http://dx.doi.org/10.1090/pspum/040.1/713067

Proceedings of Symposia in Pure Mathematics
Volume **40** (1983), Part 1

WEYL GROUPS AND CREMONA TRANSFORMATIONS[1]

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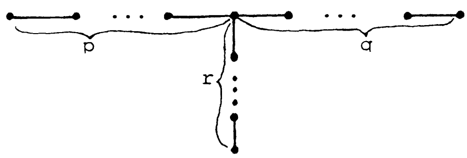
1. Introduction. Let $T_{p,q,r}$ be a nonorientable graph of the form:



It defines a lattice (a free abelian group equipped with an integral quadratic form) $Q_{p,q,r}$ of rank $p+q+r-2$ with a basis $e_1,\ldots,e_{p+q+r-2}$ corresponding to the set of vertices and $(e_i, e_j) = -2, 1, 0$ if $i = j$, e_i is joined to e_j by an edge, e_i and e_j are disjoint respectively. The orthogonal transformations s_i: $x \to x + (x, e_i)e_i$ generate a subgroup W of the orthogonal group $O(Q_{p,q,r})$. If $(p, q, r) = (p, q, 1)$, $(p, 2, 2)$, $(3, 3, 2)$, $(3, 4, 2)$ or $(3, 5, 2)$ then the lattice $-Q_{p,q,r}$ (resp. the group W) coincides with the root lattice (resp. the Weyl group) of a simple Lie algebra of type A_{p+q-1}, D_{p+2}, E_6, E_7 and E_8 respectively. If $(p, q, r) = (3, 3, 3)$, $(4, 4, 2)$ or $(3, 6, 2)$, then the group W is the affine Weyl group of the affine root system of type $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$ respectively. In the remaining cases the lattice $-Q_{p,q,r}$ (resp. the group W) is the root lattice (resp. the Weyl group) of a certain Kac-Moody infinite-dimensional Lie algebra (see [**Ka1, Mo**]). The ubiquity of the diagrams A, D and E is well known (see [**Ha**]). Other diagrams $T_{p,q,r}$ appear in the singularity theory as the "infinity curves" in a certain smoothing of a triangular surface singularity [**Pi**] or as the orthogonal summands in the Milnor lattice of exceptional unimodal critical points [**Ga**] (the lattices $\tilde{E}$ are related to certain degenerations of elliptic curves [**Ko**]). The purpose of this talk is to interpret the lattices $T_{p,q,2}$ and the corresponding Weyl groups by means of regular (in the sense clarified later) Cremona transformations of projective spaces. We follow

1980 *Mathematics Subject Classification*. Primary 14E07.

[1]Research partially supported by the National Science Foundation.

very closely the beautiful works of S. Kantor [**Kan**], A. Coble [**Co1**] and P. Du Val [**DV**] on this subject which certainly deserve to be saved from oblivion. A modern treatment of some of these results in the cases A, D, E can be found in the works of E. Brieskorn [**Br**], G. Tjurina [**Tj**] and Yu. Manin [**Ma**] (an exposition of these results is in [**Se**]). The recent work of E. Looijenga [**Lo2**] is closely related to my talk (and more relevant to the singularity theory).

2. Root systems [Bo, Ka2, Lo1]. Let V be a real vector space and V^* its dual space of linear functions. A (symmetric) *root base* in V is a finite set $B = \{\alpha_1, \dots, \alpha_r\}$ of linear independent vectors in V together with an injective map $\vee$: $B \to V^*$ such that the images $\check{\alpha}_i$ of α_i are linearly independent, $\check{\alpha}_i(\alpha_i) = -2$, $\check{\alpha}_i(\alpha_j) = \check{\alpha}_j(\alpha_i)$, $i \neq j$, are nonnegative integers. The *Dynkin diagram* of a root base is a nonorientable graph whose vertices v_i correspond to the vectors α_i and the vertices v_i, v_j are joined by $\check{\alpha}_i(\alpha_j)$ edges. The matrix $A = (a_{ij})$, where $a_{ij} = \check{\alpha}_i(\alpha_j)$, is called the *Cartan matrix* of B. It defines a structure of a lattice on the group $Q(B) = \mathbf{Z}\alpha_1 + \cdots + \mathbf{Z}\alpha_r$ by setting $(\alpha_i, \alpha_j) = a_{ij}$. The lattice $Q(B)$ is called the *root lattice* of B. For every $\alpha_i \in B$ the linear mapping s_i: $V \to V$

