t_0^{p^m-p^{m-1}} t_1^{p^{m-1}} + a_m t_1^{p^m} = 0 \tag{2.3}$$

if $m \leq n$.

It contains \mathbf{U} as the complement of the hyperplane at infinity $V(t_0)$.

Specializing formula (1.3), we obtain

$$\omega_X \cong \mathcal{O}_X(-2 - p^{\max\{m,n\}-\min\{m,n\}} + p^{\max\{m,n\}})$$

and

$$p_a(X) = \frac{1}{2}(p^{\min\{m,n\}} - 1)(p^{\max\{m,n\}} - 2) \tag{2.4}$$

(see also [1, Theorem 1.25]).

The curve has a unique unbranched nonsmooth point with coordinates $(x_0, y_0, 0) \in X(\overline{F})$, where $y_0^{p^n} + a_m x_0^m = 0$. It is regular if and only if $u^{p^n} + a_m v^{p^m}$ is reduced over F , equivalently, a_m is not a p -th-power in F . The residue field of the boundary point is $F(a_m^{1/p^n})$. It is a subextension of the splitting field of U .

Let $q : \tilde{X} \rightarrow X$ be the normalization of X over F . Recall that the annihilator ideal \mathcal{I} of $q_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X$ can be considered as an ideal sheaf on \tilde{X} . The closed subscheme \mathfrak{c} of \tilde{X} defined by this ideal is called the *conductor* of the normalization (the same name applies to the closed subscheme of X defined by the ideal \mathcal{I}). A curve X is *unbranched* if the support of \mathfrak{c} over each singular point consists of one point. It is *semi-normal* if \mathfrak{c} is reduced (e.g. has only ordinary double points as its singularities).

It is easy to see that a semi-normal unbranched curve over an algebraically closed field is regular.

Let X be a regular compactification of U and P_∞ be its boundary point. One can construct a semi-normal unbranched compactification X' by “pinching” the boundary point. This means that X is a normalization of X' and the conductor is equal to P_∞ (see [10]). The residue field of the boundary point of X' is an extension of F that is contained in the splitting extension of U . Conversely, each such extension can be realized as the residue field of the boundary point of some semi-normal compactification of U [19].

DEFINITION 2.4. U is said to be of arithmetic genus g if it admits a minimal regular compactification of arithmetic genus g . We say that U is *quasi-rational* (resp. *quasi-elliptic*) if its arithmetic genus is equal to 0 (resp. 1).

Since a regular model is unique, up to isomorphism, the arithmetic genus is a birational invariant of a one-dimensional F -wound unipotent group.

The genus formula (2.4) suggests that there must be a bound on p for a fixed g . In fact, it is known that the arithmetical genus of a regular curve can drop under an inseparable extension of the ground field only if

$$p \leq 2g + 1$$

[33] (see also a nice proof in [31, Lemma 9]). For example, genus 1 unipotent group may exist only if $p = 2, 3$.

The following Proposition was first proven by Rosenlicht [28].

PROPOSITION 2.5. *A F -wound one-dimensional unipotent group is quasi-rational if and only if it is isomorphic to a subgroup of $\mathbb{G}_{a,F}^2$ given by equation*

$$u^2 + v + av^2 = 0, \tag{2.5}$$

where a is not a square in F . In particular, a quasi-rational F -wound unipotent group exists only if $p = 2$.

Proof. Any regular curve of genus 0 over F that has a rational point in F is isomorphic to \mathbb{P}_F^1 . We know that the action of \mathbf{U} on itself extends to an action on \mathbb{P}^1 fixing the point $P_\infty = \mathbb{P}^1 \setminus \mathbf{U}$. We assume that the image of $0 \in \mathbf{U}$ has homogeneous coordinates $(t_0 : t_1)$ equal to $(0 : 1)$. Thus there is a nontrivial transformation $T : (t_0 : t_1) \mapsto (\alpha t_1 + \beta t_0 : \gamma t_1 + \delta t_0)$ that fixes P_∞ . But the residue field of any fixed point of T is of degree ≤ 2 over F . Since \mathbf{U} is F -wound, P_∞ is not a rational point (in this case the complement is \mathbb{A}_F^1). Thus \mathbf{U} is the complement of a point of degree 2. Since it is inseparable, $p = 2$. The linear system $|P_\infty|$ maps \mathbb{P}_F^1 isomorphically to a conic in \mathbb{P}_F^2 , and there exists a line defined over F that intersects the conic at one point equal to the image of P_∞ . We choose projective coordinates $(t_0 : t_1 : t_2)$ in the plane such that the equation of the line is $t_0 = 0$ and the image of P_∞ is the point $(0 : 0 : 1)$. It is easy to see that the equation of the conic can be reduced to the form $t_1^2 + t_0 t_2 + a t_2^2 = 0$. In affine coordinates $u = t_1/t_0, v = t_2/t_0$, we obtain the asserted equation of \mathbf{U} . \square

We can also find an explicit formula for the group law on a quasi-rational unipotent group. Let $(u, v) = (s/(1 + as^2), s^2/(1 + as^2))$ be a rational parameterization of \mathbf{U} . Then the group law is given in terms of the parameter s by the formula

$$s_1 \oplus s_2 = \frac{s_1 + s_2}{1 + as_1 s_2}. \tag{2.6}$$

The group is isomorphic to \mathbb{G}_a only if a is a square in F .

Let us now find equations of quasi-elliptic curves. The following proposition is due to Queen [25].

PROPOSITION 2.6. *A unipotent group is quasi-elliptic if and only if its Russell equation is one of the following:*

- (1) $p = 2$
 - (a) $u^2 + v + a_1 v^2 + a_2 v^4 = 0, a_2^{1/2} \notin F, [F(a_1^{1/2}, a_2^{1/2}) : F] = 2;$
 - (b) $u^4 + v + av^2 = 0, [F(a^{1/2}) : F] = 2;$
 - (c) $u^4 + v + av^2 + b^2 v^4 = 0, [F(a^{1/2}, b^{1/2}) : F] = 4.$

- (2) $p = 3$

$$u^3 + v + av^3 = 0, [F(a^{1/3}) : F] = 3.$$

Proof. All these equations are Russell equations, so we can use the weighted homogeneous compactification to check the genus. The genus formula gives that $p_a(X)$ is indeed equal to 1 except in case 1(c), where it is equal to 3. Since a_m is not a p th power in case $p_a(X) = 1$, the compactification is regular, hence the genus is 1. In case 1(c), a_m is a p th power, and the compactification is not regular. We use its equation from (2.2) with $n = m = 4$. In affine coordinates $x = t_0/t_2, y = t_1/t_2$, the equation is $y^4 + x^3 + a_1 x^2 + c^2 = 0$. Let $t = (y^2 + c)/x$. Then $t^2 = x + a_1$ shows that t belongs to the integral closure of the coordinate ring of the affine curve in its field of fractions. Substituting $x = t^2 + a_1$ in the equation,

we obtain equation $y^2 + t^3 + a_1t + c = 0$. This shows that the normalization of X is a regular plane cubic curve of arithmetic genus 1 with equation

$$t_0t_2^2 + t_1^3 + at_1t_0^2 + ct_0^3 = 0.$$

It contains U as an open subset $D(t_1^2 + a_1)$. The residue field of the boundary is $F(a_1^{1/2}, c^{1/2})$. If a is a square, the change $u \mapsto u + av + bv^2$ transforms the equation to $u^2 + v = 0$, so the genus is zero. If $b^{1/2} \in F(a^{1/2})$, then $b = a\alpha^2 + \beta^2$ for some $\alpha, \beta \in F$. Replacing y with $y + \beta$, we may assume that $\beta = 0$. Then setting $v' = a/(x^2 + a)$, $u' = (x + \alpha^2)/(y + \alpha x)$, we obtain an equation $v'^2 + v' + au'^4$ which is reduced to case 1(b) after multiplying it by a^{-1} and replacing v' with $a^{-1}v'$.

We refer to [25] for proving that starting from a Weierstrass equation $t_2^2t_0 + t_1^3 + at_0^2t_1^2 + bt_0^3 = 0$ of a regular genus one curve, if $p = 2$, one can reduce it to case 1(a) and a is a square, to case 1(b) if a is not square but $[F(a^{1/2}, b^{1/2}) : F] = 2$, and to case 1(c) if $[F(a^{1/2}, b^{1/2}) : F] = 4$. If $p = 3$, then $a = 0$ and, using affine coordinates $u = t_0/t_2, v = t_1/t_2$, we obtain case 2. \square

REMARK 2.7. The notion of genus is not conserved under inseparable extensions. In case 1 (c) from the previous proposition, the genus of \bar{U} is equal to one. However, after we adjoin $a_2^{1/2}$, the group becomes of genus 0.

Let X be a regular proper geometrically reduced and geometrically irreducible curve of genus $g > 0$ over a field F . The (generalized) Jacobian $\text{Jac}(X)$ is the connected component of the identity $\text{Pic}_{X/F}^0$ of the Picard scheme $\text{Pic}_{X/F}$ of X . It is a connected commutative algebraic group of dimension g . It is an abelian variety if and only if X is smooth. If X is not smooth, then $\text{Jac}(X)$ contains neither $\mathbb{G}_{a,F}$ nor $\mathbb{G}_{m,F}$ [5, 9.2, Proposition 4]. If $X_{\bar{F}}$ is a unibranch rational curve, then $\text{Jac}(X)$ is an F -wound unipotent group [1, 4.1]. If the geometric genus of $X_{\bar{F}}$ is positive, then it could be a pseudo-abelian variety [35, Example 3.1].

In particular, if $X = \bar{U}$ is a regular compactification of an F -wound unipotent group U of genus g , then $\text{Jac}(X)$ is an F -wound unipotent group of dimension g that contains U [1].

The degree homomorphism $\text{deg} : \text{Pic}_{X/F} \rightarrow (\mathbb{Z})_F$ of group schemes over F is surjective because X has a rational point. The restriction homomorphism $r : \text{Pic}_{X/F} \rightarrow \text{Pic}_{U/F}$ is also surjective because X is regular. The kernel of r is an infinite cyclic group generated by the invertible sheaf $\mathcal{O}_X(P_\infty)$. Its image under the homomorphism deg is a cyclic subgroup of $(\mathbb{Z})_F$ generated by p^k , where $p^k = \text{deg}(P_\infty)$ equals the minimal degree of the splitting extension of U . This leads to an exact sequence

$$0 \rightarrow \text{Pic}_{X/F}^0 \rightarrow \text{Pic}_{U/F} \rightarrow (\mathbb{Z}/p^k\mathbb{Z})_F \rightarrow 0$$

of algebraic groups [15, Theorem 6.10.1].

Let n' be the smallest positive integer such that $U_{F-p^{n'}}$ is isomorphic to \mathbb{G}_a or a unipotent group of genus 0. We call it the *pre-height* of U . Obviously, $n' \leq n$.

The following theorem is proved in [1, Theorem 4.4].

THEOREM 2.8. $\text{Jac}(X) = \mathbf{Pic}^0(X)$ is a $p^{n'}$ -torsion F -wound unipotent group, where n is the pre-height of \mathbf{U} .

Note that the theorem agrees with our classification of quasi-elliptic unipotent groups. Indeed, in this case the pre-height n' is always equal to 1.

Recall from [6, Definition B.3.1] that the *cckp-kernel* of a smooth connected unipotent group over F is its maximal smooth connected p -torsion central subgroup. We do not know whether the cckp-kernel of $\text{Jac}(X)$ coincides with \mathbf{U} .

