MATH 296 PROBLEMS 12

IGOR KRIZ

Regular problems:

1. Let $f : [-a, a] \to \mathbb{R}$. We say that f is *even* if $f(-x) = f(x)$ and *odd* if $f(-x) = -f(x)$. If f is Riemann-integrable on $[0, a]$, prove that

(a) $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$ is f is even

(b) $\int_{-a}^{a} f(x)dx = 0$ is f is odd.

2. Integrate:

(a) $\int \frac{\sin(x)+\cos(x)}{1+\sin(x)}dx$.

(b) $\int \frac{dx}{1+\sqrt{x^2+2}}$.

3. Use the definition of Riemann integral and "scaling" of partitions to prove that

$$\int_a^b f(x)dx = (b-a)\int_0^1 f(a+(b-a)x)dx.$$

4. The equations $u = f(x, y, z)$, $x = X(r, s, t)$, $y = Y(r, s, t)$, and $z = Z(r, s, t)$ define u as a function of r, s, and t, say, $u = F(r, s, t)$. Use the chain rule to express the partial derivatives $\frac{\partial F}{\partial r}$, $\frac{\partial F}{\partial s}$, and $\frac{\partial F}{\partial t}$ in terms of partial derivatives of f, X, Y and Z.

Challenge problems:

5. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1/n & \text{if } x = m/n \text{ where } m \text{ is an integer, } n \text{ is a positive integer and there is no integer greater than 1 dividing both } m, n \end{cases}$$

Show that

$$\int_0^1 f(x)dx = 0.$$

6. Suppose $f : [a, b] \to [0, +\infty)$ is a bounded function (i.e. there is a constant M such that $|f(x)| \leq M$ for all $x \in [a, b]$) and suppose that there is a subset $S \subset [a, b]$ such that

(1) f is continuous at every point *outside* of S, and

(2) For every $\epsilon > 0$, S can be covered by finitely many intervals with total length $< \epsilon$.

Prove that then f is Riemann-integrable.