$$s_i(x) = x + \check{\alpha}_i(x)\alpha_i$$

induces an orthogonal transformation of $Q(B)$ (called the *simple reflection* through α_i). The subgroup of the orthogonal group $O(Q(B))$ generated by the transformations s_i is called the *Weyl group* of B and is denoted by W. An s-translate ($s \in W$) of an element of B is called a *root* (a real root in the terminology of Kac-Moody algebras). The set of roots is denoted by R. The set $C = \{x \in V\colon \check{\alpha}_i(x) > 0,\ i = 1, \dots, r\}$ is called the *fundamental Weyl chamber*, an s-translate ($s \in W$) of C is called a *chamber*. The closure $\bar{C}$ of C is a convex polyhedral cone spanned by the subspace $F_B = \{x \in V\colon \check{\alpha}_i(x) = 0,\ i = 1, \dots, r\}$ and the vectors ω_i such that $\check{\alpha}_i(\omega_j) = \delta_{ij}$. The latter vectors are defined up to an element of F_B and are called the *fundamental weights*. The union of the closures of chambers, $I = \bigcup_{s \in W} s(\bar{C})$, is called the *Tits cone* of B. It is a convex cone in V. The Weyl group W acts totally discontinuously in the interior I^0 of I and $\bar{C}$ is a fundamental domain for this action. For any subset B' of B, let $F_{B'} = \{x \in V\colon \check{\alpha}_i(x) > 0,\ \alpha_i \in B - B',\ \check{\alpha}_i(x) = 0,\ \alpha_i \in B'\}$ be one of the faces of $\bar{C}$. Then the isotropy subgroup of a point $x \in \bar{C}$ is the group $W_{B'}$ generated by the s_i's belonging to the subset B' such that $x \in F_{B'}$ but $x \notin F_{B''}$ for any $B'' \supset B'$. Clearly, $W_{B'}$ is the Weyl group of the root base B' (for which the map $\vee$ is the restriction of the map $\vee$ defined for B). It follows from above that the Weyl group W acts simply transitively on the set of chambers.

3. The main example. Let f: $X \to \mathbf{P}^k$ be a birational morphism of a nonsingular projective algebraic variety X onto the k-dimensional projective space which is a composition $X = X_0 \to X_1 \to \cdots \to X_r = \mathbf{P}^k$ of monoidal transformations f_i: $X_{i-1} \to X_i$ centered at a point p_i of X_i. Let e_i' be the class of the exceptional space E_i of f_i in the 1-codimensional component $A^1(X_{i-1})$ of the Chow ring of X_{i-1} (= the Picard group of $X_{i-1} = H^2(X_{i-1}, \mathbf{Z})$). Let h_i' be the class of a line in E_i in the

1-dimensional component $A_1(X_{i-1})$ of the Chow ring of X_{i-1} $(= H_2(X_{i-l}, \mathbf{Z}))$. Let

$$e_i = (f_i \circ \cdots \circ f_1)^*(e_i') \in A^1(X), \qquad i = 1,\dots,r,$$
$$h_i = (f_i \circ \cdots \circ f_1)^*(h_i') \in A_1(X), \qquad i = 1,\dots,r,$$
$$e_0 = f^*(H), \quad h_0 = f^*(h),$$

where H (resp. h) is the class of a hyperplane (resp. a line) in $\mathbf{P}^k$. It follows from the standard intersection formulas that the set $B = \{\alpha_1,\dots,\alpha_r\}$, where

$$\alpha_1 = e_1 - e_2,\dots, \quad \alpha_{r-1} = e_{r-1} - e_r, \alpha_r = e_0 - e_1 - \cdots - e_{k+1}$$

is a root basis in the space $V = A^1(X)_{\mathbf{R}} = A^1(X) \otimes_{\mathbf{Z}} \mathbf{R}$ with respect to the set $\check{B} = \{\check{\alpha}_1,\dots,\check{\alpha}_r\} \subset A_1(X)_{\mathbf{R}} = V^*$, where

$$\check{\alpha}_1 = h_1 - h_2,\dots,\check{\alpha}_{r-1} = h_{r-1} - h_r, \check{\alpha}_r = (k-1)h_0 - h_1 - \cdots - h_{k+1}.$$

It is easy to check that its Dynkin diagram is the graph $T_{k+1,r-k-1,2}$ with the vertices numbered from the left to the right putting the last vertex at the lower end.

The simple reflection s_i $(i = 1,\dots,r-1)$ acts on the basis $e_0,\dots,e_r$ of $A^1(X)$ by leaving e_j $(j \neq i, i+1)$ fixed and permuting e_i and e_{i+1}. The simple reflection s_r transforms e_0 to $ke_0 - e_1 - \cdots - e_{k+1}$ and e_i $(i = 1,\dots,k+1)$ to $(k-1)e_0 - e_1 - \cdots - e_{i-1} - e_{i+1} - \cdots - e_{k+1}$, leaving all other e_i fixed. The closure $\overline{C}$ of the fundamental Weyl chamber is spanned by the space $F_B = \mathbf{R}K$, where $K = -(k+1)e_0 + (k-1)(e_1 + \cdots + e_r)$ is the canonical class of X, and the fundamental weights

$$\omega_1 = \frac{1}{k-1}e_0 - e_1,$$
$$\omega_2 = \frac{2}{k-1}e_0 - e_1 - e_2,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\omega_k = \frac{k}{k-1}e_0 - e_1 - \cdots - e_k,$$
$$\omega_{k+1} = \frac{k+1}{k-1}e_0 - e_1 - \cdots - e_{k+1} = e_{k+2} + \cdots + e_r \pmod{F_B},$$
$$\omega_{k+2} = e_{k+3} + \cdots + e_r,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\omega_{r-1} = e_r,$$
$$\omega_r = \frac{1}{k-1}e_0.$$

One has to make a slight modification of these formulas if $k \geqslant r$.