3. Integral Models over a Dedekind Scheme

From now on, C will denote a Dedekind scheme over an algebraically closed field \mathbb{k} of characteristic $p > 0$ obtained by localization, henselization, or completion from a complete smooth curve C over \mathbb{k} . Let K denote the field of rational functions of C , that is, the residue field of its generic point. We say that C is local if C is the spectrum of a local ring R .

DEFINITION 3.1. Let G_K be a smooth connected algebraic group over K . An *integral C -model* of G_K is a smooth flat group scheme C -scheme \mathbf{G} with general fiber isomorphic to G_K . It is called a *minimal integral model* if, for any integral model \mathbf{G}' of G_K , any morphism $G'_K \rightarrow G_K$ extends to a morphism of group schemes $\mathbf{G}' \rightarrow \mathbf{G}$.

PROPOSITION 3.2. An integral C -model \mathbf{G} of a smooth affine commutative algebraic group G_K is a commutative affine group scheme over C .

Proof. To prove the assertions, we may assume that $C = \text{Spec}(R)$, where R is a discrete valuation ring. The affine property is proven in [2, Proposition 2.3.1]. The commutativity is proven in [36, Theorem 1.2]. □

EXAMPLE 3.3. Let \mathbf{U}_K be a quasi-elliptic unipotent group over K and $\overline{\mathbf{U}}_K$ be its minimal regular compactification of arithmetic genus one. It follows from the theory of relative minimal models of two-dimensional schemes that there exists a smooth projective flat scheme $f : J \rightarrow C$ over C such that its generic fiber J_K is isomorphic to $\overline{\mathbf{U}}_K$ and any birational morphism to any such C -scheme is an isomorphism. The closure of the cusp P_∞ of \mathbf{U}_K is a regular closed subscheme \mathcal{C} of J such that the restriction of f to \mathcal{C} is an inseparable finite morphism of degree $p = 2, 3$.

Let

$$J^\sharp = \{x \in J : f \text{ is smooth at } x\}.$$

In particular, $\mathbf{U}_K = J_K^\sharp$ and $\mathcal{C} \subset J \setminus J^\sharp$. It follows from [5, 1.5, Proposition 1] that $J^\sharp \rightarrow C$ is the Néron model of \mathbf{U}_K .

EXAMPLE 3.4. Let \mathbf{U} be an F -wound unipotent group over F given by a Russell equation (2.1). Assume that F is the field K of fractions of a discrete valuation ring R with local parameter π . Then, replacing u with $u\pi^{p^s}$ and v with $v\pi^{p^{sn}}$

with sufficiently large s , we may assume that $a_i \in R$. This gives an integral model of \mathbf{U} .

It follows from the previous example that an integral model is not unique. Assume that C is local, and let G_K be any affine algebraic group over K and $G \rightarrow \text{Spec}(R)$ be its integral model over $C = \text{Spec}(R)$. For any proper subgroup H of the closed fiber G_t of G defined by an ideal I in the coordinate ring $R[G]$ of G , we consider the *blow-up* G^H of H defined to be the spectrum of the subalgebra $R[\mathfrak{m}^{-1}I]$ of $K[G] = R[G] \otimes_R K$. It is an integral model of G and any integral model of G_K that is mapped to G with the image of the closed fiber equal to H . Any integral model of G is obtained as the composition of the blow-ups and their inverses [37].

A minimal model of an abelian variety is its Néron model.

DEFINITION 3.5. Let G_K be a smooth group scheme of finite type over K . A *Néron model* of G_K is a C -model \mathbf{G} of G_K which is smooth and of finite type satisfying the following universality property (the Néronian property):

For each smooth C -scheme Y and each K -morphism $u :: Y_K \rightarrow G_K$, there is a unique C -morphism $\tilde{u} : Y \rightarrow \mathbf{G}$ extending u .

It follows from [5, 1.2, Criterion 9] that it is enough to check the weak Néronian property: for each point $c \in C$ and an étale $\mathcal{O}_{C,c}$ -algebra R' with field of fractions K' , the canonical map $G(R') \rightarrow G_K(K')$ is surjective.

PROPOSITION 3.6. *Let S be an excellent Dedekind scheme and G_K be a smooth commutative algebraic group over the field K of rational functions of S . Assume that G_K admits a regular compactification over K and does not contain nontrivial subgroups unirational over K . Then G_K admits a Néron model over S . If S is a regular algebraic curve over a field F , then the latter condition is necessary for the existence of a Néron model of G_K .*

This is [5, 10.3, Theorem 5]. It follows that any K -wound unipotent group that admits a regular compactification over K admits a Néron model over S . It is conjectured that the condition for the existence of a regular compactification can be dropped.

COROLLARY 3.7. *Let X_K be a regular geometrically reduced and irreducible proper curve of arithmetic genus $g > 0$ over K . Then the Néron model over C of its Jacobian $\text{Jac}(X_K)$ exists.*

In fact, in an earlier influential paper [26] Raynaud proves that the Néron model of $\text{Jac}(X_K)$ represents the maximal separated quotient of the relative Picard functor $\mathcal{P}_{X/S}$ by an abelian subsheaf \mathcal{E} with fibers at closed points isomorphic to $(\mathbb{Z}^{r_t})_t$, where r_t is the number of irreducible components of X_t (see also [5, 9.5, Theorem 4]).

THEOREM 3.8. *The groups scheme $\mathbf{U} = J^\sharp$ from Example 3.3 is a Néron model of the quasi-elliptic unipotent group \mathbf{U}_K .*

Proof. It suffices to assume that $C = \text{Spec}(R)$ is local. Each connected component of an étale scheme Y_K over K is isomorphic to $\text{Spec}(L)$, where L is a separable field extension of K . Since the residue field \mathbb{k} of the unique closed point of C is algebraically closed, the normalization R' of R in L has the same residue field. Thus we can apply [5, 7.2, Theorem 1] and assume that R is henselian. Then any separable irreducible K -point is isomorphic to $\text{Spec}(K)$, that is, defines a rational K -point of \mathbf{U}_K . Since J is projective over R , it defines a section of $\text{Spec}(R) \rightarrow J$. The usual properties of the intersection theory of Cartier divisors show that it intersects the closed fiber at a smooth point. Hence it defines a section of J^\sharp . This proves that $J^\sharp \rightarrow C$ is a Néron model of \mathbf{U}_K . \square

REMARK 3.9. The group $G_K = \mathbb{G}_{a,K}$ admits a trivial integral model $\mathbb{G}_{a,C}$. However, it is not a Néron model. In fact, assume that $C = \text{Spec}(R)$ is local with a local parameter u , the zero point of $G_K = \text{Spec}(K[T])$ defined by $T = 1/u$ has schematic closure defined by the ideal $(uT - 1)$ and hence does not extend to a R -point of $\mathbb{G}_{a,R}$. By Theorem 2.1 from loc.cit., any model of $\mathbb{G}_{a,K}$ with connected fibers is isomorphic to $\mathbb{G}_{a,R}$.

The quasi-rational group \mathbf{U}_K does not admit a Néron model over a local $C = \text{Spec}(R)$. Any local integral model over R is given by $u^2 + v + av^2 = 0$, where a belongs to the maximal ideal \mathfrak{m} of R [37, Theorem 2.1]. Any K -rational point $(u, v) = (s/(1 + as^2), s^2/(1 + as^2)) \neq (0, 0)$ extends to an integral point $(s_0s_1 : s_1^2 : s_0^2 + as_1^2)$ of the projectivization $y^2 + xz + ax^2 = 0$. However, it has the point $(0 : 1 : 0)$ at infinity lying in the closed fiber. So, it does not extend to a section of the open subset $z \neq 0$ isomorphic to the integral model of \mathbf{U}_K .

DEFINITION 3.10. A flat group scheme of finite type over a base scheme S is called *unipotent* if its fibers are unipotent algebraic group schemes over the residue fields of the points.

Let us recall the relationship between the relative Picard sheaf $\mathcal{P}_{X/C}$ (in étale topology) and the Néron model \mathbf{J} of $\text{Jac}(X_K)$ (see [26] and [5, 9.5]).

Let $\mathcal{P}_{X/C}^0$ be the connected component of identity of $\mathcal{P}_{X/C}$. It is a subsheaf of $\mathcal{P}_{X/C}$ which consists of elements whose restriction to all fibers belongs to the identity component of the Picard scheme of the fiber. Although $\mathcal{P}_{X/C}$ is not representable in general, $\mathcal{P}_{X/C}^0$ is representable by a separated scheme if $X \rightarrow C$ has a section. In fact, it coincides in this case with the identity component \mathbf{J}° of the Néron scheme of $\text{Jac}(X_K)$. The Néron scheme \mathbf{J} coincides with the maximal separated quotient of the subsheaf $\mathcal{P}'_{X/C}$ equal to the kernel of the total degree homomorphism $\text{deg} : \mathcal{P}_{X/C} \rightarrow (\mathbb{Z})_C$.

THEOREM 3.11. *Let X_K be a regular completion of a unipotent group of genus $g > 0$ over K . Then the Néron model \mathbf{J} of $\text{Jac}(X)$ is a unipotent group scheme over C .*

Proof. We may assume that $C = \text{Spec}(R)$, where R is a complete discrete valuation ring. It suffices to prove that \mathbf{J}° is a connected unipotent group. Since $\text{Jac}(X_K)(K) \neq \emptyset$, the set $X(C)$ is also nonempty, and hence by above \mathbf{J}° represents $\mathcal{P}_{X/C}^0$. Since there is a section, the degree homomorphism deg is surjective. We also have an exact sequence

$$0 \rightarrow \mathcal{P}_{X/C}^0 \rightarrow \mathcal{P}'_{X/C} \rightarrow (\mathbb{Z}^{r-1})_t \rightarrow 0,$$

where r is the number of irreducible components of the closed fiber X_t and $(\mathbb{Z}^{r-1})_t$ is the skyscraper sheaf associated to the group \mathbb{Z}^{r-1} . It follows that the closed fiber \mathbf{J}_t° of \mathbf{J}° is isomorphic to a subscheme of $\mathbf{Pic}_{X_t/\mathbb{k}}$ with quotient a free abelian group.

Let $r_t : \text{Pic}(X) = \mathcal{P}_{X/C}(C) \rightarrow \text{Pic}(X_t)$ be the restriction homomorphism. It is known that r_t is surjective and its kernel is a group uniquely divisible by any integer s prime to p [3, Proposition 2.1]. It follows that the s -torsion subgroup ${}_s\text{Pic}(X_t)$ of $\text{Pic}(X_t)$ is isomorphic to the s -torsion subgroup ${}_s\text{Pic}(X)$ of $\text{Pic}(X)$. Since ${}_s\text{Pic}(X_t) = {}_s\mathbf{Pic}_{X_t/\mathbb{k}}^0$, it also coincides with ${}_s\mathbf{J}_t^\circ$ (where, as always, we identify a smooth algebraic group over an algebraically closed field \mathbb{k} with its group of \mathbb{k} -points).

Let $r_K : \text{Pic}(X) \rightarrow \text{Pic}(X_K)$ be the restriction homomorphism to the generic fiber. Its kernel is a free abelian group \mathbb{Z}^r generated by irreducible components of X_t . By Theorem 2.8, $\text{Pic}(X_K)^0 = \text{Jac}(X_K)(K)$ is a $p^{n'}$ -torsion group. Let $\text{Pic}(X)^0 = r_K^{-1}(\text{Pic}(X_K)^0)$. It follows that ${}_s\text{Pic}(X)^0 = {}_s\text{Pic}(X) = {}_s\mathbf{J}_t^\circ = \{0\}$.

