THE DIFFERENT AND DISCRIMINANT OF REGULAR MAPPINGS

I. V. Dolgachev and A. N. Parshin

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The different of morphisms of surfaces onto curves is studied. A relationship is established between the discriminant and Euler characteristics of degenerate layers for morphisms without multiple components.

1. Let $f : X \to Y$ be a proper morphism of schemes over a field k. The logical problem of its degree of smoothness can be solved, as in the case of a finite morphism, by means of a suitable analog of the different and discriminant. The appropriate definitions have been given by Shafarevich in his lecture series in 1961 (their background and motivation are discussed in [6]). Let $\mathcal{O}$ be a local ring without divisors of zero, M an $\mathcal{O}$ -module of finite type, and K the quotient field of $\mathcal{O}$. If $0 \to N \to \mathcal{O}^n \to M \to 0$ and $\dim M \otimes_{\mathcal{O}} K = m$, we let

$$\vartheta(M) = \bigcup_{S_m} \Lambda^n (N + S_m),$$

where S_m runs through the submodules $\mathcal{O}^n$,having m generators and Λ^n is the nth outer degree. The ideal $\vartheta(M)$ of ring $\mathcal{O}$ is invariantly defined and is called the different of the module M. Using the projective resolvent, we can provide an analogous definition for any integral rings. The basic properties are as follows:

1) $\vartheta(M) = (1) \Leftrightarrow M$ is projective.

2) $\vartheta(M \oplus M') = \vartheta(M)\,\vartheta(M')$.

3) If $p \subset \mathcal{O}$ is a prime ideal and $\mathcal{O}_p$, M_p are localizations, then

$$\vartheta(M_p) = \vartheta(M)\,\mathcal{O}_p.$$

If now a coherent sheaf is given on the scheme X, property 3) makes it possible to define the different $\vartheta(F)$ of the sheaf F such that $\vartheta(F)_x = \vartheta(F_x)$, $x \in X$. Here $\vartheta(F)$ is a sheaf of ideals on X. Returning to the morphism $f : X \to Y$, we set $\vartheta_{X|Y} = \vartheta(\Omega^1_{X|Y})$, where $\Omega^1_{X|Y}$ is a sheaf of relative differentials (see [3]). We define the discriminant of the mapping f as the sheaf $D_{X|Y} = \mathcal{O}_X / \vartheta$ on X. If Y is a smooth irreducible scheme of dimensionality one and the common layer of the morphism f is smooth, the sheaf $f_* D_{X|Y}$ is nothing other than $\bigoplus_{y \in Y} \mathcal{O}_{Y,y} | m_y^{n_y}$, m_y, i.e., the maximum ideals of points of Y. We also call the number Σn_y the discriminant of the morphism f and denote it by $d_{X|Y}$.

LEMMA 1. Suppose that scheme Y is smooth over k, scheme X is integral, and morphism f is smooth at the common point of X. We then have an exact sequence of sheafs on X

$$0 \to f^*\Omega^1_Y \to \Omega^1_X \to \Omega^1_{X|Y} \to 0. \tag{1}$$

Proof. For any morphism the sequence

$$f^*\Omega^1_Y \to \Omega^1_X \to \Omega^1_{X|Y} \to 0$$

is exact (see [3]). According to the differential smoothness criterion of Grothendieck [3], morphism f is smooth at a point $x \in X$ if the sequence

$$0 \to (f^*\Omega^1_Y)_x \to (\Omega^1_X)_x \to (\Omega^1_{X|Y})_x \to 0$$

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is exact and the module $(\Omega^1_{X|Y})_x$ is projective. The conditions of the lemma indicate that the sheaf $f^*\Omega^1_Y$ is locally free and the kernel of the homomorphism $f^*\Omega^1_Y \to \Omega^1_X$ is a torsion sheaf. Inasmuch as a locally free sheaf does not contain a torsion subsheaf, we arrive at that which was to be proved.

COROLLARY. Given the conditions of the lemma, $\mathrm{supp}(D_{X|Y})$ is congruent with the set of points of X in which the morphism f is nonsmooth. In particular, if all layers of f have the same dimensionality, this set comprises the union of singular points of the layers of f.

2. We now assume that X is a smooth algebraic surface, Y is a smooth algebraic curve of genus q, the common layer of the morphism f is a smooth curve of genus g over a field of functions on Y, and the morphism f and schemes X, Y are defined on an algebraically closed field k. Let us assume also that the layers of f do not have multiple components.

LEMMA 2. Under these conditions we have an exact sequence of sheafs on X

$$0 \to \Omega^1_{X|Y} \otimes f^*\omega_Y \xrightarrow{\alpha} \omega_X \to D_{X|Y} \to 0, \tag{2}$$

where $\omega_X = \Omega^2_{X|k}$, $\omega_Y = \Omega^1_{Y|k}$ are canonic sheafs of X and Y, respectively.

