MATH 395 PROBLEMS 10

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Regular problems:

1. Solve
$$y' = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} y$$
for a general $\lambda \in \mathbb{R}$.

2. Which of the following sets are countable:
(a) The set of all finite sequences of rational numbers $(a_1, ...a_n)$ for all possible positive integers n
(b) The set of all real numbers not containing the digits $4, 5, 6$ in their decimal expansion. Can you describe this set?
(c) The set of all numbers which are roots of non-zero polynomials with integer coefficients.

3. Using Fubini's theorem, calculate
$$\int_{[0,\pi]\times[0,\pi]} |\cos(x+y)|dxdy$$

4. Recall that Z is the set of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ with compact support. Recall also that Z^{inc} resp. Z^{dec} is the set of all functions $f : \mathbb{R} \to \mathbb{R} \cup \{-\infty, \infty\}$ which are limits of an increasing (resp. decreasing) sequence of functions from Z. Now let
$$g(x) = \begin{cases} \frac{1}{\sqrt[3]{x}} & \text{if } x \in (0,1] \\ 0 & \text{if } x \notin (0,1]. \end{cases}$$
(a) Prove that $g(x) \in Z^{inc}$. Is $g(x) \in Z^{dec}$?
(b) Choosing an increasing sequence f_n of functions from Z converging to g, calculate
$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)dx.$$

Challenge problem:

5. *Reducing* (L) *to* $(\tilde{L})$. Let

$$y' = Ay.$$

Reduce this to the problem $(\tilde{L})$ of the form

$$y^{(n)} + a_1 y^{(n-1)} + \dots a_n y = 0$$

in the following steps:

(a) It suffices (at least theoretically) to consider the case when A is a Jordan block

$$\begin{pmatrix} \lambda & 1 & \dots & 0 \\ 0 & \lambda & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda \end{pmatrix}.$$

Why?

(b) Write down the matrix B of the problem $(\tilde{L})$ with characteristic polynomial $(x-\lambda)^n$, rewritten in the form of (L) with $y_1 = y$, $y_2 = y'$, ..., $y_n = y^{(n-1)}$. It suffices to exhibit P such that

$$P^{-1}AP = B.$$

Why?

(c) Assume you can find vectors $u_1, \dots, u_n \in \mathbb{R}^n$ such that

$$Bu_1 = \lambda u_1, \; Bu_i = \lambda u_i + u_{i-1}, \; i > 1.$$

Then you can find P. How?

(d) Find the vectors $u_1, \dots u_n$. [This is the tough part. Try it for $n = 1, 2, 3$ and try to guess the general pattern.]