The only connected commutative algebraic group over an algebraically closed field that has this property is a vector group. This proves that \mathbf{J}° is a unipotent group scheme over C . □

Let \mathcal{F} be a quasi-coherent sheaf of \mathcal{O}_S -modules over a scheme S . The affine S -scheme $\mathbb{V}(\mathcal{F}) = \text{Spec}(\text{Sym}^\bullet \mathcal{F})$ has a structure of a commutative group affine scheme over S . Its value on any S -scheme $a : T \rightarrow S$ is the group of global sections $\Gamma(T, a^* \mathcal{F}^\vee)$, where \mathcal{F}^\vee is the dual sheaf of \mathcal{O}_S -modules. It is smooth if and only if \mathcal{F} is a locally free sheaf. In this case it is called a *vector group scheme*. A homomorphism of vector group schemes $\mathbb{V}(\mathcal{E}_1) \rightarrow \mathbb{V}(\mathcal{E}_2)$ corresponds to homomorphisms of locally free sheaves $\mathcal{E}_2 \rightarrow \mathcal{E}_1$ or $\mathcal{E}_1^\vee \rightarrow \mathcal{E}_2^\vee$. The closed embeddings correspond to surjective homomorphisms.

Let G be a commutative group scheme of finite type over a base S . Recall that its identity component G° is the open group subscheme whose fibers are the identity components of the fibers of G . We set

$$\pi_0(G) = G/G^\circ$$

and call it the group of connected components. Its fiber over a closed point $s \in S$ is the group $\pi_0(G_s)$ of connected components of G_s .

We also recall the definition of the Lie algebra $\text{Lie}(G)$ [8; 21]. For any affine scheme $T = \text{Spec}(A)$, let $T_\varepsilon = \text{Spec}(A[\varepsilon]/(\varepsilon^2))$ with canonical projection $T_\varepsilon \rightarrow T$ defined by $a \mapsto a$ and a canonical section $T \rightarrow T_\varepsilon$ defined by $a + b\varepsilon \mapsto a$. For any group S -scheme G , let $\mathcal{L}ie(G)$ be the sheaf of

abelian groups defined on the category of affine S -schemes by $Lie(G)(A) = \text{Ker}(G(T_\varepsilon) \rightarrow G(T))$. If $\text{Spec}(R)$ is affine and $G = \text{Spec}(\mathcal{O}[G])$ is affine, then we have $Lie(G)(A) = \text{Der}_R(\mathcal{O}[G], A)$. The sheaf $Lie(G)$ has a natural structure of an \mathcal{O}_S -module defined by $r \cdot (a + b\varepsilon) = a + rb\varepsilon$. We denote this \mathcal{O}_S -module by $\text{Lie}(G)$. If $e : S \rightarrow G$ is the zero section of G , then

$$\text{Lie}(G)^\vee \cong e^* \Omega_{G/S}^1. \tag{3.1}$$

For example, for any locally free sheaf \mathcal{E} of \mathcal{O}_S -modules, we have $\Omega_{\mathbb{V}(\mathcal{E})/S}^1 \cong a^* \mathcal{E}$, where $a : \mathbb{V}(\mathcal{E}) \rightarrow S$ is the natural projection, and hence $e^*(\Omega_{\mathbb{V}(\mathcal{E})/S}^1) = \mathcal{E}$ so that

$$\text{Lie}(\mathbb{V}(\mathcal{E})) = \mathcal{E}^\vee.$$

The formation $G \rightarrow \text{Lie}(G)$ is a covariant functor on the category of group schemes over S . It is left exact.

THEOREM 3.12. *Let C be a local or global base, \mathbf{U}_K be a genus $g > 0$ unipotent group over K , and \mathbf{U} be its Néron model. Then \mathbf{U}° is isomorphic to a closed subgroup of a smooth group scheme \mathbf{V} locally isomorphic to a vector group scheme (not necessary a vector group scheme).*

Proof. Localizing at a closed point $t \in C$, we get a unipotent group of genus g over the field of fractions K_t of a discrete valuation ring $R = \mathcal{O}_{C,t}$. We use Russell’s equation (2.1) for \mathbf{U}_{K_t} . If π is a local parameter in R , then we replace (u, v) with $(\pi^{p^s} u, \pi^{p^{sn}} v)$ for sufficiently large integer s , and canceling by $\pi^{p^{sn}}$, we may assume that all $a_i \in R$. It follows from [37, Theorem 2.8] that all smooth integral connected models of \mathbf{U}_{K_t} over R are given by such an equation. In particular, the identity component \mathbf{U}_R° of the Néron model \mathbf{U}_R is given by such an equation. By the Néronian property, it is a minimal integral model over R . The equation allows one to embed \mathbf{U}_R° into the vector group scheme $\mathbb{G}_{a,R}^2 = \mathbb{V}(\mathcal{O}_R^{\oplus 2})$.

Now assume that C is global. Let $\mathfrak{V} = (V_i)_{i \in I}$ be a finite open affine cover of C such that the restriction \mathbf{U}_i° of \mathbf{U}_C° to each $V_i \in \mathfrak{V}$ is given by a Russell equation with coefficients in $\mathcal{O}_C(V_i)$

$$\Phi_i(u_i, v_i) = u_i^{p^n} + v_i + a_1^{(i)} v_i^p + \dots + a_m^{(i)} v_i^{p^m} = 0.$$

Since any derivation of $\mathcal{O}(V_i)[u_i, v_i]/(\Phi_i)$ is a derivation of $\mathcal{O}(V_i)[u_i, v_i]$ vanishing on v_i , the $\mathcal{O}(V_i)$ -module $\text{Lie}(\mathbf{U}_i) = \text{Lie}(\mathbf{U}_i^\circ)$ is generated by $\frac{\partial}{\partial u_i}$.

For brevity of notation, let \mathcal{L} be the invertible sheaf on C equal to $\text{Lie}(\mathbf{U}_C)$. Let (c_{ij}) be the transition functions of \mathcal{L} , so that $u_i = c_{ij}^{-1} u_j$.

The transition functions for \mathbf{U}_C from (v_i, u_i) to (v_j, u_j) must be p -polynomials with coefficients in $\mathcal{O}_C(V_i \cap V_j)$. Suppose $n \leq m$. Then

$$v_i = c_{ij}^{-p^n} v_j, \tag{3.2}$$

$$u_i = c_{ij}^{-1} u_j + c_{ij}^{-p^n} \alpha_{ij}^{(1)} v_j + \dots + c_{ij}^{-p^m} \alpha_{ij}^{(m)} v_j^{p^{m-n}}, \tag{3.3}$$

and

$$a_k^{(i)} = c_{ij}^{p^{n+k}-p^n} a_k^{(j)} - (\alpha_{ij}^{(k)})^{-p^n}, \tag{3.4}$$

where $\alpha_{ij}^{(k)} = 0$ for $k < n$. If $n > m$, then $\alpha_{ij}^{(k)}$ are all zeros.

We can view $(1, a_1^{(i)}, \dots, a_m^{(i)})$ as a section of a vector bundle \mathcal{A} of rank $m + 1$ given as an extension

$$0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{A} \rightarrow \mathcal{L}^{\otimes p^n - p^{n+1}} \oplus \dots \oplus \mathcal{L}^{\otimes p^n - p^{m+n}} \rightarrow 0 \tag{3.5}$$

with transition functions inverse to the transition functions (3.4).

We will use that the étale cohomology of a vector group scheme $\mathbb{V}(\mathcal{E})$ is isomorphic to the abelian group of the Zariski cohomology of \mathcal{E}^\vee [22, Chapter III, Proposition 3.7].

Let \mathbf{V} be the group scheme locally isomorphic to \mathbb{G}_a^2 with transition functions defined before. The group scheme \mathbf{V} fits into an extension of commutative group schemes

$$0 \rightarrow \mathbb{V}(\mathcal{L}^{\otimes -p^n}) \rightarrow \mathbf{V} \rightarrow \mathbb{V}(\mathcal{L}^{\otimes -1}) \rightarrow 0 \tag{3.6}$$

given by the projection $(u, v) \rightarrow u$. It induces an exact cohomology sequence

$$\begin{aligned} 0 \rightarrow H^0(C, \mathcal{L}^{\otimes p^n}) &\rightarrow \mathbf{V}(C) \rightarrow H^0(C, \mathcal{L}) \\ &\rightarrow H_{\text{et}}^1(C, \mathcal{L}^{\otimes p^n}) \rightarrow H_{\text{et}}^1(C, \mathbf{V}) \rightarrow H^1(C, \mathcal{L}) \rightarrow 0. \end{aligned}$$

In the case $n > m$, exact sequence (3.6) splits and we get

$$H_{\text{et}}^i(C, \mathbf{V}) \cong H^i(C, \mathcal{L}) \oplus H^i(C, \mathcal{L}^{\otimes p^n}).$$

The local embedding of \mathbf{U}_i° into \mathbb{G}_{a, V_i}^2 glues together to obtain an exact sequence

$$0 \rightarrow \mathbf{U}^\circ \rightarrow \mathbf{V} \xrightarrow{\mu} \mathbb{V}(\mathcal{L}^{\otimes -p^n}) \rightarrow 0. \tag{3.7}$$

□

REMARK 3.13. Note that \mathbf{V} is a vector group only if $m = n$. In this case the transition matrices are

$$\begin{pmatrix} c_{ij}^{p^n} & -\alpha_{ij} c_{ij}^{p^n} \\ 0 & c_{ij} \end{pmatrix}.$$

The vector group scheme \mathbf{V} is equal to $\mathbb{V}(\mathcal{E}^\vee)$, where \mathcal{E} fits in an extension

$$0 \rightarrow \mathcal{L}^{\otimes p^n} \rightarrow \mathcal{E} \rightarrow \mathcal{L} \rightarrow 0.$$

It always splits, that is, we may assume that $\alpha_{ij} = 0$, if $H^1(C, \mathcal{L}^{\otimes 1-p^n}) = 0$. For example, this happens if $(1 - p^n) \deg(\mathcal{L}) > 2g(C) - 2$. If \mathcal{E} splits, then we take $\alpha_{ij}^{(k)} = 0$, and hence \mathcal{A} also splits.

4. Global Integral Models of a Quasi-Rational Unipotent Group

Let U_K be a quasi-rational unipotent group over the field of rational functions K of a complete smooth algebraic curve C over an algebraically closed field \mathbb{k} . Its regular compactification is isomorphic to \mathbb{P}_K^1 . A regular relatively minimal model of \mathbb{P}_K^1 over C is a minimal ruled surface $f : X \rightarrow C$, where $X = \mathbb{P}(\mathcal{E})$ is a projective line bundle. This shows that we can try to construct an integral affine connected model U of U_K over C as an open subset of X whose complement is equal to the closure \mathcal{C} of the point P_∞ on the generic fiber. The intersection of \mathcal{C} with each closed fiber consists of one point taken with multiplicity 2, that is, \mathcal{C} is an inseparable bisection of X . In fact, starting from an inseparable point P_∞ of degree 2, we take any minimal ruled surface $f : X \rightarrow C$, throw away the closure of P_∞ , and the complement will be an integral affine connected model of U_K . We call such a model an *affine minimal ruled surface*.

