

ENRIQUES SURFACES: WHAT IS LEFT?

IGOR DOLGACHEV *

Abstract. *In this talk I would like to comment on some open problems in the theory of Enriques surfaces. I am not assuming that the ground field k is the field of complex numbers. One of my main problems is to get rid of the transcendental techniques in the study of Enriques surfaces.*

1. WHAT IS KNOWN?

The main references here are [B-P-vdV], [C-D 3], [Ba], [Pe]. The full account is not expected. Only highlights are discussed.

Recall that an Enriques surface is a nonsingular minimal projective surface with numerically trivial canonical class and the second Betti number equal to 10. Together with $K3$ -surfaces, abelian surfaces and hyperelliptic surfaces, they occupy the class of algebraic surfaces of Kodaira dimension zero. If $p = \text{char}(k) \neq 2$, an Enriques surface F can be characterized equivalently by the conditions that $p_g = 0$, $2K_F \sim 0$. The unramified cover corresponding to the torsion element K_F in the Picard group of F is a $K3$ -surface, called the $K3$ -cover of F . Conversely, every $K3$ -surface with a fixed-point-free involution is isomorphic to the $K3$ -cover of an Enriques surface. So, if $p \neq 2$, the theory of Enriques surfaces becomes a part of the theory of $K3$ -surfaces.

a) «*Very old*» results. The first construction of a surface with $p_g = 0$ but $2K_F = 0$ was given by Federico Enriques in 1896 [En 3]. Together with another example of Guido Castelnuovo, given in the same year (this time $2K_F > 0$), these were the first examples of non-rational surfaces with $p_g = 0$. Other constructions and

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properties of Enriques surfaces were later presented in Enriques's book [En 3]. For instance, one finds there that every such surface contains an elliptic pencil with two double multiple fibres and that it admits a rational map of degree 2 onto the projective plane. Enriques found the ramification curve of this double cover and showed that it may degenerate if the surface contains a smooth rational curve intersecting a general member of some elliptic pencil at 2 points (a *special* Enriques surface). He showed that a generic surface admits a birational model isomorphic to a surface of degree 6 passing doubly through the edges of the coordinate tetrahedron. Again, as was first noticed by Castelnuovo, it may degenerate. Another observation is the fact that an Enriques surface can be obtained as the quotient of a $K3$ -surface (a regular surface with trivial canonical class) by a fixed-point-free involution. Finally he indicated a proof that every generic Enriques surface has infinitely many automorphisms and raised a question about the existence of one with only finitely many automorphisms [En 2].

In 1901 Gino Fano [Fa 1] discovered a model of an Enriques surface as a surface of degree 10 in \mathbb{P}^5 . He showed that the Reye congruence of lines in \mathbb{P}^3 (the variety of lines contained in a subpencil of a generic web of quadrics) is an Enriques surface. In the Plücker embedding it is represented by certain surface of degree 10 in \mathbb{P}^5 . These surfaces depend on 9 parameters, one less than the number of moduli for Enriques surfaces. Much later Fano proved that a generic surface can be embedded into \mathbb{P}^5 as a surface of degree 10 not necessarily lying on a quadric [Fa 3]. In 1910 Fano answered a question of Enriques by constructing a first example of an Enriques surface with finitely many automorphisms [Fa 2].

b) «Old» results. During the sixties, when the theory of algebraic surfaces was brought to the light of modern mathematics, Enriques's results were reconstructed, by using modern techniques, in the thesis of Michael Artin [Ar 1] at M.I.T. and Boris Averbukh [AS], Chapter IX, [Av], at Moscow. In the seventies, while extending Enriques's classification of surfaces to the case of characteristic $p > 0$, Enrico Bombieri and David Mumford studied Enriques surfaces in characteristic 2 (first giving the right definition) [B-M]. They showed that Enriques surfaces are divided into 3 classes according to the structure of their Picard scheme. The τ -part of the latter is a group scheme of order 2, and as such, is isomorphic to μ_2 , α_2 , or $\mathbb{Z}/2$. Thus we speak now about μ_2 -surfaces, α_2 -surfaces and $\mathbb{Z}/2$ -surfaces. For every $\mathbb{Z}/2$ -surface, $p_g = 0$, $H^1(F, \mathcal{O}_F) = 0$ and $2K_F \sim 0$, as in the case $p \neq 2$. For this reason, they are called *classical* Enriques surfaces. For surfaces from the remaining two classes $H^1(F, \mathcal{O}_F) = k$ and $K_F \sim 0$. The double cover of an Enriques surface corresponding to the τ -part of the Picard scheme is a principal cover with respect to the dual group $\mu_2^* \cong \mathbb{Z}/2$, $\alpha_2^* \cong \alpha_2$, $(\mathbb{Z}/2)^* \cong \mu_2$. Only for μ_2 -surfaces the double cover is unramified and

is isomorphic to a $K3$ -surface. In the remaining two cases, the cover is purely inseparable, the corresponding surface is singular, and birationally isomorphic to either a $K3$ -surface or a rational surface. Bombieri and Mumford proved also that every Enriques surface contains a genus 1 fibration (elliptic or quasi-elliptic if $p = 2$).