We can interpret geometrically some of these weights in the following way. Given a hypersurface of degree m in $\mathbf{P}^k$ with multiple points at $p_1,\dots,p_r$ of order $m_1,\dots,m_r$ respectively, then its proper inverse transform in X belongs to the class $me_0 - m_1e_1 - \cdots - m_re_r$. Thus, the weight ω_{k-1} represents the linear system of hyperplanes passing through the points $p_1,\dots,p_{k-1}$. The weight ω_{r-1} represents the exceptional divisor $E_r = f^{-1}(p_r)$. The $(k-1)$-multiple of ω_r represents the linear system of hyperplanes in $\mathbf{P}^k$. We will return to this interpretation later.

4. Cremona isometries ($k = 2$). Let X be a nonsingular projective surface, $A^1(X)$ its Picard group. An automorphism σ of $A^1(X)$ is called a *Cremona isometry* (cf. [**Lo2**]) if the following three properties are satisfied:

(C1) σ preserves the intersection form in $A^1(X)$;

(C2) σ leaves the canonical class of X fixed;

(C3) σ leaves the semigroup of effective classes invariant.

Let Cris(X) denote the group of Cremona isometries. An example of a Cremona isometry is the automorphism of $A^1(X)$ induced by an automorphism of X. Such a Cremona isometry is called *effective*. The set of effective Cremona isometries forms a subgroup $\mathrm{Cris}_{\mathrm{ef}}(X)$ of Cris(X).

We will be interested here only in the case where X is a rational surface, even assuming that $X = \Sigma(p_1,\ldots,p_r)$ is obtained from $\mathbf{P}^2$ by blowing up r points (some of them may be infinitely near). Notice that in the case where X is a del Pezzo surface (i.e. the anticanonical linear system of X is ample) condition (C3) follows from (C1) and (C2). This is an easy application of the Riemann-Roch theorem.

Choose a root base in $A^1(X)$ as in the previous section, where $X_i = \Sigma(p_1,\ldots,p_{r-i})$ and f_i: $X_{i-1} \to X_i$ is the canonical projection. Let W be the corresponding Weyl group and R be the set of roots. A root $\alpha \in R$ is called *nodal* (cf. [**Lo2**]) if it defines an effective class. Let R^n be the set of nodal roots. Its elements describe the "discriminant conditions" on the points $p_1,\ldots,p_r$. For example, $\alpha_i = e_i - e_{i+1} \in R^n$ if and only if the point p_{i+1} is infinitely near to p_i. Or, $\alpha_r = e_0 - e_1 - e_2 - e_3 \in R^n$ if and only if the points p_1, p_2 and p_3 lie on a line. Or, the root $\alpha = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6$ (the maximal root in the root system E_6) belongs to R^n if and only if the six points $p_1,\ldots,p_6$ lie on a conic.

THEOREM 1. *Assume that $p_1,\ldots,p_r$ are in general position. Then* $W = $ Cris(X).

PROOF. First, let us show that $W \subset$ Cris(X). It suffices to show that all the simple reflections s_i belong to Cris(X). Let $i = r$. Then, for every $v = me_0 - m_1e_1 - \cdots - m_re_r$,

$$s_r(v) = (2m - m_1 - m_2 - m_3)e_0 - (m - m_2 - m_3)e_1 - (m - m_1 - m_3)e_2 - (m - m_1 - m_2)e_3 - m_4e_4 - \cdots - m_re_r.$$

It is immediately verified that s_r preserves the intersection form and the canonical class $K = -3e_0 + e_1 + \cdots + e_r$. Assume that v is effective. Then v is the class of the proper inverse transform of a certain curve $C \subset \mathbf{P}^2$ of degree m with m_i-multiple points at p_i, $i = 1,\ldots,r$. Let T be the standard quadratic transformation based on the points p_1, p_2, p_3. Then T^{-1} has fundamental points at certain points q_1, q_2, q_3 and T transforms the point p_i to the line $\overline{q_jq_k}$ (i, j, $k = 1, 2, 3$, pairwise distinct). Let $T(p_j) = q_j$, $j \neq 1, 2, 3$. Then T^{-1} transforms a curve C' of degree m with m_i-multiple points at q_i $(i = 1,\dots,r)$ to a curve $\overline{C}$ of degree $2m - m_1 - m_2 - m_3$ with $(m - m_2 - m_3)$-multiple point at p_1, $(m - m_1 - m_3)$-multiple point at p_2, $(m - m_1 - m_3)$-multiple point at p_3 and m_j-multiple point at p_j, $j > 3$. Clearly, the proper inverse transform of $\overline{C}$ represents the class $s_r(v)$. To see why the above curve C' exists, we argue as follows. Since the points $p_1,\dots,p_r$ are in general position, the conditions for a curve of degree m to pass through the points p_i with multiplicities m_i are independent. Since at least one such curve exists, we get $m(m+3) - m_1(m+1) - \cdots - m_r(m_r+1) > 0$. Therefore, by counting constants, we find a curve C' of degree m with m_i-multiple points at q_i, $i = 1,\dots,r$.