Let us remind some general facts about minimal ruled surfaces $\mathbb{P}(\mathcal{E})$ [14, Chapter V, §2]. For any invertible sheaf \mathcal{N} , we have $\mathbb{P}(\mathcal{E}) \cong \mathbb{P}(\mathcal{E} \otimes \mathcal{N})$. One can normalize \mathcal{E} by tensoring it with an appropriate \mathcal{N} such that $H^0(C, \mathcal{E}) \neq \{0\}$ but, for any invertible sheaf \mathcal{N} of negative degree on C , one has $H^0(C, \mathcal{E} \otimes \mathcal{N}) = \{0\}$. Assume that \mathcal{E} is normalized. The inclusion $\mathcal{O}_C \rightarrow \mathcal{E}$ defined by a section is a saturated subsheaf of \mathcal{E} , that is, the quotient sheaf has no torsion, and defines an extension of locally free sheaves

$$0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{E} \rightarrow \mathcal{L}_0 \rightarrow 0. \tag{4.1}$$

Any surjection $\alpha : \mathcal{E} \rightarrow \mathcal{N}$ onto an invertible sheaf \mathcal{N} defines a section $s_\alpha : C \cong \text{Proj}(\text{Sym}^\bullet \mathcal{N}) \rightarrow X = \text{Proj}(\text{Sym}^\bullet \mathcal{E})$, and conversely any section arises in this way. The sheaf $s_\alpha^*(\text{Ker}(\alpha) \otimes \mathcal{N}^{\otimes -1})$ is isomorphic to the conormal sheaf $\mathcal{O}_E(-E)$, where $E = s_\alpha(C) \subset X$.

Let $s_0 : C \rightarrow \mathbb{P}(\mathcal{E})$ be the section corresponding to the surjection $\mathcal{E} \rightarrow \mathcal{L}_0$. We will often identify it with its image E_0 in X . It is called the *exceptional section*. We have

$$\mathcal{O}_{E_0}(E_0) \cong \mathcal{L}_0. \tag{4.2}$$

In particular,

$$e := -E_0^2 = \text{deg}(\mathcal{L}_0) = \text{deg}(\mathcal{E}).$$

LEMMA 4.1. *For any section of $s : C \rightarrow E \subset \mathbb{P}(\mathcal{E})$,*

$$E^2 \geq E_0^2.$$

If $E \neq E_0$, then the equality happens only in the case where the projective bundle is trivial, that is, $\mathcal{E} \cong \mathcal{O}_C \oplus \mathcal{O}_C$.

Proof. Let $\alpha : \mathcal{E} \rightarrow \mathcal{N}$ be a surjection corresponding to E . Then either the restriction of α to the subsheaf \mathcal{O}_C of \mathcal{E} is the zero map, or it is an injection. In the former case, α factors through an isomorphism $\mathcal{L}_0 \rightarrow \mathcal{N}$ and hence defines the same section. In the latter case, $\text{deg} \mathcal{N} \geq 0$ and the kernel of α is mapped isomorphically onto \mathcal{L}_0 . Hence exact sequence (4.1) splits and $\mathcal{E} \cong \mathcal{O}_C \oplus \mathcal{L}_0$. We also have $\text{deg} \mathcal{E} = \text{deg} \mathcal{L}_0 + \text{deg} \mathcal{N}$ gives $\text{deg} \mathcal{N} = 0$ and, since it has a section, we get

$\mathcal{N} \cong \mathcal{O}_C$. The conormal bundle of the section E defined by \mathcal{N} is isomorphic to \mathcal{L}_0 , hence $E^2 = -E_0^2$. Since $E^2 \geq E_0^2$, we get $E_0^2 = E^2 = 0$, and hence $\mathcal{L}_0 \cong \mathcal{O}_C$. This proves that $\mathcal{E} \cong \mathcal{O}_C^{\oplus 2}$. \square

PROPOSITION 4.2. *Let \mathbf{U} be a quasi-rational unipotent group and $P_\infty \in P_K^1$ be its boundary point. There exists a unique (up to isomorphism) integral affine minimal ruled model $f : \mathbb{P}(\mathcal{E}) \rightarrow C$ such that the zero section is the exceptional section, the closure of P_∞ is disjoint from the zero section, and the restriction of f to it is isomorphic to the Frobenius cover $C^{(2)} \rightarrow C$.*

Proof. Recall that an elementary transformation elm_x with center at a closed point $x \in \mathbb{P}(\mathcal{E})$ consists of blowing up the point x followed by blowing down the proper transform of the fiber X_t where $t = f(x)$. If x does not lie on the special section \mathcal{O} , the image of \mathcal{O} has self-intersection equal to $\mathcal{O}^2 + 1$. The new projective line bundle is isomorphic to $\mathbb{P}(\text{Ker}(\phi_x))$, where ϕ_x is the surjective map from \mathcal{E} to the sky-scraper sheaf $(\mathbb{k})_t$ defined by the point x . The normalized locally free sheaf \mathcal{E}' is an extension

$$0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{E}' \rightarrow \mathcal{L}_0 \rightarrow 0$$

with $\text{deg } \mathcal{E}' = \text{deg } \mathcal{E} + 1$. Suppose that \mathcal{E} is not smooth. After a sequence of elementary transformations, we arrive at a new minimal model $f' : X' \rightarrow C$ of $\bar{\mathbf{U}}$ with smooth closure \mathcal{E}' of P_∞ .

Note that, if we localize X at a closed point $t \in C$, in other words, replace C with a local base $R = \text{Spec}(\mathcal{O}_{C,t})$, we obtain an integral model of $\mathbf{U}_{Q(R)}$ isomorphic to the affine surface $u^2 + v + av^2 = 0$ and the localized X is its projective closure $y^2 + zx + ax^2 = 0$ in \mathbb{P}_R^1 . The localized bisection \mathcal{E}_c is the closed subscheme $V(z)$. It is smooth if and only if a_s generates the maximal ideal of R . So, in the global equation of \mathbf{U} discussed in the previous section, the differentials da_i of local coefficients a_i glue together to a differential

$$da \in H^0(C, \mathcal{L}^{\otimes -2} \otimes \Omega_{C/\mathbb{k}}^1)$$

of a section of $\mathcal{L}^{\otimes -2}$ which does not vanish anywhere. In particular, we obtain

$$\mathcal{L}^{\otimes 2} = \text{Lie}(\mathbf{U})^{\otimes 2} \cong \omega_{C/\mathbb{k}}. \tag{4.3}$$

We know that the zero section \mathcal{O} of \mathbf{U} is disjoint from the curve at infinity \mathcal{E} . Suppose that we have another section E with this property. Since their restrictions to the generic fiber are linearly equivalent, we have

$$\mathcal{O} \sim E_0 + f^*(D) \tag{4.4}$$

for some divisor class D on C of some degree m . In particular, $\mathcal{O}^2 = E_0^2 + 2m$ shows that $m \geq 0$. Intersecting both sides of (4.4) with \mathcal{E} , we obtain $0 = \mathcal{E} \cdot E_0 + 2m$, hence $m = 0$ and $\mathcal{E} \cdot E_0 = 0$. Thus E_0 is also disjoint from \mathcal{E} . On the other hand, E_0 and \mathcal{O} are two sections of the group scheme \mathbf{U} , hence they must differ by a translation automorphism. Changing the zero section, we may assume that

$$E_0 = \mathcal{O}, \quad \mathcal{L} = \mathcal{L}_0.$$

Applying Lemma 4.1, we obtain

$$\mathbf{U}(C) = \{0\}, \tag{4.5}$$

unless the fibration $X \cong C \times \mathbb{P}^1 \rightarrow C$ is trivial. However, in this case we have a base-point free pencil $|\mathcal{O}|$ of sections linearly equivalent to \mathcal{O} , so they all intersect the curve \mathcal{C} , a contradiction.

We can also find the divisor class of the curve \mathcal{C} . Since the restrictions of \mathcal{C} and $2\mathcal{O}$ to the generic fiber are linearly equivalent, we have

$$\mathcal{C} \sim 2\mathcal{O} + f^*(D)$$

for some divisor D on C . Since \mathcal{C} and \mathcal{O} are disjoint, restricting to \mathcal{O} , we obtain

$$\mathcal{O}_C(D) \cong \mathcal{L}^{\otimes -2} \cong \omega_C^{\otimes -1}.$$

Thus

$$\mathcal{C} \sim 2\mathcal{O} - f^*(K_C). \tag{4.6}$$

Comparing it with the formula for the canonical class [14, Chapter V, §2, Lemma 2.10]

$$\omega_X \cong \mathcal{O}_X(-2\mathcal{O}) \otimes f^*(\omega_C \otimes \mathcal{L}), \tag{4.7}$$

we find

$$\mathcal{O}_X(\mathcal{C}) \cong \omega_X^{-1} \otimes f^*(\mathcal{L}). \tag{4.8}$$

It remains to prove the uniqueness statement. Recall that the map \mathcal{C} is isomorphic to the Frobenius map. We have an exact sequence

$$0 \rightarrow \mathcal{O}_C \rightarrow f_*\mathcal{O}_{\mathcal{C}} \rightarrow \mathcal{N} \rightarrow 0$$

for some invertible sheaf \mathcal{N} . By applying the direct image functor to an exact sequence

$$0 \rightarrow \mathcal{O}_X(-\mathcal{C}) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{\mathcal{C}} \rightarrow 0,$$

using (4.8) and the relative duality $f_*\omega_{X/C} \cong (R^1 f_*\mathcal{O}_X)^\vee = 0$ and $R^1 f_*\omega_{X/C} \cong (f_*\mathcal{O}_X)^\vee = \mathcal{O}_X$, we obtain that

$$\mathcal{N} \cong \omega_{C/\mathbb{k}} \otimes \mathcal{L}^{\otimes -1} \cong \mathcal{L}^{\otimes -1}.$$

This shows that the isomorphism class of the invertible sheaf \mathcal{L} is uniquely determined by C . In fact it is isomorphic to the sheaf $\mathcal{B}_{C/\mathbb{k}}^1$ equal to the image of $d : \mathbf{F}_*\mathcal{O}_C \rightarrow \mathbf{F}_*\omega_{C/\mathbb{k}}$. Thus \mathcal{L} and hence $\mathbb{P}(\mathcal{E})$ is uniquely determined by the base curve C . □

EXAMPLE 4.3. Assume $C = \mathbb{P}^1$. A minimal ruled surface over \mathbb{P}^1 is isomorphic to one of the surfaces $\mathbf{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n)), n \geq 0$. We have $\text{Lie}(\mathbf{U}) = \mathcal{O}_{\mathbb{P}^1}(-n)$ and $\text{deg Lie}(\mathbf{U}) = -n$. It follows from (4.6) that $\mathcal{C} \in |2n\mathfrak{f} + 2\mathfrak{e}|$, where \mathfrak{f} is the divisor class of a fiber and \mathfrak{e} is the divisor class of the special section \mathcal{O} with self-intersection $-n$. The canonical class of \mathbf{F}_n is equal to $-(n+2)\mathfrak{f} - 2\mathfrak{e}$. The adjunction formula shows that $\text{deg } \omega_{\mathcal{C}} = 2n - 4$ that agrees with the formula for the canonical class of \mathcal{C} from above. Since \mathcal{C} is smooth, it is of genus $g(C) = 0$. The cover $\mathcal{C} \rightarrow C$ is just the Frobenius morphism. The canonical class formula shows that $n = 1$ and $X = \mathbf{F}_1$.

The surface \mathbf{F}_1 is obtained by blowing up one point p_0 in the projective plane. The projection $f : \mathbb{P}(\mathcal{E}) \rightarrow C$ is given by the pencil of lines through the point p_0 . Choose projective coordinates $(x_0 : x_1 : x_2)$ such that $p_0 = (1 : 0 : 0)$ and consider a conic Q with equation $x_1x_2 + x_0^2 = 0$. Then any line from the pencil is tangent to the conic, hence the pre-image of Q in \mathbf{F}_1 defines an inseparable bisection and its complement is the Néron model of a quasi-rational curve U_K . Let t be the parameter of the pencil of lines so that $x_2 + tx_1 = 0$. Then the generic fiber of the pencil is isomorphic to $\text{Proj}(\mathbb{k}(t)[t_0, t_1])$ and the complement of its intersection with the conic is equal to the open subset $D_+(tx_1^2 + x_0^2)$. It is equal to the affine spectrum of the homogeneous localization of the graded ring $\mathbb{k}(t)[t_0, t_1]$ with respect to $f = tx_1^2 + x_0^2$. The ring is generated by $u = x_0x_1/f, v = x_1^2/f, w = x_0^2/f$ with relations $tv + w = 1$ and $vw + u^2 = 0$. This gives the equation $u^2 + v + tv^2 = 0$ of U_K that agrees with a general equation (2.5) of a quasi-rational unipotent group.