c) *New results.* It is difficult to trace out the reason for the sudden explosion in the research on Enriques surfaces during the eighties. My feeling is that the following three papers have played some important role in this. The first is the epochal paper of Ilia Piatetski-Shapiro and Igor Shafarevich [PS-S] on the periods of $K3$ -surfaces. This paper gave us a very important tool for the study of moduli and automorphisms of complex Enriques surfaces and demonstrated the importance of applications of the arithmetic of integral quadratic forms to the study of $K3$ -surfaces (though the latter can be traced back to Francesco Severi, see for example [Se]). Using these methods Eiji Horikawa and Slava Nikulin described the moduli of Enriques surfaces by means of the periods of the corresponding $K3$ -covers [Ho], [Ni 3]. Nikulin, and independently, Wolf Barth and Chris Peters were able to compute the automorphism group of a generic Enriques surface. In fact, Nikulin showed how to compute the automorphism group of any Enriques surface in terms of the period point of the corresponding $K3$ -cover [Ni 3]. They revealed the importance of an isomorphism between the Picard lattice of a complex Enriques surface modulo the torsion part and the lattice E_{10} , an even unimodular lattice of signature (1,9). The latter is isomorphic to the direct sum of the root lattice of type E_8 (with minus sign) and the standard hyperbolic plane H . It can also be described by the Dynkin diagram of type E_{10} (or $T_{2,3,7}$), whose importance in the theory of surface singularities was earlier noted in the works of many people.

The second paper is by Bernard Saint-Donat [SD], it contains a systematic study of linear systems on $K3$ -surfaces. Afterwards similar results in the case of an Enriques surface were obtained by Francois Cossec [Co 1], [Co 2]. He was the first to realize that the arithmetic of the lattice E_{10} can also be applied to the study of linear systems. For the first time the word «generic» was excluded from the statement of many classical theorems on projective models of Enriques surfaces. For example, Cossec proved (first in the case $p \neq 2$), that every nodal Enriques surface (i.e. a surface with a smooth rational curve, called a *nodal curve*) is special in the sense of Enriques. Nevertheless, even a nodal surface admits a non-degenerate model as a double cover, or as a surface of degree 6 passing doubly through the edges of a tetrahedron, or as a quotient of a $K3$ -surface of degree 8 in \mathbb{P}^5 , or as a surface of degree 10 in \mathbb{P}^5 with at most rational double points. Later these results were extended to characteristic 2 and presented in [C-D 3].

Finally, the third crucial paper is by Artin [Ar 2]. It develops the theory of

periods of supersingular $K3$ -surfaces in positive characteristic. It excited the interest of Shafarevich in the study of $K3$ -surfaces in characteristic p . Thus, he and Alexei Rudakov proved the absence of non-trivial regular vector fields on $K3$ -surfaces [R-S 1] and later the unirationality of supersingular $K3$ -surfaces in characteristic 2 [R-S 2]. This provoked a similar study of Enriques surfaces in characteristic 2 [Bl], [Cr], [Ek], [Il], [Ka], [La 1], [La 2].

Much work was done on automorphisms of Enriques surfaces. We have already mentioned the work of Barth, Peters, and Nikulin. A purely geometric computation of the automorphism group of a generic nodal Enriques surface over a field of characteristic > 17 was given in [C-D 2]. In [Do 1] I gave an example of an Enriques surface with only finitely many automorphisms, believing that it is the first one until I discovered the existence of Fano's example. Using the period description of Enriques surfaces, all Enriques surfaces with finitely many automorphisms were later classified by Nikulin [Ni 3]. Finally, S. Kondō [Kō] obtained this classification from a purely geometric point of view. An explicit construction of each of the seven possible families (one of them was first omitted by Nikulin) was given. Genus 1 fibrations (elliptic and quasi-elliptic) on Enriques surfaces were studied in detail in [C-D 3]. We know all possible types of degenerate fibres, the number of the orbits of the automorphism group in the set of elliptic fibrations on generic complex Enriques surfaces [B-P] and generic nodal Enriques surfaces ($p = 0$ or $p > 17$) [C-D 1].

Special attention was given to a class of Enriques surfaces obtained as Reye congruences [Co 3], [G-H] (see below). It is proven that every nodal Enriques surface of degree 10 in \mathbf{P}^5 is isomorphic to a Reye congruence. Also the theory of Reye congruences was extended to the case of characteristic 2 [C-D 4].