Next, let $i = 1$ (the remaining cases are considered similarly). Then

$$s_1(v) = me_0 - m_2e_1 - m_1e_2 - m_3e_3 - \cdots - m_re_r.$$

Obviously, s_1 satisfies conditions (C1) and (C2) of the definition of a Cremona isometry. Assume that v is effective and is represented by a curve C as above. Let T be a projective transformation which interchanges the points p_1 and p_2 and maps the remaining points $p_3,\dots,p_r$ to some points $q_3,\dots,q_r$ respectively. Since the points $p_1,\dots,p_r$ are in general position, one can argue as above and find a curve C' of degree m with an m_1-multiple point at p_1, an m_2-multiple point at p_2, and an m_i-multiple point at q_i, $i = 3,\dots,r$. Then $T^{-1}(C')$ is the curve whose proper inverse transform represents the class $s_1(v)$.

Conversely, let σ be a Cremona isometry. Then σ maps the classes e_i $(i = 1,\dots,r)$ to the classes e_i' represented by exceptional curves of the first kind (this easily follows from the adjunction formula). Blowing them down, we get another birational morphism $f'\colon X \to \mathbf{P}^2$. Then, the composition $T = f^{-1} \circ f'$ is a birational automorphism of $\mathbf{P}^2$ based among the points $p_1,\dots,p_r$. If T is projective, then $e_1',\dots,e_r'$ is a permutation of the set $e_1,\dots,e_r$. This implies that σ is induced by a composition of the simple reflections $s_1,\dots,s_{r-1}$, which generate the symmetric group on the set $e_1,\dots,e_r$. If T is the standard quadratic transformation based on the points p_1, p_2, p_3, then e_1 is mapped to $e_0 - e_2 - e_3$, e_2 to $e_0 - e_1 - e_3$, e_3 to $e_0 - e_1 - e_2$, e_i to e_i $(i > 3)$, e_0 to $2e_0 - e_1 - e_2 - e_3$. It is immediately checked that this mapping coincides with the simple reflection s_r. In the general case we represent T as a composition of the standard quadratic transformation based at p_1, p_2 and p_3 and projective transformations. This can be done by using the Max Noether theorem which easily follows in our case from the lemma proven in the next section.

Remark. The assertion of the theorem was certainly known to Coble and others. But it never was stated so explicitly. There are also some hints (see [**Co1**, p. 22; or **DV**]) that the following stronger assertion is true. Let W^n be the subgroup of W which leaves the set of nodal roots invariant. Then $W^n = \mathrm{Cris}(X)$. In this form the theorem was proven by E. Looijenga [**Lo2**] in the case when the points $p_1,\dots,p_r$ lie on a cubic curve.

5. Exceptional curves. The following result is the key point in proving the Max Noether theorem cited above.

LEMMA (NOETHER'S INEQUALITY). *Let C be an irreducible curve of degree m in $\mathbf{P}^2$ passing through points $p_1,\dots,p_r$ with multiplicities $m_1 \geqslant \cdots \geqslant m_r$. Suppose that $m_2 \neq 0$, the proper inverse transform $\bar{C}$ of C in $\Sigma(p_1,\dots,p_r)$ is a nonsingular rational curve and $-2 \leqslant (\bar{C}^2) \leqslant 1$. Then $m_1 + m_2 + m_3 > m$.*

PROOF (FOLLOWS [**Go**]). We may assume that $m > 1$. Clearly,

$$m_1 \geqslant \rho = \frac{m_1^2 + \cdots + m_r^2}{m_1 + \cdots + m_r} = \frac{m^2 - (\bar{C}^2)}{3m - 2 - (\bar{C}^2)}.$$

Let $m_1 = \rho + \delta$, $\delta \geqslant 0$. Then

$$m_2 \geqslant \rho' = \frac{m_2^2 + \cdots + m_r^2}{m_2 + \cdots + m_r} = \rho - \delta\frac{m_1}{m_2 + \cdots + m_r}.$$

Let $m_2 = \rho' + \delta'$, $\delta' \geqslant 0$. Then

$$m_3 \geqslant \rho'' = \frac{m_3^2 + \cdots + m_r^2}{m_3 + \cdots + m_r} = \rho' - \delta'\frac{m_2}{m_3 + \cdots + m_r}.$$

Let K be the canonical class of $X = \Sigma(p_1,\dots,p_r)$. We have

$$\nu = (\bar{C}^2) = m^2 - m_1^2 - \cdots - m_r^2, \quad (\bar{C}K) = -3m + m_1 + \cdots + m_r = -2 - \nu.$$

Since $m_1 + m_2 \leqslant m$ (pass a line through the points p_1 and p_2), we obtain

$$m_2 + \cdots + m_r = 3m - 2 - \nu - m_1 \geqslant 2m - 1 - \nu,$$
$$m_3 + \cdots + m_r = 3m - 2 - \nu - m_1 - m_2 \geqslant m - \nu.$$

Therefore,

$$\rho' \geqslant \rho - \delta\frac{m-1}{2m-1-\nu}, \quad \rho'' \geqslant \rho' - \delta'\frac{m-1}{m-\nu},$$

and

$$\begin{aligned} m_1 + m_2 + m_3 &\geqslant (\rho + \delta) + (\rho' + \delta') + \rho'' \geqslant \rho + \delta + \rho' + \delta' + \rho' - \delta'\frac{m-1}{m-\nu} \\ &\geqslant \rho + \delta + 2\rho - \delta\frac{2m-2}{2m-1-\nu} + \delta'\left(1 - \frac{m-1}{m-\nu}\right) \\ &= 3\rho + \delta\frac{1-\nu}{2m-1-\nu} + \delta'\frac{1-\nu}{m-\nu} \geqslant \frac{3m^2 - 3\nu}{3m-2-\nu} > m. \end{aligned}$$

