MATH 395 PROBLEMS 11

IGOR KRIZ

Regular problems:

1. Calculate
$$\int_Q f(x+y)dxdy$$
where $Q = [0,2] \times [0,2]$ and $f(t)$ denotes the greatest integer $\leq t$. [Find some way to visualize the function; the fundamental theorem of calculus won't work.]

2. Prove that if S is a countable set, then the set of all finite subsets of S is countable.

3. Using theorems from class (your notes), prove that if $f_n : [a,b] \to \mathbb{R}$ are continuous functions and $f_n \nearrow f$, then f is continuous if and only if $f_n \rightrightarrows f$.

4. (a) Calculate
$$\int_0^1 \ln(x)dx,$$
and justify using Lebesgue integral.

(b) Prove, more generally, that if $f : [a,b] \to [0,\infty)$ (or $f : [a,b] \to (-\infty, 0]$ and f is continuous on $[a,b)$, then
$$\int_a^b f = \lim_{x\to a} \int_a^x f$$
where $\int$ denotes Lebesgue integral. [Hint: express f as an increasing (resp. decreasing) limit of continuous functions.]

Challenge problems:

5. The Cantor set C is the set of all $x \in [0, 1]$ which are *not* of the form

$$\frac{k + \alpha}{3^n}$$

for any $k \in \mathbb{Z}$, $\alpha \in (1/3, 2/3)$. Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 \text{ if } x \in C \\ 0 \text{ else.} \end{cases}$$

Is f Lebesgue-integrable? If yes, what is the integral equal to?

6. Prove that the set of all subsets of $\mathbb{N}$ is not countable. [Find a method analogous to the one used for the real numbers.]