EXAMPLE 4.4. If $g(C) = 1$, then $\mathcal{L} \cong \mathcal{O}_C$ or $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_C$. In the former case, taking cohomology in an exact sequence

$$0 \rightarrow \mathcal{O}_C \rightarrow \mathbf{F}_* \mathcal{O}_C \rightarrow \mathcal{L}^{\otimes -1} \rightarrow 0,$$

we obtain that $\mathbf{F} : H^1(C, \mathcal{O}_C) \rightarrow H^1(C, \mathcal{O}_C)$ is the zero map, hence C is super-singular. The ruled surface is a well-known elliptic ruled surface defined by the nonsplit exact sequence

$$0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{E} \rightarrow \mathcal{O}_C \rightarrow 0$$

with the extension class $1 \in H^1(C, \mathcal{O}_C) \cong \mathbb{k}$. In the latter case, we have

$$\mathcal{E} \cong \mathcal{O}_C \oplus \mathcal{L}.$$

and C is an ordinary elliptic curve. If $g(C) > 1$, $\text{deg } \mathcal{L} = g - 1 \geq 0$ and tensoring by $\mathcal{L}^{\otimes -1}$, we see that the exact sequence cannot split (otherwise \mathcal{E} is not normalized).

5. Torsors of Unipotent Groups of Genus g : Local Case

Recall that isomorphism classes of separable *torsors* (=principal homogeneous spaces) of any algebraic group G over a field F form a group isomorphic to the pro-finite Galois cohomology group $H^1(\text{Gal}(F^{\text{sep}}/F), G(F^{\text{sep}}))$, where F^{sep} is the separable closure of F . If we view G as an abelian sheaf that it represents in étale topology, then

$$H^1(\text{Gal}(F^{\text{sep}}/F), G(F^{\text{sep}})) = H_{\text{ét}}^1(F, G).$$

In this way we can consider torsors of any group scheme over K where we replace the étale topology with the fpqc-topology (flat topology for short). If G is a smooth group scheme, then the flat cohomology coincides with étale cohomology [22, Chapter III, Theorem 3.9].

We have already mentioned in Section 2 that the unipotent group $\mathbb{G}_{a,F}^r$ does not have nontrivial torsors. However, an F -wound unipotent group may have.

Let U_K be given by a separable p -polynomial $\Phi(x_1, \dots, x_{r+1})$. We have an exact sequence of abelian sheaves in étale topology

$$0 \rightarrow U_K \rightarrow \mathbb{G}_{a,F}^{r+1} \xrightarrow{\Phi} \mathbb{G}_{a,F} \rightarrow 0. \tag{5.1}$$

Passing to cohomology, we get an isomorphism of abelian groups

$$H^1(F, U_K) \cong \mathbb{G}_a(F)/\Phi(\mathbb{G}_a(F)^{r+1}) = F/\Phi(F^{\oplus r+1}).$$

Explicitly, if U_K is given by a p -polynomial Φ , an equation of a torsor is

$$\Phi(x_1, \dots, x_{r+1}) + f = 0,$$

where $f \in F$. The torsor is trivial (i.e. has a rational point in F) if and only if $f \in \Phi(F^{\oplus r+1})$.

In this section we will study a special case when U_K is a genus g unipotent group and $F = K$ is strictly local, that is, $K = \mathbb{k}((t))$ is the quotient ring of $R = \mathbb{k}[[t]]$ and \mathbb{k} is algebraically closed. We denote by $v : K \setminus \{0\} \rightarrow \mathbb{Z}$ the discrete valuation of K .

We start with the following.

PROPOSITION 5.1. *Let U be a quasi-rational unipotent group over a strictly local field K . Then its any torsor is trivial.*

Proof. Of course, this immediately follows from the fact that the field K is a C_1 -field [30]; however, we give a proof that serves as a warm-up for our study of cases $g > 0$. We choose a minimal model of U_K defined by the equation

$$u^2 + v + av^2 = 0, \quad a \in R \setminus R^2.$$

Replacing u with $t^s u + c$ and v with t^{2s} , we may assume that $a = t$.

Let $f \in K$ represent a nonzero element from $K/\Phi(K \oplus K)$. By taking $\Phi((x, 0))$, we may assume that f is not a square. By Hensel's lemma, the equation $h(v) = v + tv^2 = f$ can be solved for any $f \in R$. Thus f can be represented by a negative Laurent polynomial $f = \sum_{k=0}^n c_k t^{-2k-1}$. Since

$$h(ct^{-i}) = ct^{-i} + c^2 t^{-2i},$$

we see that any t^{-2k-1} belongs to $\text{Im}(\Phi)$. Thus there are no nontrivial torsors. \square

If U_K is a quasi-elliptic unipotent group over strictly local K of characteristic 3, then a set of unique representatives of $K/\Phi(K \oplus K)$ was found by W. Lang [17, Theorem 2.1].

PROPOSITION 5.2. *Assume $p = 3$. Let X be a nontrivial torsor of a quasi-elliptic unipotent group U_K . Then X is isomorphic to an affine curve over K given by one of the following equations:*

$$u^3 + v + t^k v^3 + t^{-k} q_n(t^{-3}),$$

where $k = 1, 2, 4, 5$ and $q_n(T)$ is a polynomial of some degree n .

The case $p = 2$ turns out to be more complicated [7, 4.8]. We have the following.

PROPOSITION 5.3. Assume $p = 2$. Let X be a nontrivial torsor of a quasi-elliptic unipotent group U_K . Then X is isomorphic to an affine curve over K given by one of the following equations:

- (1) $u^2 + v + tv^4 + t^{-1}q_n(t^{-4}) = 0,$
- (2) $u^2 + v + t^3v^4 + t^{-3}q_n(t^{-4}) = 0,$
- (3) $u^2 + v + t^5v^4 + t^{-5}q_n(t^{-4}) = 0,$
- (4) $u^2 + v + t^{2s+1}\varepsilon^2v^2 + tv^4 + t^{-1}q_n(t^{-4}) = 0,$
- (5) $u^2 + v + t^{2s+1}\varepsilon^2v^2 + t^{2k+1}v^4 + t^{-5}q_n(t^{-4}) = 0, k = 1, 2,$
- (6) $u^4 + v + (\varepsilon^4 + t\varepsilon_2^4 + t^2\varepsilon_3^4 + t^3\varepsilon_3^4)v^2 + t^{-1}q_n(t^{-4}) = 0,$
- (7) $u^4 + v + tv^2 + t^{-2}q_n(t^{-4}) = 0,$
- (8) $u^4 + v + t^2(\varepsilon^4 + t\varepsilon_2^4 + t^2\varepsilon_3^4 + t^3\varepsilon_3^4)v^2 + t^{-3}q_n(t^{-2}),$
- (9) $u^4 + v + t^3v^2 + t^{-6}q_n(t^{-4}) = 0,$

where $q_n(T)$ is a polynomial of some degree n , and ε_n are some units.

Here the representatives are determined uniquely from the equation of U_K only in cases (1), (2), (3), (7), and (9).

REMARK 5.4. Let X_t be the closed fiber of a relatively minimal model of a nontrivial torsor over R . Considered as a Cartier divisor on regular scheme X , it is equal to $p\bar{X}_t$ for some divisor \bar{X}_t contained in X_t . A fiber is called tame if $f : X \rightarrow C$ is cohomologically flat [26] or, equivalently, the normal sheaf $\mathcal{O}_{\bar{X}_t}(\bar{X}_t)$ is an element of order p in $\text{Pic}(\bar{X}_t)$. It is conjectured by W. Lang that X_t is tame if and only $n = 0$. The number n is conjecturally related to the length of the torsion subgroup of $H^1(X, \mathcal{O}_X)$.

6. Torsors of Unipotent Groups of Genus g : Global Case

In this section we assume that C is a smooth complete curve over an algebraically closed field of characteristic $p > 0$. Let $K = \mathbb{k}(C)$ be its field of rational functions. Let U_K be a genus g unipotent group over K considered as an abelian sheaf in the étale topology of K .

Let us first get rid of the case $g = 0$. Let X_K be a torsor of U_K . Since X_K and U_K become isomorphic after a separable extension of K , a regular compactification \bar{X}_K is a genus 0 curve and hence isomorphic to a conic over K . By Tsen’s theorem the global field K is a C_1 -field, hence every separable torsor is trivial [30]. This also follows from the fact that $\text{Br}(K) = \{0\}$.

From now on we assume that $g > 0$. We denote by U its Néron model. We also consider U as an abelian sheaf in the étale topology of C .

Let

$$\iota : \eta = \text{Spec}(K) \hookrightarrow C$$

be the inclusion morphism of the generic point η of C . The Leray spectral sequence for the morphism ι and the sheaf U_K gives an exact sequence

$$\begin{aligned} 0 \rightarrow H_{\text{ét}}^1(C, \iota_*U_K) \rightarrow H^1(K, U_K) \rightarrow H^0(C, R^1\iota_*U_K) \\ \rightarrow H^2(C, \iota_*U_K) \rightarrow H^2(K, U_K). \end{aligned} \tag{6.1}$$

By the definition of Néron model, the sheaf $\iota_*\mathcal{U}_K$ is represented by the Néron model \mathbf{U} of \mathcal{U}_K . The fiber $(R^1\iota_*\mathcal{U})(t)$ of $R^1\iota_*\mathcal{U}$ at a closed point $t \in C$ is isomorphic to $H^1(K_t, \mathcal{U}_{K_t})$. The homomorphism

$$\text{loc}_t : H^1(K, \mathcal{U}_K) \rightarrow H^0(C, R^1\iota_*\mathcal{U}) \rightarrow (R^1\iota_*\mathcal{U})(t) = H^1(K_t, \mathcal{U}_{K_t})$$

assigns to an isomorphism class $[X]$ of a torsor X of \mathcal{U}_K the isomorphism class of the torsor X_{K_t} of \mathcal{U}_{K_t} . Since each torsor X splits over some separable extension L of K , $\text{loc}_t([X]) = 0$ for any point t which is not ramified in L/K . Let \hat{R}_t be the formal completion of R_t and \hat{K}_t be its field of fractions. We have an isomorphism

$$H^1(K_t, \mathcal{U}_{K_t}) \cong H^1(\hat{K}_t, \mathcal{U}_{\hat{K}_t})$$

[9], so we are in a situation from the previous section. Passing to cohomology in exact sequence (5.1), we obtain $H^2(K, \mathbf{U}) = 0$ [22]. So, we get an exact sequence

$$\begin{aligned} 0 \rightarrow H_{\text{et}}^1(C, \iota_*\mathcal{U}) &\rightarrow H^1(K, \mathcal{U}_K) \\ &\rightarrow \bigoplus_{t \in C} H^1(\hat{K}_t, \mathcal{U}_{\hat{K}_t}) \rightarrow H_{\text{et}}^2(C, \iota_*\mathcal{U}) \rightarrow 0. \end{aligned} \tag{6.2}$$

The group $\text{III}(\mathbf{U}, K) := H_{\text{et}}^1(C, \mathbf{U})$ is an analog of the Tate–Shafarevich group of an abelian variety. The group $H_{\text{et}}^2(C, \mathbf{U})$ is the group of obstructions to define a global torsor in terms of local torsors.