2. WHAT IS LEFT?

A. Moduli

1. *Local moduli.* A universal Kuranishi family of complex Enriques surfaces exists and is smooth of dimension 10 (see [B-P-vdV]). If $p \neq 2$, or $p = 2$ and F is a μ_2 -surface, the local deformation functor of an Enriques surface F is prorepresentable by the ring $k[[t_1, \dots, t_{10}]]$ (see [La 1]). We do not know whether classical or α_2 -surfaces have local moduli. For the latter surfaces there is a nice construction of rigidified local moduli [Ek].

2. *Polarized moduli.* The existence of the moduli space of polarized Enriques surfaces is known only in a few cases (same as for $K3$ -surfaces, by the way).

An element $h \in \text{Pic}(F)$, $h^2 > 0$, is called a *polarization* if h is numerically effective (nef) and the corresponding linear system is base-point-free. One can

prove that the latter condition is satisfied if and only if h is numerically effective and satisfies:

$$(*) \quad |h \cdot f| \geq 2 \text{ for every isotropic vector } f \in \text{Pic}(F).$$

For every $h' \in \text{Pic}(F)$, $h'^2 > 0$, satisfying $(*)$ there exists a unique polarization h such that $\sigma(h') = h$ for some automorphism σ of the Picard group $\text{Pic}(F)$ which preserves the intersection form. Also it is known that every h can be represented by a sum of classes of irreducible curves of arithmetic genus 1 or 0. We say that h is *non-special* if h can be represented by a sum of irreducible curves of arithmetic genus 1 only, we say that h is *special* otherwise. A *polarized Enriques surface* is a pair (F, h) consisting of an Enriques surface and a polarization h . Two polarized surfaces (F, h) and (F', h') are isomorphic if there exists an isomorphism of surfaces $f: F \rightarrow F'$ such that $f^*(h') = h$. One defines naturally a family of polarized Enriques surfaces, defines the corresponding functor, and asks about the existence of the corresponding coarse moduli scheme.

Let $\pi: X \rightarrow S$ be a family of polarized Enriques surfaces. Since $R^1\pi_*G_m$ is locally constant, we can identify the Picard groups of all surfaces X_s with the fixed quadratic group $P = E_{10} \oplus \mathbb{Z}/2$ (we assume here that $p \neq 2$). Clearly the polarization classes $h_s \in \text{Pic}(X_s)$ belong to the same orbit with respect to the action of the orthogonal group $O(P)$ of $P(\cong (\mathbb{Z}/2)^{10} \rtimes O(E_{10}))$.

PROBLEM 1. *Is it true that for every positive integer k there exists a non-special polarization h of F with $h^2 = 2k \geq 4$?*

Note that special polarizations of degree ≥ 4 always exist and there are no polarizations h with $h^2 = 2$.

PROBLEM 2. *Does the coarse moduli variety \mathcal{M}_{2k}^{ns} of Enriques surfaces with non-special polarization of degree $2k$ exist?*

PROBLEM 3. *Is it true that the number of irreducible components of \mathcal{M}_{2k}^{ns} is equal to the number of orbits in the set of polarizations of degree $2k$ with respect to the group $O(P)$?*

We will discuss the cases $d = 2k = 4, \dots, 10$.

$d = 4$. There is only one orbit of polarization vectors h of degree 4. Each non-special polarization can be represented by a divisor of type $E_1 + E_2$, where $|2E_1|$ is a genus 1 pencil, and $E_1 \cdot E_2 = 2$. Every Enriques surface admits

such a polarization [C-D 3]. The complete linear system $|h|$ defines a morphism of degree 4 onto \mathbb{P}^2 . If $p \neq 2$, the branch curve of this map was described by A. Verra [Ve 1]. We do not know the answer to Problem 2 in this case.

Note that the inverse image of h in the $K3$ -cover X defines a linear system which maps X birationally onto a complete intersection of three quadrics in \mathbb{P}^5 with at most double rational points as singularities [Co 1], [Ve 2].

$d = 6$. There is only one orbit of polarization vectors h of degree 6. Each non-special polarization can be represented by a divisor of type $E_1 + E_2 + E_3$, where $|2E_i|$ is a genus 1 pencil, and $E_1 \cdot E_j = 1$, $i \neq j$. Every Enriques surface ($p \neq 2$) admits such a polarization [C-D 3]. Assume

$$(*) \quad |E_i + E_j - E_k| = \emptyset \text{ for all distinct } i, j \text{ and } k.$$

Then the complete linear system $|h|$ defines a birational morphism onto a surface \overline{F} of degree 6 in \mathbb{P}^3 . If h satisfies (*), then \overline{F} is an Enriques sextic, i.e. a surface of degree 6 with double lines along the edges of the coordinate tetrahedron. Certainly, this is always true for unnodal Enriques surfaces. If h does not satisfy (*), $|h|$ maps F onto a symmetric cubic surface in \mathbb{P}^3 . In this case the polarization $h + K_F$ satisfies (*), and $|h + K_F|$ maps F onto a degenerate Enriques sextic corresponding to a degenerate tetrahedron (note here a mistake in Corollary 4.7.9 in [C-D 3], where the word «non-degenerate» must be deleted). If $p \neq 2$, this allows us to construct \mathcal{M}_6^{ns} as a geometric quotient of the space of Enriques sextics. This is again an irreducible rational variety of dimension 10.