COROLLARY 1. *Let $X = \Sigma(p_1,\dots,p_r)$, where $p_1,\dots,p_r$ are in general position (maybe $R^n = \varnothing$ is sufficient, see the remark in* §4). *There exists a bijection between the set of exceptional curves of the first kind on X and the orbit $W\omega_{r-1}$, where ω_{r-1} is the fundamental weight defined as in* §3. *In particular, the Weyl group W acts transitively on the set of exceptional curves.*

PROOF. By Theorem 1, $W = \mathrm{Cris}(X)$. Take the exceptional curve E_r represented by the class $e_r = \omega_{r-1}$. Then, for every $s \in W$, $s(E_r)$ is an exceptional curve. Hence, $W\omega_{r-1}$ is contained in the set of exceptional curves. Let E be an exceptional curve on X and $me_0 - m_1'e_1 - \cdots - m_r'e_r$ be its class in $A^1(X)$.

Applying the elements of the subgroup S_r generated by the simple reflections $s_1, \ldots, s_{r-1}$, we will transform E to an exceptional curve E' represented by the class $me_0 - m_1e_1 - \cdots - m_{r'}e_{r'}$ ($r' \leqslant r$), where $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_{r'} > 0$. Then $E' = \overline{C}$ in the notation of the lemma. If $r \leqslant 2$, then an elementary argument shows that either $r = 1$ and $(m, m_1) = (0, 1)$, or $r = 2$ and $(m, m_1, m_2) = (1, 1, 1)$. In the first case, applying elements of S_r, we transform E' to the curve E_r. In the second case, $s_r(E') = e_3$, and again applying elements of S_r we will transform e_3 to e_r. If $r > 2$, then by the lemma $m_1 + m_2 + m_3 > m$. Applying the reflection s_r we will transform the class of E' to the class $(2m - m_1 - m_2 - m_3)e_0 - (m - m_2 - m_3)e_1 - (m - m_1 - m_3)e_2 - (m - m_1 - m_2)e_3 - m_4e_4 - \cdots - m_{r'}e_{r'}$. Since $2m - m_1 - m_2 - m_3 < m$, repeating this process, we come to the case $r \leqslant 2$. Thus, any exceptional curve can be transformed to the curve E_r.

REMARK. This result is also repeatedly mentioned in the classic literature (see, for example [**DV**]). In [**Lo2**] Looijenga proves this result in the case where $p_1, \ldots, p_r$ lie on a cubic, replacing W by the group $\mathrm{Cris}(X)$. His proof is completely different from the above proof.

COROLLARY 2. *Under the assumption of Corollary* 1 *there are no nodal curves on* X (*i.e. nonsingular rational curves* C *with* $(C^2) = -2$).

PROOF. Using the same argument as in the previous proof, we will transform such a curve to a curve of the class $e_0 - e_1 - e_2 - e_3$. But the existence of such a curve would imply that the points p_1, p_2 and p_3 lie on a line. This contradicts the assumption of general position.

COROLLARY 3. *Under the assumptions of Corollary* 1, *the number of exceptional curves of the first kind on* X *is finite if and only if* $r < 9$.

PROOF. If $r < 9$, then W is finite and everything is finite. If $r = 9$, then by Corollary 1 the number of exceptional curves is equal to the index of the isotropy subgroup of ω_8 in W. As was explained in §2, the group W_{ω_8} is the Weyl group $W_{B'}$, where $B' = B - \{\alpha_8\}$. Clearly, $W_{B'}$ is isomorphic to the Weyl group of the root system of type E_8 and, hence, finite. Since W is infinite for $r > 8$, we get the result in this case. If $r > 9$, then we realize $\Sigma(p_1, \ldots, p_r)$ as the blow up of the point p_r on $\Sigma(p_1, \ldots, p_{r-1})$. By induction, we may assume that $\Sigma(p_1, \ldots, p_{r-1})$ has infinitely many exceptional curves. Since none of them can pass through p_r (otherwise, we have a nodal curve on $\Sigma(p_1, \ldots, p_r)$ that would contradict Corollary 2), their proper inverse transforms in $\Sigma(p_1, \ldots, p_r)$ are an infinite set of exceptional curves on X.

Let $T\colon \mathbf{P}^2 \to \mathbf{P}^2$ be a birational automorphism of $\mathbf{P}^2$ with the fundamental points $p_1, \ldots, p_r$. It is given by a two-dimensional linear system (a net) of curves of certain degree m passing through $p_1, \ldots, p_r$ with certain multiplicities $m_1, \ldots, m_r$ respectively. A proper inverse transform of this system in $X = \Sigma(p_1, \ldots, p_r)$ is represented by a class $me_0 - m_1e_1 - \cdots - m_re_r$. Such a class is represented by a nonsingular rational curve of self-intersection 1. Conversely, given an effective class $v = me_0 - m_1e_1 - \cdots - m_re_r$ with $v^2 = 1$ and $vK = -3m + m_1 + \cdots + m_r = -3$, it represents a linear system which defines a birational automorphism of $\mathbf{P}^2$ with fundamental points among $p_1, \ldots, p_r$ (a homaloidal net). The corresponding vector $(m, m_1, \ldots, m_r)$ is called the *type* of a homaloidal net (or, of a birational automorphism). The number m is called the *degree* of T.