THEOREM 6.1. *Let C be a complete curve over \mathbb{k} , \mathcal{U}_K be a genus g unipotent group over K , and X_K be its regular compactification. Then $\text{Jac}(X_K)(K)$ is a finite group killed by $p^{n'}$, where n' is the pre-height of \mathcal{U}_K (see Section 3).*

Proof. Let $f : X \rightarrow C$ be a regular relatively minimal model of X_K . Here X is a smooth projective surface over \mathbb{k} . Its Picard variety $\mathbf{Pic}_{X/\mathbb{k}}^0$ is an abelian variety. Since \mathbb{k} is algebraically closed, we may identify any variety over \mathbb{k} with its set of \mathbb{k} -points. Let $\text{Pic}^0(X) = \mathbf{Pic}_{X/\mathbb{k}}^0(\mathbb{k})$. It is a subgroup of $\mathbf{Pic}_{X/\mathbb{k}}(\mathbb{k}) = \text{Pic}(X)$ and the quotient $\text{NS}(X) = \text{Pic}(X)/\text{Pic}^0(X)$ is the Néron–Severi group of the surface X . It known to be a finitely generated abelian group. The pull-back morphism

$$f^* : \text{Jac}(C) = \mathbf{Pic}_{C/\mathbb{k}}^0 \rightarrow \mathbf{Pic}_{X/\mathbb{k}}^0$$

is a homomorphism of abelian varieties. By the Poincaré reducibility theorem (see [23], Chapter 4, §19), there exists an abelian variety A over \mathbb{k} and an isogeny

$$A \times \text{Jac}(C) \rightarrow \mathbf{Pic}_{X/\mathbb{k}}^0.$$

Since $\text{Jac}(X_K)$ is an affine algebraic group, the image of A under the restriction morphism $r_K : \mathbf{Pic}_{X/\mathbb{k}}^0 \rightarrow \mathbf{Pic}_{X_K/K}^0 = \text{Jac}(X_K)$ is equal to zero. However, the kernel of r_η is generated by $f^*\text{Jac}(C)$ and the subgroup of divisor classes of irreducible components of fibers of f . The latter group is finitely generated abelian group, and hence $A = 0$. We have $f^*\text{Jac}(X) \subset \text{Pic}^0(X)$, and hence $\text{Jac}(X_K)(K)$ is a finite generated abelian group. On the other hand, by Theorem 2.8, $\text{Jac}(X_K)$ is killed by $p^{n'}$. This proves the assertion. □

COROLLARY 6.2. *Let U_K be a unipotent group of genus $g > 0$ over the field K of rational function of an irreducible algebraic curve over an algebraically closed field. Then $U(K)$ is an elementary p -group.*

Let $f : X \rightarrow C$ be a regular relatively minimal model of X_K over C . By definition, it does not contain (-1) -curves in fibers. If $g(C) > 0$, then it also does not contain other (-1) -curves because they cannot map surjectively onto C . This X is a minimal smooth algebraic surface over \mathbb{k} . If $g = 1$, then it is a quasi-elliptic surface of Kodaira dimension ≤ 1 . If $g > 1$, then it is a surface of general type.

Let \mathcal{C} be the closure of the boundary point P_∞ in X_K . It is a purely inseparable cover of C of degree p^k , where p^k is the minimal degree of the splitting field of U_K . Let

$$X^\sharp = \{x \in X : f \text{ is smooth at } x\}.$$

The generic fiber X_K^\sharp is equal to U_K . Since $X \setminus X^\sharp$ is closed, the closure \mathcal{C} of P_∞ is contained in the complement $X \setminus X^\sharp$.

Let X_0^\sharp be the open subset of X^\sharp obtained by throwing away the irreducible components of fibers that do not intersect the zero section \mathcal{O} . The restriction of α to X_0^\sharp defines a quasi-finite birational morphism of smooth affine schemes $\alpha_0 : X_0^\sharp \rightarrow U^\circ$. Applying the Zariski main theorem, we infer that α_0 is an isomorphism. Thus we can consider U° as an open subscheme of X^\sharp whose complement consists of irreducible components of fibers that do not intersect \mathcal{O} .

We know that the Neron model \mathbf{J} of $\text{Jac}(X_K)$ is a maximal separated quotient of $\mathcal{P}_{X/C}$ by a discrete subsheaf supported at the points t with reducible fiber X_t . In particular its Lie algebra $\text{Lie}(\mathbf{U})$ is isomorphic to the Lie algebra functor of $\mathcal{P}'_{X/C}$ that, in its turn, is isomorphic to $R^1 f_* \mathcal{O}_X$ [21, Proposition 1.3]. The closed embedding $U_K \hookrightarrow \text{Jac}(X_K)$ defines a morphism $U \rightarrow \mathbf{J}$, and hence an injective homomorphism of the Lie algebras sheaves

$$\text{Lie}(U) \hookrightarrow \text{Lie}(\mathbf{J}) \cong R^1 f_* \mathcal{O}_X. \tag{6.3}$$

PROPOSITION 6.3. *Let $f : X \rightarrow C$ be a relatively minimal model of X_K of genus $g > 0$ over a smooth projective curve C over \mathbb{k} . Suppose that the Picard scheme of X is reduced. Then*

$$H^0(C, \text{Lie}(U)) = \{0\}.$$

Proof. Since the generic fiber is geometrically irreducible, $f_* \mathcal{O}_X \cong \mathcal{O}_C$. The Leray spectral sequence for $f : X \rightarrow C$ and the sheaf \mathcal{O}_X gives an exact sequence

$$\begin{aligned} 0 \rightarrow H^1(C, \mathcal{O}_C) &\rightarrow H^1(X, \mathcal{O}_X) \\ &\rightarrow H^0(C, R^1 f_* \mathcal{O}_X) \rightarrow H^2(C, \mathcal{O}_C) = 0. \end{aligned} \tag{6.4}$$

The linear spaces $H^1(C, \mathcal{O}_C)$ and $H^1(X, \mathcal{O}_X)$ are identified with the Lie algebras of the Picard schemes $\mathbf{Pic}_{C/\mathbb{k}}^0$ and $\mathbf{Pic}_{X/\mathbb{k}}^0$. Although the former scheme is always reduced, the latter one may not be reduced if $p_g(X) > 0$. The map between these spaces corresponds to the pull-back morphism of abelian varieties $f^* : \mathbf{Pic}_{C/\mathbb{k}}^0 \rightarrow$

Pic $_{X/\mathbb{k}}^0$. As in the proof of Theorem 6.1, using the Poincaré reducibility theorem, we show that f^* must be an isogeny, hence $H^0(C, R^1 f_* \mathcal{O}_X) = 0$.

Now the assertion follows from the fact that $\mathbf{U}^\circ \subset \mathbf{J}$, hence $\text{Lie}(\mathbf{U}) \subset \text{Lie}(\mathbf{J})$. □

EXAMPLE 6.4. If X is a quasi-hyperelliptic surface, that is, quasi-elliptic surface with no reducible fibers, then C is an elliptic curve and $\text{deg } \mathcal{L} = \text{deg } R^1 f_* \mathcal{O}_X = 0$. In fact $\mathcal{L}^{\otimes 4} \cong \mathcal{O}_C$ if $p = 2$ and $\mathcal{L}^{\otimes 6} \cong \mathcal{O}_C$ if $p = 3$ [4]. In particular, it may occur that $\mathcal{L} \cong \mathcal{O}_C$, in which case $H^0(C, \mathcal{L}) = \mathbb{k}$.

Since $R^i f_* \mathcal{O}_X = 0, i > 0$, the Leray spectral sequence degenerates at page E_2 and gives an isomorphism

$$H^1(C, R^1 f_* \mathcal{O}_X) \cong H^2(X, \mathcal{O}_X), \tag{6.5}$$

and the equality

$$\chi(X, \mathcal{O}_X) = -\text{deg } R^1 f_* \mathcal{O}_X. \tag{6.6}$$

If $g > 1$, the surface X is of general type, and a recent work of Yi Gu shows that $\chi(X, \mathcal{O}_X) > 0$ [12; 13] (eliminating a few possible cases where it may be not true from [31]). Thus in this case, $\text{deg } R^1 f_* \mathcal{O}_X < 0$. However, this does not imply that $\text{deg } \mathcal{L} < 0$ since $R^1 f_* \mathcal{O}_X$ is not necessary a semi-stable vector bundle. In fact, it may occur that $H^0(C, \mathcal{L}) \neq \{0\}$ and is isomorphic to the Lie algebra $H^0(X, \Theta_{X/\mathbb{k}})$ of regular vector fields on X [18, Theorem 4.1]. This happens for some Raynaud surfaces, which we will discuss in the next example.

EXAMPLE 6.5. Assume that \mathbf{U}_K of genus $g > 0$ has the Russell equation $u^{p^n} + v + av^{p^n} = 0$. We can compactify the group in the usual way by homogenizing the equation $y^{p^n} + z^{p^n-1}x + ax^{p^n} = 0$. In the open set $x \neq 0$, the affine equation is $Y^{p^n} + Z^{p^n-1} + a = 0$. Its singular points are the zeroes of the differential $\omega = z^{p^n-2}dz + da$. Thus a singular point has coordinate z equal to zero and da must vanish at this point. To get a smooth compactification $f : X \rightarrow C$ would require to blow up this point that will force the fiber passing through the singular point to be reducible.

Suppose that $f : X \rightarrow C$ has no reducible fibers. Formula (3.4) implies that da is well defined as a section

$$da \in H^0(C, \mathcal{L}^{\otimes -p^n(p-1)} \otimes \omega_{C/\mathbb{k}})$$

that does not vanish anywhere. In this case, we also see that the closure \mathcal{C} of P_∞ is locally given by equation $Z = 0$, and it has a smooth point in each fiber. Thus \mathcal{C} is smooth and $f : \mathcal{C} \rightarrow C$ is the iterated Frobenius map $\mathbf{F}^n : C \rightarrow C$. We have

$$\mathcal{L}^{p^n(p-1)} \cong \omega_{C/\mathbb{k}}.$$

It implies that

$$\text{deg } \mathcal{L} = \frac{2g(C) - 2}{p^n(p - 1)},$$

which is positive if $g(C) > 1$.

If $g = 1$, then we see that $\mathcal{L}^{\otimes 4} \cong \mathcal{O}_C$ if $p = 2$ (resp. $\mathcal{L}^{\otimes 6} \cong \mathcal{O}_C$ if $p = 3$) and C in an elliptic curve. This agrees with the classification of quasi-elliptic surfaces with no reducible fibers from [4]. If $\mathcal{L} \cong \mathcal{O}_C$, then $\mathbf{Pic}_{X/\mathbb{k}}$ is not reduced and $\dim_{\mathbb{k}} H^0(C, \mathcal{L}) = 1$.

Assume $n = 1$. The injection $\mathcal{O}_C \rightarrow \mathbf{F}_*\mathcal{O}_C$ corresponds to the inclusion $\mathcal{O}_C^p \rightarrow \mathcal{O}_C$ upstairs. Taking the differential $d : \mathcal{O}_C \rightarrow \mathcal{B}_{C/\mathbb{k}}^1$ with values in the sheaf of locally exact regular 1-forms on C , we obtain an exact sequence

$$0 \rightarrow \mathcal{O}_C \rightarrow \mathbf{F}_*\mathcal{O}_C \rightarrow \mathcal{B} \rightarrow 0,$$

where $\mathcal{B} = \mathbf{F}_*\mathcal{B}_{C/\mathbb{k}}^1$ is a locally free sheaf of rank $p - 1$. The sheaf $\mathcal{L}^{\otimes p}$ is equal to $\mathbf{F}_*\mathcal{L}$ and the section da defines an injection $\mathcal{L}^{\otimes p-1} \rightarrow \mathcal{B}$. The curve C admitting an invertible subsheaf \mathcal{N} such that $\mathcal{N}^{\otimes p} \cong \omega_{C/\mathbb{k}}$ is called a *Tango curve*, and one says that the sheaf \mathcal{N} defines a *Tango structure* on C [32]. Thus we see that $\mathcal{L}^{\otimes p-1}$ defines a Tango structure on C . The surface $X \rightarrow C$ is an example of a *Raynaud surface*. These surfaces provide counter-examples in characteristic $p > 0$ to the Kodaira vanishing theorem [27] as well as examples of surfaces of general type with of nonzero regular vector fields [18].