$d = 8$. There are two orbits of polarization vector h of degree 8. Each non-special polarization belonging to the first orbit can be represented by a divisor of type $2E_1 + 2E_2$, where $|2E_i|$ is a genus 1 pencil, and $E_1 \cdot E_2 = 1$. Every Enriques surface admits such a polarization [C-D 3]. The complete linear system $|h|$ defines a morphism of degree 2 onto a non-degenerate symmetric quartic Del Pezzo surface \mathcal{Q} . If $p \neq 2$, this is a unique (up to isomorphism) quartic Del Pezzo surface with 4 ordinary double points. The branch locus consists of the singular locus of \mathcal{Q} and a curve B with simple singularities which is cut out by a quadric not passing through the four nodes of \mathcal{Q} . This allows one to construct $(\mathcal{M}_8^{ns})_1$ as a geometric quotient of the space of such curves B by the automorphism group of \mathcal{Q} ([C-D 4]). This is an irreducible rational variety of dimension 10. If $p = 2$, the map $F \rightarrow \mathcal{Q}$ could be inseparable, and the construction of $(\mathcal{M}_8^{ns})_1$ is more complicated ([C-D 4]). The second orbit is represented by the divisors $E_1 + 2E_2$, where $|2E_i|$ is a genus 1 pencil, and $E_1 \cdot E_2 = 2$. Every non-special polarization of degree 4 represented by the divisor $E_1 + 2E_2$ where $E_1 \cdot E_2 = 2$, defines two

non-special polarizations of degree 8 belonging to the second orbit. They are of the form $E_1 + 2E_2$ or $2E_1 + E_2$. This shows that every Enriques surface admits non-special polarizations of degree 8 of both types. Each polarization of this form defines a birational map onto a non-normal octic surface in \mathbf{P}^4 . Not much known about this surface. Thus, if \mathcal{M}_8^{ns} exists, it must consist of two components; one is isomorphic to $(\mathcal{M}_2^{ns})_1$ and the other, $(\mathcal{M}_2^{ns})_2$, must be a two-sheeted cover of \mathcal{M}_4^{ns} .

$d = 10$. There are two orbits of polarization vectors h of degree 10. Each non-special polarization belonging to the first orbit can be represented by a divisor of type $2E_1 + E_2 + E_3$, where $|2E_i|$ is a genus 1 pencil, and $E_i \cdot E_j = 1$, $i \neq j$. Such a polarization always exists and defines a birational map from F to a non-normal surface of degree 10 in \mathbf{P}^5 . We do not know much about this surface. The second orbit is represented by the divisors $E_1 + E_2 + E_3$, where $|2E_i|$ is a genus 1 pencil, and $E_1 \cdot E_2 = 1$, $E_1 \cdot E_3 = E_2 \cdot E_3$. Such a polarization always exists and defines a birational map from F onto a normal surface of degree 10 in \mathbf{P}^5 which has at most double rational singularities. We do not know yet how to prove the existence of \mathcal{M}_{10}^{ns} . An approach to this problem will be discussed later.

3. *Global moduli.* Contrary to the case of $K3$ -surfaces, where the moduli space of non-polarized surface does not exist as an algebraic variety, one can parametrize the isomorphism classes of non-polarized Enriques surfaces by using the periods of the corresponding $K3$ -covers. The corresponding variety \mathcal{M}_E is a quasi-projective variety isomorphic to an open subset of the factor of a bounded domain of type IV by a discrete group of automorphisms. This was first shown by Horikawa [Ho] and Nikulin [Ni 1] (cf. also [B-P-vdV], Chapter VIII, [Na]).

PROBLEM 4. *Is there a purely geometric analog (a coarse moduli space) of the moduli space \mathcal{M}_E , or, at least, of some open Zariski subset of it?*

For every family of Enriques surfaces $X \rightarrow S$ the period mapping $p : S \rightarrow \mathcal{M}_E$ is defined. In particular, if \mathcal{M}_d^{ns} exists, the period map defines a quasi-finite forgetful mapping:

$$p_d : \mathcal{M}_d^{ns} \rightarrow \mathcal{M}_E.$$

PROBLEM 5. *Define an appropriate compactification of both spaces \mathcal{M}_d^{ns} and \mathcal{M}_E , such that the period map extends to a proper morphism of the compactifications.*

Clearly, a compactifications of \mathcal{M}_d^{ns} must include the space of special polarized

surfaces. I expect that the ramification divisor of the period map will parametrize nodal Enriques surfaces and their degenerations.

