MATH 296 PROBLEMS 14

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Regular problems:

1. Determine if the following series converge, and if they converge absolutely:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}} \tag{1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n \ln^2(n+1)} \tag{2}$$

[Use the integral criterion, and the behaviour of series with alternating signs.]

2. A function is *periodic* with *period* a if $f(x+a) = f(x)$ for all x.

(a) If f is periodic with period a and integrable on $[0,a]$, show that

$$\int_0^a f = \int_b^{b+a} f \text{ for all } b.$$

(b) Find a function f such that f is not periodic, but f' is. [Hint: Choose a periodic g for which it can be guaranteed that $\int_0^x g$ is not periodic.]

(c) Suppose that f' is periodic with period a. Prove that f is periodic if and only if $f(a) = f(0)$.

3. Calculate

$$\int_0^{1/2} \sqrt{\frac{1-x}{1+x}} dx.$$

4. Show that the infinite series

$$\sum_{n=0}^{\infty} (\sqrt{n^a - 1} - \sqrt{n^a})$$

converges for $a > 2$ and diverges for $a = 2$. [Hint: $(a+b)(a-b) = a^2 - b^2$]

Challenge problems:

5. A series

$$\sum_{n=1}^{\infty} a_n$$

is called *Cesaro summable* with Cesaro sum ℓ if

$$\lim_{n\to\infty} \frac{s_1 + ... + s_n}{n} = \ell$$

where $s_k = a_1 + ... + a_k$.

(a) Prove that a convergent series is always Cesaro summable, and that its Cesaro sum is automatically equal to the ordinary sum.

(b) Find a series which is Cesaro summable, but not convergent in the ordinary sense.

6. Prove that every positive rational number q can be written as a finite sum of *distinct* numbers of the form $1/n$ where n is a positive integer For example,

$$\frac{27}{31} = \frac{1}{2} + \frac{1}{3} + \frac{1}{27} + \frac{1}{1674}.$$

[Use the divergence of the series $\sum 1/n$ and after you get close enough, keep subtracting the largest possible number $1/n$.]