COROLLARY 4. *Under the assumption of Corollary* 1, *the number of types of birational automorphisms with the fundamental points among the points* $p_1, \ldots, p_r$ *is finite if and only if* $r < 9$.

PROOF. Again, as in the proof of Corollary 1, we transform the class $me_0 - m_1 - \cdots - m_re_r$ corresponding to a type of a birational transformation T to the class $e_0 = \omega_r$. Thus, the set of types is bijective to the orbit $W\omega_r$. Since the isotropy subgroup of ω_r is always finite (it is isomorphic to $W_{B-\{\alpha_r\}} = W(A_{r-1}) = S_r$), we get that this orbit is finite if and only if W is finite, i.e. $r < 9$.

REMARKS. 1. In the cases E_6 and E_7 the W-orbit of ω_{r-1} is the weight diagram for the corresponding irreducible representation of a simple Lie algebra of type E_r (see [**Hum**]). This representation occurs in the space $\mathbf{R}^{[W:\ W']}$, where W' is the isotropy subgroup of ω_{r-1}. For example, when $r = 6$ we have a 27-dimensional irreducible representation of a simple Lie algebra of type E_6. It is known (see E. Cartan's thesis) that there exists an invariant *cubic* homogeneous form in this representation. Similarly, for E_7 we have a 56-dimensional representation with an invariant *quartic* form. Now, recall that classically $W(E_6)$ (resp. $W(E_7)$) is related to a *cubic* surface: $\Sigma(p_1, \ldots, p_6)$ is a nonsingular cubic surface if the points $p_1, \ldots, p_r$ are in general position, $W(E_6)$ is the group of 27 lines on a cubic surface (resp. to a *quartic* curve in $\mathbf{P}^2$: $\Sigma(p_1, \ldots, p_7)$ is a double cover of $\mathbf{P}^2$ branched along a quartic curve, $W(E_7)$ is the group of bitangents to this curve). Is there any relation between the two constructions?

2. The set $\{s(\omega_{r-1}),\ s \in W\}$ representing exceptional curves lying on $\Sigma(p_1, \ldots, p_r)$ is the set of vertices of a certain convex polytope in the affine hyperplane $\{v \in V\colon vK = -1\}$. The latter is equipped with a metric, induced by the hyperbolic metric in V, which is of Minkowski type for $r > 9$. The Weyl group W is the full group of symmetries of this polytope. We refer to works of Coxeter for the properties of these polytopes (see [**Cox**]). Some of them are summarized in [**DV**].

6. Cremona isometries (general case). In the case $k > 2$ Max Noether's theorem on decomposition of birational automorphisms of the plane has no analogue. One can still define a standard "quadratic" transformation as a transformation given in projective coordinates by the formula $x_i' = x_0 \cdots x_{i-1}x_{i+1} \cdots x_k$, $i = 0, \ldots, k$. It is given by the linear system of hypersurfaces of degree k passing through the points $(0, \ldots, 1, 0, \ldots, 0)$ with multiplicities $k - 1$. A composition of such a transformation with a projective automorphism of $\mathbf{P}^k$ is called a *standard Cremona transformation* of $\mathbf{P}^k$. Together with projective automorphisms the standard Cremona transformations generate a subgroup (proper, if $k > 2$) of the Cremona group Cr_k of birational automorphisms of $\mathbf{P}^k$. Its elements are called *regular Cremona transformations* of $\mathbf{P}^k$. Such a transformation $T\colon \mathbf{P}^k \to \mathbf{P}^k$ has a fundamental set which is completely determined by a finite set of points $p_1,\ldots,p_r$ (in the sense that all other base conditions follow from assigning certain multiplicities at the given set of points). The inverse transformation T^{-1} has the same property; its fundamental set is determined by a finite set $q_1,\ldots,q_r$ of the *same* number of points. This property is not satisfied by a general Cremona transformation. We will call the above finite sets of points the *fundamental points* of T. Let $f_1\colon X_1 = \Sigma(p_1,\ldots,p_r) \to \mathbf{P}^k$, $f_2\colon X_2 = \Sigma(q_1,\ldots,q_r) \to \mathbf{P}^k$ be the blowings up of the fundamental points of T and T^{-1} respectively. Then there exists a pair of birational morphisms $g_1\colon X \to X_1$ and $g_2\colon X \to X_2$ such that $T \circ f_1 \circ g_1 = f_2 \circ g_2$ (as birational mappings). Let us state an analogue of the theorem of §4 without defining the group of Cremona isometries. The latter can be done using the notion of the "limit Picard group" of $\mathbf{P}^k$ which is rather technical to give here (see [**Is, Ma**]).