In the case $(\mathcal{M}_8^{ns})_1$ this problem has been solved in the works of Horikawa [Ha], Jayant Shah [Sh] and Hans Sterk [St].

PROBLEM 6. *Is there a geometric analog of the map $p_d: \mathcal{M}_d^{ns} \rightarrow \mathcal{M}_E$ living in any characteristic (or, at least, when $p \neq 2$)?*

Here I shall try to convince you that this is not just a mere dream. It is known (over \mathbb{C}) that

$$p: (\mathcal{M}_8^{ns})_1 \rightarrow \mathcal{M}_E$$

is of degree $2^7 \cdot 17 \cdot 31$ [B-P]. One observes that

$$2^7 \cdot 17 \cdot 31 = 2^3(2^4 + 1)2^4(2^5 - 1)$$

and recalls that the number $2^3(2^4 + 1)$ is equal to the number of even theta characteristics on a curve of genus 4, and the number $2^4(2^5 - 1)$ is equal to the number of odd theta characteristics on a curve of genus 5. This suggests that one may try to represent the map p as a composition:

$$(\mathcal{M}_2^{ns})_1 \xrightarrow{f} X_4 \xrightarrow{\varphi} \mathcal{M}_5',$$

where X_4 (resp. \mathcal{M}_5') stands for the 10-dimensional moduli space of pointed curves of genus 4 (resp. some subvariety of codimension 2 of the moduli space of curves of genus 5). The degree of the first (resp. the second) map must be equal to $2^3(2^4 + 1)$ (resp. $2^4(2^5 - 1)$). Let us indicate how one can construct such maps.

We define f as follows. Let (F, h) be a representative of a point of $(\mathcal{M}_8^{ns})_1$. We assume that F is unnodal. As we remarked above, $|h|$ defines a map of degree 2 onto a 4-nodal quartic Del Pezzo surface \mathcal{Q} . Its branch curve is a canonical curve B of genus 5 with two vanishing theta characteristics. Let \mathcal{N} be the net of quadrics passing through B and $H(\mathcal{N}) \subset \mathcal{N}$ be the Hessian curve of \mathcal{N} parametrizing the singular quadrics from \mathcal{N} . Since \mathcal{N} contains two quadrics of rank 3 (defining \mathcal{Q}), $H(\mathcal{N})$ is a quintic with two nodes. Its normalization is a curve C of genus 4. The quintic model of C is given by the linear system $|K_C - q|$, where q is the point residual to the line passing through the nodes of $H(\mathcal{N})$. It is known that the net \mathcal{N} can be uniquely reconstructed from its Hessian curve and an even theta characteristic on its normalization [Be]. Forgetting the theta characteristic, we obtain a point on the variety X_4 . This defines a map

f . To define the map φ , we take a representative (C, q) in the image of f , and consider the birational map $\mathbf{P}^3 \rightarrow \mathbf{P}^4$ defined by the linear system of cubics passing through the canonical model of C in \mathbf{P}^3 . It is known that the image of \mathbf{P}^3 is a cubic hypersurface X given by the equation:

$$x_4 Q(x_0, x_1, x_2, x_3) + F(x_0, x_1, x_2, x_3) = 0$$

where $C = \{Q = F = 0\}$ (see [Be], [Ty]). The point $q = (a_0, a_1, a_2, a_3, a_4) \in C$ defines the line $\ell(q)$ on X which joins the singular point $(0, \dots, 0, 1)$ with the point $(a_0, a_1, a_2, a_3, a_4)$. Projecting from $\ell(q)$, we obtain a conic bundle $\bar{X} \rightarrow \mathbf{P}^2$. Its discriminant curve is a quintic D' with a cusp (the image of the singular point $(1, 0, \dots, 0)$ of X). One easily shows that the normalization D of D' is a trigonal curve of genus 5 with a vanishing theta constant (see [S-V]). Also it is known that the cubic hypersurface can be uniquely reconstructed (up to isomorphism) from the discriminant curve D' as above and an odd theta characteristic on its normalization [Be]. Forgetting the theta characteristic, we obtain the map φ of the needed degree with the image equal to the subvariety $\mathcal{M}'_{5, \text{trig}}$ of trigonal curves of genus 5 with a vanishing theta constant.

In this way we have constructed an unramified map of degree $2^7 \cdot 17 \cdot 31$:

$$\phi : (\mathcal{M}_8^{\text{un}})_1 \rightarrow \mathcal{M}'_{5, \text{trig}}$$

from the moduli space of degree 8 polarized unnodal Enriques surfaces onto the moduli space of trigonal curves of genus 5 with a vanishing theta constant. This leads us to the following:

CONJECTURE 1. *The fibres of ϕ are isomorphic Enriques surfaces. In particular, if $k = \mathbb{C}$, the map ϕ factors through an isomorphism between the open subset of \mathcal{M}_E parametrizing isomorphism classes of unnodal Enriques surfaces and the variety $\mathcal{M}'_{5, \text{trig}}$.*

Note that one can prove that the variety $\mathcal{M}'_{5, \text{trig}}$ is rational (see [Do 2]). This may eventually lead to the proof that \mathcal{M}_E is rational.