Recall that a smooth projective surface X over \mathbb{k} is called *supersingular* in the sense of Shioda if its second Betti number $b_2(X)$ computed in étale topology coincides with the Picard number $\rho(X)$, the rank of the Néron–Severi group $\text{NS}(X)$. It is known that a Shioda-supersingular surface is *supersingular* in the sense of Artin, which means that the formal completion of its Brauer group $\text{Br}(X)$ is isomorphic to the formal completion of the vector group \mathbb{G}_a^r . The number r is equal to $\dim_{\mathbb{k}} H^2(X, \mathcal{O}_X) = p_g(X)$. The converse is true for Artin-supersingular K3 surfaces (which conjecturally are all unirational) but is not known for surfaces of general type.

Since our surface X is uniruled, it is Shioda-supersingular [20, Theorem 7.3].

LEMMA 6.6. *The torsion group $\text{Tors}(\text{NS}(X))$ of the Néron–Severi lattice of X is a finite p -group. It is trivial if $g = 1$ and $\chi(X, \mathcal{O}_X) > 0$.*

Proof. For any commutative finite flat group scheme G , there is a natural isomorphism $H_{\mathbb{F}}^1(X, G) \cong \text{Hom}(G^D, \mathbf{Pic}_{X/\mathbb{k}})$, where $H_{\mathbb{F}}$ stands for the flat cohomology and G^D is the Cartier dual group scheme [26, Proposition 6.2.1]. A nontrivial invertible sheaf \mathcal{L} of finite order n in $\text{Pic}(X)$ defines an embedding of the constant group scheme $(\mathbb{Z}/n\mathbb{Z})_X$ into $\mathbf{Pic}_{X/C}$. It corresponds to an element α in $H_{\mathbb{F}}^1(X, \mu_{n,X})$, where $\mu_{n,X}$ is the kernel of $x \mapsto x^n$ for the multiplicative group scheme $\mathbb{G}_{m,X}$. In its turn, the cohomology class α defines an isomorphism class of $\mu_{n,X}$ -torsor Y_α on X in the flat topology. Since each irreducible fiber X_t of f is homeomorphic to \mathbb{P}^1 , the étale cohomology of μ_{n,X_t} is trivial for $(n, p) = 1$. Thus the restriction of Y_α to an open subset of X containing the generic fiber X_K is trivial. Hence the restriction of \mathcal{L} to X_K is trivial, and therefore $\mathcal{L} \cong \mathcal{O}_X(D)$ for some divisor D contained in fibers of f . It is known that a divisor class that intersects with zero each irreducible component of a closed fiber is linearly equivalent

to a rational linear combination of the divisor classes X_t . Since f has a section, no fiber is multiple, and hence D is an integer linear combination of the classes of fibers, hence comes as the pull-back of a divisor class in $\text{Pic}^0(C)$. Such a class belongs to $\text{Pic}^0(X)$.

To prove the second assertion we use Riemann–Roch theorem and Serre’s duality theorem on X , which gives that $\omega_X \otimes \mathcal{L}^{\otimes -1}$ has a section defined by an effective divisor $D' \sim K_X - D$, where $\mathcal{L} \cong \mathcal{O}_X(D)$. The restriction of K_X and D to each irreducible component of a fiber is of degree 0, hence D' is a linear combination of fibers, and therefore $D' = f^*(d')$ for some effective divisor class on C . Since $K_J = f^*(k)$ for some divisor class on C , we obtain that $D = f^*(k - d)$ where $k - d$ is a torsion divisor class on C . Hence $D \in f^*\text{Jac}(C)$, and thus it is algebraically equivalent to zero. \square

THEOREM 6.7. *Let $f : X \rightarrow C$ be a regular relatively minimal model of a regular compactification \bar{U} of a unipotent group of genus $g > 0$ and \mathbf{J} be the Néron model of $\text{Jac}(X_K)$. Then there is an isomorphism of abelian groups*

$$H_{\text{et}}^1(C, \mathbf{J}) \cong \mathbb{k}^{p_g(X)} \oplus \text{Tors}(\text{NS}(X)),$$

where $p_g(X) = \dim_{\mathbb{k}} H^2(X, \mathcal{O}_X)$.

Proof. It is known that $H^1(C, \mathbf{J})$ is isomorphic to the Brauer group $\text{Br}(X)$ of X that coincides with the cohomological Brauer group $H_{\text{et}}^2(X, \mathbb{G}_m)$ [11]. Since the general fiber of $f : X \rightarrow C$ is geometrically rational curve, the surface X is a unirational surface, hence its second Betti number computed in ℓ -adic cohomology coincides with the rank of the Picard group. The computation of the Brauer group of a smooth surface gives an isomorphism of abelian groups

$$\text{Br}(X) \cong \mathbb{k}^{p_g(X)} \oplus \text{Tors}(\text{NS}(X))$$

(see [7, 0.10]). \square

We use the notations from the proof of Theorem 3.12. We denote by μ the restriction of the homomorphism of group $\mathbf{V} \rightarrow \mathbb{V}(\mathcal{L}^{\otimes -p^n})$ to the subgroup $\mathbb{V}(\mathcal{L}^{\otimes -p^n})$. The homomorphism μ is surjective in étale topology. Let

$$G = \text{Ker}(\mu).$$

Now everything is ready to prove our main theorem.

THEOREM 6.8. *Let \mathbf{U} be the Néron model of a unipotent group \mathbf{U}_K of genus $g > 0$. Assume $H^0(C, \text{Lie}(\mathbf{U})) = \{0\}$. Then*

- (i) $\mathbf{U}^\circ(C) = H_{\text{et}}^0(C, \mathbf{U}^\circ) \cong H_{\text{et}}^0(C, G) = G(C)$.
- (ii) *There is an exact sequence*

$$0 \rightarrow H_{\text{et}}^1(C, G) \rightarrow H_{\text{et}}^1(C, \mathbf{U}^\circ) \rightarrow H^1(C, \mathcal{L}) \rightarrow 0.$$

- (iii) *There is an exact sequence*

$$0 \rightarrow \pi_0(\mathbf{U})/\mathbf{U}(C) \rightarrow H_{\text{et}}^1(C, \mathbf{U}^\circ) \rightarrow H_{\text{et}}^1(C, \mathbf{U}) \rightarrow 0.$$

- (iii) $H_{\text{et}}^2(C, \mathbf{U}) \cong H_{\text{et}}^2(C, G)$.

Proof. By assumption, $H^0(C, \mathcal{L}) = 0$, thus $H_{\text{et}}(C, \mathbf{V}) \cong H^1(C, \mathcal{L}^{\otimes p^n})$, hence $\text{Ker}(H_{\text{et}}^0(C, \mathbf{V}) \xrightarrow{H^0(\mu)} H^0(C, \mathcal{L}^{\otimes p^n}))$ coincides with $H_{\text{et}}^0(C, G)$. Now the first assertion follows immediately from exact sequence (3.7). Taking cohomology in (3.7), we get an exact sequence

$$0 \rightarrow H_{\text{et}}^1(C, \mathbf{U}^\circ) \rightarrow H_{\text{et}}^1(C, \mathbf{V}) \rightarrow H^1(C, \mathcal{L}^{\otimes p^n}) \rightarrow H_{\text{et}}^2(C, \mathbf{U}^\circ) \rightarrow 0. \quad (6.7)$$

Let us look at the homomorphism $H_{\text{et}}^1(C, \mathbf{V}) \rightarrow H^1(C, \mathcal{L}^{\otimes p^n})$. The group G is the subgroup of $\mathbb{V}(\mathcal{L}^{\otimes -p^n})$ equal to the kernel of the projection $\mathbf{U}^\circ \rightarrow \mathbb{V}(\mathcal{L})$. The image of $H_{\text{et}}^1(C, \mathbf{U}^\circ)$ in $H_{\text{et}}^1(C, \mathbf{V})$ is equal to an extension of $H^1(C, \mathcal{L})$ with kernel $H_{\text{et}}^1(C, G)$. This is the kernel of $H_{\text{et}}^1(C, \mathbf{V}) \rightarrow H^1(C, \mathcal{L}^{\otimes p^n})$. Thus we obtain an exact sequence

$$0 \rightarrow H_{\text{et}}^1(C, G) \rightarrow H_{\text{et}}^1(C, \mathbf{U}^\circ) \rightarrow H^1(C, \mathcal{L}) \rightarrow 0 \quad (6.8)$$

and an isomorphism

$$H_{\text{et}}^2(C, \mathbf{U}^\circ) \cong H_{\text{et}}^2(C, G).$$

Next we relate \mathbf{U}° and \mathbf{U} by means of the exact sequence coming from the definition of $\pi_0(\mathbf{U})$

$$0 \rightarrow \mathbf{U}^\circ \rightarrow \mathbf{U} \rightarrow \pi_0(\mathbf{U}) \rightarrow 0.$$

Here $\pi_0(\mathbf{U}) = \mathbf{U}/\mathbf{U}^\circ$ is supported at a finite set of points in C , and so it can be identified with its group of sections. Applying cohomology, we find

$$H_{\text{et}}^2(C, \mathbf{U}) = H_{\text{et}}^2(C, \mathbf{U}^\circ) \cong H_{\text{et}}^2(C, G),$$

as asserted. We also get an exact sequence

$$0 \rightarrow \pi_0(\mathbf{U})/\mathbf{U}(C) \rightarrow H_{\text{et}}^1(C, \mathbf{U}^\circ) \rightarrow H_{\text{et}}^1(C, \mathbf{U}) \rightarrow 0. \quad (6.9)$$

□

REMARK 6.9. Suppose $H^0(C, \mathcal{L}) \neq \{0\}$. This implies that $\{0\} \neq H^0(C, \mathcal{L}^{\otimes p^n}) \subset H_{\text{et}}^0(C, \mathbf{V})$, but the kernel $\mathbf{U}^\circ(C)$ of the map $\alpha : H_{\text{et}}^0(C, \mathbf{V}) \rightarrow H^0(C, \mathcal{L}^{\otimes p^n})$ must be a finite group. This happens, for example, if $f : X \rightarrow C$ is a quasi-hyperelliptic surface and $\mathcal{L} \cong \mathcal{O}_C$. In this case

$$H_{\text{et}}^0(C, \mathbf{V}) = H^i(C, \mathcal{L}) = H^i(C, \mathcal{L}^{\otimes p^n}) = \mathbb{G}_a(\mathbb{k}), \quad i = 0, 1,$$

and $\mathbf{U}(C)$ is a nontrivial finite group isomorphic to $\text{Tors}(\text{NS}(X))$. We know that $\text{deg } \mathcal{L} > 0$ is possible, for example, for Raynaud surfaces from Example 6.5.

The following lemma is taken from [22, III, §4, Lemma 4.13] where one can also find references to its proof.