THEOREM 2. *Let $X = \Sigma(p_1,\ldots,p_r)$, where $p_1,\ldots,p_r$ are in general position in $\mathbf{P}^k$, $f\colon X \to \mathbf{P}^k$ be the corresponding projection, and W the Weyl group of the root base in $V = A^1(X)_{\mathbf{R}}$ constructed in* §3. *For every $s \in W$ there exists a birational morphism $g_1\colon X' \to X$ and $g_2\colon X' \to \mathbf{P}^k$ such that g_2 blows down the classes $g_1^*(s(e_i))$ $(i \neq 0)$ to points and the mapping $g_2 \circ g_1^{-1} \circ f^{-1}$ is a regular Cremona transformation. Conversely, given a regular Cremona transformation $T\colon \mathbf{P}^k \to \mathbf{P}^k$ with fundamental points among the points $p_1,\ldots,p_r$, there exists an element $s \in W$ which induces T in the way described above.*

COROLLARY (NOETHER'S INEQUALITY). *Let T be a regular Cremona transformation of $\mathbf{P}^k$ with fundamental points $p_1,\ldots,p_r$ in general position. Let $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_r \geqslant 0$ be the multiplicities of a general member of the corresponding linear system $p_1,\ldots,p_r$ respectively and m be its degree. Then*

$$m_1 + \cdots + m_{k+1} > (k-1)m$$

unless T is a projective automorphism.

PROOF. Let v be the class of the proper inverse transform of a general member of the linear system defining T, then $v = me_0 - m_1e_1 - \cdots - m_re_r$. Let $s \in W$ be an element of the Weyl group which defines T in the way described in the theorem. Then $s(e_0)$ is represented by the inverse transform of a hyperplane (under $g_1^{-1} \circ g_2$), which is equal to v. Thus, e_0 and v lie in the same W-orbit. However, e_0 lies in the closure of the fundamental Weyl chamber described by the inequalities

$$m_1 - m_2 \geqslant 0,\ldots,m_{r-1} - m_r \geqslant 0, (k-1)m - m_1 - \cdots - m_{k+1} \geqslant 0.$$

Since the closure of the fundamental Weyl chamber is a fundamental domain, v cannot lie there unless it is equal to e_0. This shows that the last inequality is satisfied only if $v = e_0$, i.e. T is a projective automorphism.

REMARK. Surprisingly enough I could not find this result in the classic literature on birational transformations of projective spaces (see, for example [**Hud**]).

7. Weyl groups as groups of Cremona transformations. Let $P_k = \mathrm{PGL}(k)$ be the group of projective automorphisms of $\mathbf{P}^k$. It acts in the diagonal way on the product $(\mathbf{P}^k)^r$. Assume $r \geqslant k+2$, then the quotient of an open subset of $(\mathbf{P}^k)^r$ is a rational variety of dimension $k(r-k-2)$. In fact, if $p_1,\dots,p_r$ are in general position, then we may translate the first $k+2$ of them to the $k+2$ fixed points in $\mathbf{P}^k$ and then normalize the projective coordinates of each of the remaining $r-k-2$ points by requiring that all of them have the same last coordinate. In this way we get exactly $(r-k-2)k+1$ independent homogeneous coordinates.

Let W_r^k be the Weyl group of the root base in $A^1(\Sigma(p_1,\dots,p_r))_{\mathbf{R}}$ constructed in §3 (it corresponds to the Dynkin diagram $T_{k+1,r-k+1,2}$). A given ordered set of points $p_1,\dots,p_r$ in general position in $\mathbf{P}^k$ defines an isomorphism of W_r^k (considered as an abstract group, not related to any root system) to the corresponding Weyl group. Take an element $s \in W$, it defines a regular Cremona transformation $T\colon \mathbf{P}^k \to \mathbf{P}^k$ with the fundamental points among the p_i's. Let $q_1,\dots,q_r$ be the points to which the exceptional divisors $s(e_i)$ are blown down (see Theorem 2). Of course, the ordered set $q_1,\dots,q_r$ is defined up to a P_k-translate. Also, if we replace the original r-tuple $(p_1,\dots,p_r)$ by a P_k-translate, we get the same r-tuple $(q_1,\dots,q_r)$ (again up to a P_k-translate). In this way we obtain a homomorphism (*a Coble representation*)

$$c_r^k\colon W_r^k \to \mathrm{Cr}_{k(r-k-2)}.$$

To see why the action of W_r^k on $(\mathbf{P}^k)^r/P_k$ described above is really realized by Cremona transformations, we check it first for the simple reflections s_i $(i \neq r)$ and then for s_r. The first reflections s_i correspond to permutations of points $p_1,\dots,p_r$ and this certainly defines a Cremona transformation on $(\mathbf{P}^k)^r$. The reflection s_r corresponds to the standard Cremona transformation with the fundamental points $(0,\dots,1,\dots,0)$.

LEMMA. *Let $(p_1,\dots,p_r) \in (\mathbf{P}^k)^r/P_k$. Then, it is fixed by an element $s \in W$ (with respect to the Coble representation) if and only if s is induced by an automorphism of $X = \Sigma(p_1,\dots,p_r)$.*

PROOF. By definition $c_r^k(s)(p_1,\dots,p_r) = (q_1,\dots,q_r)$, where $s(e_i)$ are blown down to q_i. If s is induced by an automorphism g, then the composition $f' \circ g$, where $f'\colon X \to \mathbf{P}^k$ is the above blowing down, blows down e_i to the point q_i. Thus $(q_1,\dots,q_r)$ is a P_k-translate of $(p_1,\dots,p_r)$, and s fixes $(p_1,\dots,p_r)$. Conversely, if $c_r^k(s)$ maps $(p_1,\dots,p_r)$ to a projective equivalent m-tuple $(q_1,\dots,q_r)$, then we may assume that the $s(e_i)$'s are blown down to the same points p_i. Since any blowings up of the same set of points are isomorphic, we get an automorphism of X, which sends the classes e_i to the classes $s(e_i)$. This automorphism induces s.