Another interesting problem is to extend the map ϕ to a finite map between certain compactifications of the corresponding spaces.

Now I shall discuss another approach to global moduli. This time we consider a polarization of degree 10 which defines a birational morphism $i : F \rightarrow \bar{F} \subset \mathbf{P}^5$ onto a normal surface of degree 10. Assume f is an embedding, for example, F is unnodal. Let $h = f^*(\mathcal{O}_{\mathbf{P}^5}(1))$, then one shows that

$$3h \sim E_1 + \dots + E_{10},$$

where $|2E_i|$ is a genus 1 pencil, and $E_i \cdot E_j = 1$, $i \neq j$. The divisors E_i are defined uniquely up to a permutation and the change $E_i \rightarrow E_i + K_F$. Assume $p \neq 2$. The choice of an ordered sequence of the divisors E_i 's as above, defines an isomorphism of quadratic groups $\varphi : P = E_{10} \oplus \mathbb{Z}/2 \rightarrow \text{Pic}(F)$, such that $\varphi(\Delta) = h$ for some fixed vector Δ in E_{10} and $\varphi(f_i) = E_i$, where (f_1, \dots, f_{10}) is a sequence of isotropic vectors in E_{10} such that $f_i \cdot f_j = 1$, $i \neq j$. Conversely, every such isomorphism defines a unique set (E_1, \dots, E_{10}) as above (see [C-D 3], Chapter 2, § 5). Define a marked Enriques surface as a pair (F, φ) , where F is an Enriques surface and $\varphi : P \rightarrow \text{Pic}(F)$ is an isomorphism of quadratic groups such that $\varphi(\Delta)$ is a polarization of degree 10. Two marked Enriques surfaces (F, φ) and (F', φ') are called isomorphic if there exists an isomorphism $f : F \rightarrow F'$ such that $f^* \circ \varphi = \varphi'^*$. Note that, if E_i is as above, $h \cdot E_i = 3$. Therefore the image of the E_i in \mathbb{P}^5 is a plane cubic curve. Let π_i be the plane containing this curve. The linear system $|E_i + K_F|$ is non-empty and consists of an isolated curve E_i' , the curves $2E_i$ and $2E_i'$ are the two double fibres of the genus 1 pencil $|2E_i|$. Let π_i be the plane containing the image of E_i' in \mathbb{P}^5 . Thus a marked unnodal Enriques surface defines two ordered sets of 10 planes (π_1, \dots, π_{10}) and $(\pi_{-1}, \dots, \pi_{-10})$ satisfying: (i) $\dim \pi_i \cap \pi_j = 0$ if $i + j \neq 0$; (ii) $\pi_i \cap \pi_{-i} = \emptyset$. We call such a set of 20 planes in \mathbb{P}^5 a *double-ten*. Let

$$X \subset G(3, 6)^{10}$$

be the subvariety of the tenth Cartesian power of the Grassman variety $G(3, 6)$ of planes \mathbb{P}^5 parametrizing the 10-tuples of planes which can be extended to a double ten. One easily verifies that every point of X is stable with respect to the natural action of $PGL(6)$ on X . Let

$$\bar{X} = X/PGL(6).$$

CONJECTURE 2. *The variety \bar{X} is a coarse moduli space for marked Enriques surfaces (F, φ) such that the corresponding polarization $\varphi(\Delta)$ defines an embedding into \mathbb{P}^5 .*

Assume this is true. Let $O(P)$ be the orthogonal group of P . We know that $O(P) \cong (\mathbb{Z}/2)^{10} \rtimes O(E_{10})$. Let $O(E_{10}) = O(E_{10})' \times \{\pm 1\}$, the subgroup $O(E_{10})'$ is the Weyl group of the lattice E_{10} . The group $O(P)' = (\mathbb{Z}/2)^{10} \rtimes O(E_{10})'$ acts naturally on the markings, hence it must define an algebraic action on \bar{X} . Every generator $(0, \dots, 1, \dots, 0) \in (\mathbb{Z}/2)^{10}$ acts naturally on X by

$$(\pi_1, \dots, \pi_{10}) \rightarrow (\pi_1, \dots, \pi_{i-1}, \pi_{-i}, \pi_{i+1}, \dots, \pi_{10}).$$