LEMMA 6.10. *Let V be a finite-dimensional linear space of dimension d over \mathbb{k} and $\phi : V \rightarrow V$ be a p^k -linear map (i.e. $\phi(\lambda x) = \lambda^{p^k} \phi(x)$ for any $\lambda \in \mathbb{k}$). Let $V = V_{\text{ss}} \oplus V_{\text{nil}}$, where ϕ is bijective on V_{ss} and ϕ is nilpotent on V_{nil} . Then $\phi - \text{id}$ is surjective on V and the kernel of $\phi - \text{id}$ is a vector space over \mathbb{F}_{p^k} of dimension equal to $\dim V_{\text{ss}}$.*

Note that the dimension of V_{ss} can be computed as follows. Choose a \mathbb{k} -basis \underline{e} of V and let $\phi(\underline{e}) = A\underline{e}$ for some matrix A . Then

$$\dim_{\mathbb{k}} V_{ss} = \text{rank}(A \cdot A^{(p)} \cdot \dots \cdot A^{(p^{d-1})}), \tag{6.10}$$

where $A^{(p^k)}$ denotes raising the entries of A in p^k th power and $d = \dim_{\mathbb{k}} V$. The number $\dim_{\mathbb{k}} V_{ss}$ is called the *stable rank* of the matrix A .

COROLLARY 6.11. *Let $u^{p^n} + v + a_1v^p + \dots + a_mv^{p^m} = 0$ be the equation of the Néron model \mathbf{U} of a unipotent group \mathbf{U}_K of genus $g > 0$. Suppose that $a_1 = \dots = a_{m-1} = 0$. Let*

$$\alpha : \mathcal{L}^{\otimes p^n} \rightarrow \mathcal{L}^{\otimes p^n}$$

be the map given by $v \mapsto v + av^{p^m}$ and

$$H^1(\alpha) : H^1(C, \mathcal{L}^{\otimes p^n}) \rightarrow H^1(C, \mathcal{L}^{\otimes p^n})$$

be the corresponding map on cohomology. Then $H^1(\alpha)$ is surjective, and its kernel is a vector space over \mathbb{F}_{p^m} of dimension equal to the stable rank r of the p^m -linear map $\phi = H^1(\alpha) - \text{id}$.

In particular,

$$H^1(C, G) \cong (\mathbb{Z}/p^m\mathbb{Z})^r, \quad H_{\text{et}}^2(C, \mathbf{U}) = H^2(C, G) = 0.$$

Assume $\text{Tors}(\text{NS}(X)) = \{0\}$. The images of the divisor classes of a fiber of $f : X \rightarrow C$ and of a section \mathbf{O} generate a sublattice of $\text{NS}(X)$ isomorphic to the integral hyperbolic plane \mathbf{U} . It splits as an orthogonal summand of $\text{NS}(X)$. Let $\text{NS}^0(X)$ be its orthogonal complement. The image of the restriction homomorphism $\text{NS}^0(X) \rightarrow \text{Pic}(X_K)$ is isomorphic to the group of sections $\mathbf{U}(C)$ and its kernel is the sublattice $\text{NS}_{\text{fib}}^0(X)$ generated by components of fibers not intersection \mathbf{O} . Let us consider a chain of lattices and the corresponding dual lattices

$$\text{NS}_{\text{fib}}^0(X) \subset \text{NS}(X)^0 \subset \text{NS}^0(X)^\vee \subset \text{NS}_{\text{fib}}^0(X)^\vee.$$

The discriminant group $\text{NS}_{\text{fib}}^0(X)^\vee/\text{NS}_{\text{fib}}^0(X)$ of the lattice $\text{NS}_{\text{fib}}^0(X)$ is isomorphic to the group $\pi_0(\mathbf{U})$ [26, 8.1.2] and the discriminant group $\text{NS}^0(X)^\vee/\text{NS}^0(X)$ of the lattice $\text{NS}^0(X)$ is isomorphic to the discriminant group $D(\text{NS}(X))$ of $\text{NS}(X)$. This gives us a chain of finite abelian groups

$$\mathbf{U}(C) \subset \mathbf{U}(C)' \subset \pi_0(\mathbf{U})$$

with quotients $\mathbf{U}(C)'/\mathbf{U} \cong D(\text{NS}(X))$ and $\pi_0(\mathbf{U})/\mathbf{U}(C)' \cong \mathbf{U}(C)$.

Comparing it with exact sequences (6.8) and (6.9), we dare to make the following conjecture.

CONJECTURE 6.12. *The intersection $H_{\text{et}}^1(C, G)_0$ of the subgroups $H_{\text{et}}^1(C, G)$ and $\pi_0(\mathbf{U})/\mathbf{U}(C)$ of $H_{\text{et}}^1(C, \mathbf{U}^\circ)$ from assertions (ii) and (iii) of the theorem splits the exact sequence*

$$0 \rightarrow D(\text{NS}(X)) \rightarrow \pi_0(\mathbf{U})/\mathbf{U}(C) \rightarrow \mathbf{U}(C) \rightarrow 0.$$

The group $H_{\text{et}}^1(C, \mathbf{U})$ is isomorphic to $H^1(C, \mathcal{L})$ and fits in an extension

$$0 \rightarrow H_{\text{et}}^1(C, G)/H_{\text{et}}^1(G, G)_0 \rightarrow H_{\text{et}}^1(C, \mathbf{U}) \rightarrow H^1(C, \mathcal{L}) \rightarrow 0.$$

Note that $H^1(C, \mathcal{L})$ is a vector p -torsion group that may contain a finite elementary p -group with quotient isomorphic to $H^1(C, \mathcal{L})$.

EXAMPLE 6.13. Assume $g = 1, C = \mathbb{P}^1$, and $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^1}(-k)$. The linear space $H^1(C, \mathcal{L}^{\otimes p}) \cong H^1(C, \mathcal{O}_{\mathbb{P}^1}(-pk))$ has a natural basis formed by negative Laurent monomials $e_i = t_0^{-i} t_1^{pk-i}, i = 1, \dots, pk - 1$ [14, III,§5]. Assume $p = 3$ and let $u^3 + v + a_{6k}v^3 = 0$ be the equation of \mathbf{U} , where $a_{6k} \in H^0(C, \mathcal{L}^{\otimes 6}) = H^0(C, \mathcal{O}_{\mathbb{P}^1}(6k))$ is a binary form of degree $6k$. Write $a_{6k} = \sum_{i=0}^{6k} c_i t_0^i t_1^{6k-i}$. Let $A = (c_{ij})$ be a matrix with entries defined by

$$\left(\begin{matrix} a_{6k}(t_0, t_1) \\ t_0^{3i} t_1^{9k-3i} \end{matrix} \right)' = \sum_{j=1}^{3k-1} c_{ij} e_j, \quad i = 1, \dots, 3k - 1.$$

Here $()'$ means that we eliminate from the Laurent polynomial all monomials $t_0^i t_1^j$ with nonnegative i or j . We compute the entries c_{ij} of A and obtain that

$$A = \begin{pmatrix} c_{3-1} & c_{6-1} & \cdots & c_{3d-1} \\ c_{3-2} & c_{6-2} & \cdots & c_{3d-2} \\ \vdots & \vdots & \vdots & \vdots \\ c_{3-d} & c_{6-d} & \cdots & c_{3d-d} \end{pmatrix}.$$

where $c_j = 0, j < 0$, and $d = 3k - 1$. The group $H^1(C, G)$ is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^{\oplus r}$, where r is the stable rank of A (see (6.10)). The matrix A coincides with the Hasse–Witt matrix that computes the p -rank of the hyperelliptic curve H of genus d given by equation $t_2^2 + a_{6k}(t_0, t_1) = 0$ (i.e. the maximal r such that $(\mathbb{Z}/p\mathbb{Z})^r$ embeds in its Jacobian). Of course, in our case, the polynomial $a_{6k}(t_0, t_1)$ may degenerate and does not define any hyperelliptic curve. Note that the projection $\pi : H \rightarrow \mathbb{P}^1$ is a separable double cover ramified over $V(a_6)$ that gives an exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^1} \rightarrow \pi_* \mathcal{O}_H \rightarrow \mathcal{O}_{\mathbb{P}^1}(-3k) \rightarrow 0$$

and an isomorphism $H^1(H, \mathcal{O}_H) = H^1(\mathbb{P}^1, \pi_* \mathcal{O}_H) \cong H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-3k))$. The matrix A describes the action of the Frobenius on the basis (e_1, \dots, e_{3k-1}) of $H^1(H, \mathcal{O}_H)$.

Assume $k = 1$. The surface X is a rational quasi-elliptic surface with a section. The classification of such surfaces is known, in particular, the group $\mathbf{U}(C)$ is known in each case (see [7, 4.9] for exposition of this classification and the references to the original results). The conjecture is checked in characteristic $p = 2, 3$ by explicit computation of the group $H^1(C, G)$ (see [7, 4.8]). Note that in this case $D(\text{NS}(X)) = \{0\}$ and $H^1(C, \mathcal{L}) = 0$, so that the Tate–Shafarevich group $H_{\text{et}}^1(C, \mathbf{U})$ is trivial that agrees with Theorem 6.7.

Under the same assumption, but taking $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(-2)$, we get a quasi-elliptic K3 surface $f : X \rightarrow \mathbb{P}^1$. In this case Theorem 6.7 gives that $H_{\text{et}}^1(C, \mathbf{U}) \cong H^1(C, \mathcal{L}) \cong \mathbb{G}_a(\mathbb{k})$.

The following two examples were suggested to me by T. Katsura. It is known that the Fermat quartic surface $x^4 + y^4 + z^4 + w^4 = 0$ in characteristic 3 is a supersingular K3 surface with Artin invariant σ equal to 1. It admits a quasi-elliptic fibration with Weierstrass equation $y^2 + x^3 + t_0^2 t_1^2 (t_0^8 + t_1^8) = 0$ [16]. The Russell equation of \mathbf{U}° is

$$u^3 + v + t_0^2 t_1^2 (t_0^8 + t_1^8) v^3 = 0.$$

The quasi-elliptic fibration has 10 reducible fibers of Kodaira's type IV with $\pi_0(\mathbf{U}) \cong (\mathbb{Z}/3\mathbb{Z})^{\oplus 10}$ and the Mordell–Weil group $\mathbf{U}(C)$ is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^{\oplus 4}$. The discriminant group $D(\text{NS}(S))$ is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^{\oplus 2}$. We compute the Hasse–Witt matrix A and find that $H^1(C, G) \cong (\mathbb{Z}/3\mathbb{Z})^{\oplus 4} \cong \mathbf{U}(C)$.

However, if we take a K3 surface given by Weierstrass equation

$$y^2 + x^3 + t_0^2 t_1^{10} + t_0^5 t_1^7 + t_0^8 t_1^4 + t_0^{10} t_1^2 = 0,$$

we obtain that its Mordell–Weil group $\mathbf{U}(C)$ is an elementary 3-group of rank 2, the quasi-elliptic fibration contains 10 reducible fibers of type IV. Thus its Artin invariant σ is equal to 3, so that $D(\text{NS}(S))$ is an elementary 3-group of rank 6 and $\pi_0(\mathbf{U})$ is an elementary 3-group of rank 10. Computing the Hasse–Witt matrix A , we find that its stable rank equals 4. Thus $\mathbf{U}(C) \cong (\mathbb{Z}/3\mathbb{Z})^{\oplus 2}$ is isomorphic to a proper subgroup of $H_{\text{et}}^1(C, G) \cong (\mathbb{Z}/3\mathbb{Z})^{\oplus 4}$. Katsura finds an explicit isomorphism from a certain subgroup $H_{\text{et}}^1(C, G)'$ of $H_{\text{et}}^1(C, G)$ to $\mathbf{U}(C)$.

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