Proof. Let $a : f^*\Omega^1_Y \subset \Omega^1_X$ be the canonic embedding determined by the exact sequence (1). Multiplying it tensorially by the sheaf Ω^1_X, we obtain

$$a' : f^*\Omega^1_Y \otimes \Omega^1_X \subset \Omega^1_X \otimes \Omega^1_X.$$

The composition of a' with the mapping of outer degree

$$\Lambda : \Omega^1_X \otimes \Omega^1_X \to \Omega^2_{X|k} = \omega_X$$

defines a homomorphism

$$\alpha' : \Omega^1_X \otimes f^*\omega_Y \to \omega_X.$$

Since the sheaf $f^*\omega_Y$ is invertible, it is readily seen that

$$f^*\omega_Y \otimes f^*\omega_Y \subset \ker \alpha'.$$

Applying the exact sequence (1), we obtain the homomorphism

$$\alpha : \Omega^1_{X|Y} \otimes f^*\omega_Y \to \omega_X.$$

Let $Z = \mathrm{supp}(D_{X|Y})$. The assumption regarding morphism f indicates that codim $Z \geq 2$; on the other hand, α is an isomorphism on $X-Z$. This shows that $\mathrm{supp}(\ker \alpha) \subset Z$. But the exact sequence (1) implies

$$\mathrm{dp}\Omega^1_{X|Y} \leqslant 1,$$

so that, consequently,

$$\mathrm{dep\,th}_Z \Omega^1_{X|Y} \geqslant 1$$

(X is a regular scheme), whence it follows that $\ker \alpha = 0$. It is readily inferred from the definition of $\vartheta_{X|Y}$ that

$$\mathrm{Im}\alpha = \vartheta_{X|Y} \otimes \omega_X,$$

and, since codim $Z \geq 2$,

$$\mathrm{coker}\,\alpha = \omega_X / \vartheta_{X|Y} \otimes \omega_X = D_{X|Y} \otimes \omega_X = D_{X|Y}.$$

COROLLARY.

1) $\mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_{X|Y}, \mathcal{O}_X) \simeq \omega_X^{-1} \otimes f^*\omega_Y$;

2) $\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_{X|Y}, \mathcal{O}_X) = D_{X|Y}$;

3) $\mathrm{Ext}^i_{\mathcal{O}_X}(\Omega^1_{X|Y}, \mathcal{O}_X) = 0, \quad i > 1$.

Proof. Inasmuch as a regular ring is a Gorenstein ring, it follows (see [1]) that

$$\mathrm{Ext}^i_{\mathcal{O}_X}(D_{X|Y}, \mathcal{O}_X) = \begin{cases} 0, & i \neq 2, \\ D_{X|Y}, & i = 2. \end{cases}$$

Now it is only required to apply the functor Hom $(., \mathcal{O}_X)$ to sequence (2).

Remark. Relation 2) is also fulfilled for multiple components. In this case the sheaf $\mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_{X|Y}, \mathcal{O}_X)$ is also locally free, being specifically equal to

$$\omega_X^{-1} \otimes f^*\omega_Y \otimes \mathcal{O}_X(D), \text{ where } D = \sum_{y \in Y} X_y - (X_y)_{\mathrm{red}}.$$

The following theorem has been advanced as a hypothesis by Shafarevich.

THEOREM.

$$d_{X|Y} = \sum_{y \in Y} \chi(X_y) - \chi(F),$$

where F is the common layer of morphism f, and χ is the l-adic Euler-Poincaré covering characteristic $(l, \mathrm{char}\ k) = 1$.

Proof. From (1) we obtain the exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_{X|Y}, \mathcal{O}_X) \to \mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \to \mathrm{Hom}_{\mathcal{O}_X}(f^*\omega_Y, \mathcal{O}_X) \to \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_{X|Y}, \mathcal{O}_X) \to 0.$$

Invoking the corollary to Lemma 2, we rewrite this sequence as

$$0 \to \omega_X^{-1} \otimes f^*\omega_Y \to T_X \to f^*T_Y \to D_{X|Y} \to 0, \tag{3}$$

where T_Z is the tangential sheaf of the scheme Z. It is seen at once that

$$c_2(D_{X|Y}) = d_{X|Y},$$

where c_2 is the Chen second class of sheaf $D_{X|Y}$. Applying the properties of the Chen character ch(F) of a sheaf F (see [2]), we obtain

$$\mathrm{ch}(\omega_X^{-1} \otimes f^*\omega_Y) = 1 + f^*c_1(Y) - c_1(X) + \frac{1}{2}c_1(X)^2 - (2-2g)(2-2q),$$
$$\mathrm{ch}(T_X) = 2 - c_1(X) + \frac{1}{2}c_1(X)^2 - c_2(X),$$
$$\mathrm{ch}(f^*T_Y) = 1 - f^*c_1(Y),$$
$$\mathrm{ch}(D_{X|Y}) = d_{X|Y}.$$

Sequence (3) and the additivity of ch yield

$$d_{X|Y} = c_2(X) - (2-2g)(2-2q).$$

Now we need only make use of the formula

$$\chi(X) = \chi(F)\cdot\chi(Y) + \sum_{y \in Y} \chi(X_y) - \chi(F)$$

and the formula

$$c_2(X) = \chi(X),$$

which follows from Lefschetz' theorem on stationary points of morphisms (see [4]).

Remarks. 1. If k = C, then by the congruence theorem and computation of the Euler characteristic for arbitrary curves the analogous theorem holds when χ is replaced by the ordinary topological Euler characteristic.

2. It is readily seen that the sheaf $\vartheta_{X|Y}$ is precisely the Jacobian sheaf of Hironaka, which plays an important role in determining the equisingularity of a morphism of schemes (see [5]).

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