It is known that $O(E_{10})'$ is generated by the permutation group \sum_{10} and a reflection s_0 which acts on E_{10} by the formula:

$$x \rightarrow x + (x \cdot \alpha)\alpha,$$

where $\alpha = \Delta - f_1 - f_2 - f_3$. The subgroup $\sum_{10} \subset O(P)'$ acts naturally on \bar{X} via permutations of the factors of $G(3, 6)^{10}$. To define the action of the generator s_0 we consider the linear system of quadrics through the planes π_1, π_2 and π_3 . Assuming F is unnodal, the three planes span \mathbf{P}^5 , and one checks that this system is homaloidal, i.e. defines a birational transformation $T: \mathbf{P}^5 \rightarrow \mathbf{P}^5$. Choosing an appropriate basis of this linear system, we obtain that $T^2 = \text{identity}$. The image of F under T is an Enriques surface isomorphic to F embedded via $\varphi(s_0(\Delta))$. In this way the whole group $O(P)' = (Z/2)^{10} \rtimes W(E_{10})$ acts on the open subset $(\bar{X})^{un} \subset \bar{X}$ parametrizing the isomorphism classes of marked unnodal Enriques surfaces. One checks that the kernel of this action is a subgroup of finite index (see below). Thus the quotient space

$$(\bar{X})^{un}/O(P)'$$

parametrizes isomorphism classes of unnodal Enriques surfaces. Again, we do not know how to extend this quotient to construct the moduli space of all Enriques surfaces.

The following observation suggests that we are moving in the right direction. Every set of 10 planes π_1, \dots, π_{10} as above defines a maximal isotropic subspace in $\mathbf{P}(\wedge^3(K^6))$ with respect to the symplectic form $\wedge^3(K^6) \times \wedge^3(K^6) \rightarrow \wedge^6(K^6) \cong k$. The variety of such subspaces is the 45-dimensional homogeneous space Sp_{20}/P , where P is a maximal parabolic subgroup. The 35-dimensional group $PGM(6)$ acts on an open subset of Sp_{20}/P with a 10-dimensional quotient. It is plausible that our space $X \subset G(3, 6)^{10}$ is isomorphic to some 2^{10} -cover of an open subspace of Sp_{20}/P .

B. Automorphisms

The main problem here is to prove the known results about automorphisms of complex Enriques surfaces without using the theory of periods of $K3$ -surfaces. For example, we do not know how to reprove the Barth-Peters-Nikulin theorem on automorphisms of unnodal Enriques surfaces in the case $p \neq 0$. One possible approach to this problem is as follows. We can prove that the 2-level congruence subgroup $W(E_{10})(2)$ is equal to the normal closure of the involution $W_0 \in O(E_{10})'$ defined by $-1_{E_8} \oplus 1_U$. This involution is induced by an automorphism of an unnodal surface obtained from a representation of F as a double cover of a

4-nodal Del Pezzo quartic surface. From this one deduces easily that the whole normal closure of W_0 in $O(E_{10})'$ is realized by automorphisms of F . Next we apply a non-trivial group-theoretical result of Looijenga that says that this closure is equal to the whole subgroup $W(E_{10})(2)$ (see [C-D 3], Theorem 2.10.1). To prove that there are no more automorphisms for generic F , we would like to argue as follows. Suppose the space \tilde{X} exists and the group $O(P')$ acts regularly on it via its action on the markings. Then the subgroup $W(E_{10})(2)$ lies in the kernel (since the latter is a normal subgroup containing W_0), hence

$$W(E_{10})(2) \subset \text{Aut}(F)$$

for every unnodal Enriques surface F . The action of $O(P')$ factors through the quotient group

$$O(P')/W(E_{10})(2) \cong (\mathbb{Z}/2)^{10} \rtimes O^+(10, F_2).$$

Then it is easy to verify, knowing the structure of this group, that none of its normal subgroups can act trivially. Thus for some open subset \cup of \tilde{X} the action of this group is fixed-point-free (provided that p does not divide $\#O^+(10, F_2)$), hence the Enriques surfaces parametrized by this subset have the automorphism group isomorphic to $W(E_{10})(2)$. What is lacking in this «proof» is the absence of the construction of the space \tilde{X} (Conjecture 2).

Another interesting problem on automorphisms of Enriques surfaces is the classification of Enriques surfaces with only finitely many automorphisms. As we mentioned already, for complex surfaces such a classification was produced by Nikulin and Kondō. It is easy to see that it covers the cases of all characteristics except 2. In the latter case we have some additional cases and it is very interesting to find them all. Note that every Enriques surface F with finitely many automorphisms defines a crystallographic polyhedron in the 9-dimensional hyperbolic space. Its reflection group is isomorphic to the subgroup of $O(\text{Pic}(F)/\text{Tors})$ generated by the reflections defined by the classes of nodal curves on F . One may ask whether every such polyhedron is obtained from some Enriques surface. For this we have to take into consideration the case $p = 2$.