COROLLARY. *The homomorphism* c_r^2: $W_r^2 \to \mathrm{Cr}_{2r-8}$ *is injective, except in the following cases*:

(a) $r \leqslant 4$, *where* c_r^2 *is trivial*;

(b) $r = 5$, *where* $\mathrm{Ker}(c_5^2) = (\mathbf{Z}/2)^4$;

(c) $r = 7$, *where* $\mathrm{Ker}(c_7^2) = \mathbf{Z}/2$;

(d) $r = 8$, *where* $\mathrm{Ker}(c_8^2) = \mathbf{Z}/2$.

PROOF. This follows from the following facts: a general $\Sigma(p_1,\dots,p_r)$ does not have nontrivial automorphism for $r > 8$ (see [**Co3**, p. 355; and **Gi**])[2]; a general $\Sigma(p_1,\dots,p_7)$ (isomorphic to a double cover of $\mathbf{P}^2$ branched along a quartic curve) has only one nontrivial automorphism whose order is 2; a general $\Sigma(p_1,\dots,p_8)$ (isomorphic to a double cover of a singular quadric in $\mathbf{P}^3$) has only one nontrivial automorphism, also of order 2; a general cubic surface does not have nontrivial automorphisms; a general $\Sigma(p_1,\dots,p_5)$ (isomorphic to the intersection of two quadrics $\Sigma x_i^2 = 0$ and $\Sigma a_i x_i^2 = 0$ in $\mathbf{P}^4$) has its group of automorphisms isomorphic to $(\mathbf{Z}/2)^4$.

EXAMPLE. For $r = 5$, the image of c_5^2 is isomorphic to the symmetric group S_5 which acts birationally on $\mathbf{P}^2$ by means of automorphisms of the Clebsch cubic surface $\Sigma x_i = 0$, $\Sigma x_i^3 = 0$ in $\mathbf{P}^4$.

REMARK. In the cases $r = 7$ and 8, the kernel of c_r^2 is generated by the unique element of the Weyl group W which acts on the root lattice by the formula $x \to -x$ (it is the element of maximal length in W, see [**Bo**]). The corresponding birational automorphism of $\mathbf{P}^2$ is the Geiser ($r = 7$) or the Bertini ($r = 8$) involution.

If c_r^k is restricted to an invariant subset of $(\mathbf{P}^k)^r$ corresponding to points in a certain special position, then the kernel of the corresponding representation may present an interesting normal subgroup of W, sometimes of finite index. We give two examples of this kind which can be found in [**Co1**] (see also [**Gi**]). In the first, one takes the set of 9-tuples in $\mathbf{P}^2$, which are the base points of a pencil of cubic curves. The kernel of c_9^2 in this case is of finite index in $W_9^2 = W(\tilde{E}_8)$, it is isomorphic to a certain double extension of the free abelian subgroup of rank 8 in $W(\tilde{E}_8)$. The quotient group is isomorphic to $W(E_8)/\pm 1$, a double extension of a *simple* group. In the second, one takes the set of 10-tuples which can be realized as the ten nodes of a rational sextic plane curve. In this case, the kernel of the representation of W_{10}^2 on this set is the congruence subgroup mod 2 (i.e. the subgroup of W_{10}^2 which acts identically on the root lattice mod 2). The quotient is a double extension of a *simple* group of order $2^{20} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17 \cdot 31$. It is isomorphic to the (even) orthogonal group $O^+(10,2)$. Recall that for every positive g, the symplectic group $\mathrm{Sp}(2g,2)$ contains two orthogonal subgroups $O^+(2g,2)$ and $O^-(2g,2)$ of index $2^{g-1}(2^g+1)$ and $2^{g-1}(2^g-1)$ respectively. In [**Co2**] A. Coble

[2]As was pointed out to me by M. Gizatullin the proof of Coble requires a justification. In [**Gi**] this result is proven only for $r = 9$.

proved that the reduction of $W_r^k \bmod 2$ is always isomorphic to one of the groups $\mathrm{Sp}(2g, 2)$, $O^+(2g, 2)$, $O^-(2g, 2)$, or their extensions by 2-elementary groups. For example, for $k = 2$, $r = 7$ (resp. $k = 2$, $r = 8$) we get a well-known isomorphism $W(E_7)/\pm 1 = \mathrm{Sp}(6, 2)$ (resp. $W(E_8)/\pm 1 = O^+(8, 2)$). A modern approach to classifying the groups W_r^k mod any prime has been outlined to me by R. Griess (see **[Gr]**). Of course, the appearance of the groups $\mathrm{Sp}(2g, 2)$ suggests to look for some relations to theta functions of algebraic curves of genus g. In fact, there is a deep relation between the theory outlined in this talk and the theory of moduli of algebraic curves. But this is another story (see **[Co1]**).

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