MATH 395 PROBLEMS 2

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Regular problems:

1. Calculate:

$$det\begin{pmatrix} 1 & 0 & 3 & 1 \\ 2 & 2 & 6 & 0 \\ -1 & 0 & -3 & 1 \\ 4 & 1 & 12 & 0 \end{pmatrix}.$$

2. Find an explicit formula for

(a) The inverse of a general 3×3 matrix

(b) The solution of a general system of two linear equations with two unknowns. [Use determinants and Cramer's rule.]

3. Find the values of x for which $det(A) = 0$, where

$$A = \begin{pmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{pmatrix}.$$

4. Find the sign of the permutation

(a) $[2, 3, \dots, n]$

(b) $[n, n-1, n-2, ..., 1]$.

Challenge problem:

5. Let $p(x) = x^m + a_1 x^{m-1} + ... + a_m x^0$ and $q(x) = x^n + b_1 x^{n-1} + ... + b_n x^0$. The *resultant* $R(p, q)$ of the polynomials $p(x), q(x)$ is the determinant of the $(m + n) \times (m + n)$ matrix

$$\begin{pmatrix} 1 & a_1 & \dots & a_m & & \\ & 1 & a_1 & \dots & a_m & \\ & & \dots & & & \\ & & 1 & a_1 & \dots & a_m \\ 1 & b_1 & \dots & b_n & & \\ & 1 & b_1 & \dots & b_n & \\ & & \dots & & & \\ & & 1 & b_1 & \dots & b_n \end{pmatrix}.$$

Let $\zeta_1, ..., \zeta_m$ be the roots of the polynomial $p(x)$, let $\xi_1, ..., \xi_n$ be the roots of the polynomial $q(x)$. Then we have

$$R(p, q) = \prod_{i<j} (\xi_i - \zeta_j) \tag{1}$$

(a) Prove formula (1) in the case $n = 1$. [Quite easy from definition.]

(b) Prove formula (1) for all m, n. [Much harder.]

(c) The *discriminant* of a polynomial $p(x)$ is the resultant of $p(x)$ and its derivative: $R(p, p')$. Write down the discriminant of polynomials of degree $2, 3$ explicitly.

(d) Prove that the discriminant of a polynomial $p(x)$ is non-zero if and only if $p(x)$ has no root of multiplicity > 1.

(e) Prove that the discriminant of a polynomial is the square of the Vandermonde determinant of its roots. [What can you say about two polynomials with the same roots and multiplicities?]