C. Nodal Enriques surfaces

1. *The set of nodal curves.* The set of nodal curves on an Enriques surface F has many interesting properties. For example, one can prove, if $p \neq 2$, that any effective divisor D with $D^2 = 2$ congruent to a nodal curve modulo $2(\text{Pic}(F))$ contains a nodal curve among its irreducible components. This was remarked by Looijenga, and can be easily seen by considering the $K3$ -cover of an Enriques surface.

It is a challenging problem to prove this fact without appealing to $K3$ -surfaces, for example to give a proof in the case $p = 2$.

One says that a nodal surface F is *general* if every two nodal curves are congruent modulo $2(\text{Pic}(F))$. This implies, among other things, that every elliptic pencil on F has at most one reducible fibre which consists of no more than two irreducible components. Another remarkable implication is the fact that every two nodal curves R and R' on a general nodal Enriques surfaces are fibre-equivalent. The latter means that one can find a sequence of genus 1 pencils $|E_1|, \dots, |E_k|$ and a sequence of nodal curves R_1, \dots, R_{k-1} such that

$$R + R_1 \in |E_1|, R_1 + R_2 \in |E_2|, \dots, R_{k-2} + R_{k-1} \in |E_{k-1}|, R_{k-1} + R' \in |E_k|.$$

More generally, following Nikulin, one may define the nodal invariant of an Enriques surface as the subset $\Delta(F)$ of $\bar{N}(F) = N(F)/2N(F) \cong \mathbb{F}_2^{10}$ of the classes of nodal curves modulo $2N(F)$, where $N(F) = \text{Pic}(F)/\text{Tors}$. The space $\bar{N}(F)$ inherits a natural quadratic form q defined by $q(\bar{x}) = \frac{1}{2}x^2 \bmod 2$, where $x \in N(F)$, $\bar{x} = x \bmod 2N(F)$. It is immediately seen that $\Delta(F) \subset q^{-1}(1)$. Assume that $\Delta(F)$ consists of two elements \bar{x} and \bar{y} . Then $\bar{x} \cdot \bar{y} = 0$ or 1 . In the first case every elliptic pencil on F may contain two reducible non-multiple fibres, each of which has no more than two irreducible components, or may contain one reducible multiple fibre which consists of two irreducible components. In the second case F may contain a genus one fibration with one reducible fibre with no more than three irreducible components.

PROBLEM 7. *Describe the set of possible nodal invariants. In each case find a geometric characterization in terms of degenerate fibres of elliptic pencils.*

Another definition of the nodal invariant of an Enriques surfaces can be given in terms of the Picard lattice of the $K3$ -cover [Ni 3]. This time it is a pair consisting of a negative definite root lattice and a finite abelian group. We do not know its analog in the case $p = 2$.

Finally let me remark that nodal curves on an Enriques surface are analogous to discriminant conditions on the sets of nodes of a ten-nodal plane sextic (see [D-0]), the blowing-up of which is a rational surface which is a degeneration of an Enriques surface.

2. Reye congruences. An example of a nodal Enriques surface is a Reye congruence of lines in \mathbb{P}^3 . Note that webs of quadrics in \mathbb{P}^3 depend on 9 parameters, the same as the number of parameters of all nodal Enriques surfaces. One can characterize a class of webs for which the Reye congruence is a normal surface with at most double

rational points as its singularities. Then, one proves that its minimal resolution is an Enriques surface. Call such webs good webs. Then the most interesting problem is:

PROBLEM 8. *Is every nodal Enriques' surface isomorphic to a nonsingular minimal model of the Reye congruence of a good web of quadrics in \mathbb{P}^3 ?*

There are various partial results towards a solution of this problem. For example, we know that for every Enriques surface F there is a birational morphism $f : F \rightarrow \bar{F} \subset \mathbb{P}^5$ onto a surface of degree 10 with at most double rational points as its singularities. If \bar{F} lies on a quadric, then it is isomorphic to a Reye congruence. Or, if f is an isomorphism and \bar{F} is nodal, it is isomorphic to a Reye congruence. Note that a general nodal Enriques surface is isomorphic to a nonsingular Reye congruence [C-D 4].

D. Characteristic 2

Here there is a lot of unsolved problems. The most notorious of them is the following (cf. [La 1]):

PROBLEM 9. *Are there non-zero regular vector fields on classical Enriques surfaces?*

T. Ekedahl showed that there are none, unless the surface admits a quasi-elliptic fibration.

PROBLEM 10. *Is any Enriques surface liftable to characteristic 0?*

This is true if $p \neq 2$ or F is a μ_2 -surface [La 2].

PROBLEM 11. *Study the moduli spaces (local, polarize, global) of Enriques surfaces in characteristic 2.*

I hope that the reader of this rather informal report will get the impression that the world of Enriques surface which has been explored by numerous geometers for almost a hundred years has still many blank spots worthy of further exploration.

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