MATH 395 PROBLEMS 3

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Regular problems:

1. Find the eigenvalues of

(a) $\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$

(b) $\begin{pmatrix} 7 & 0 & -4 \\ 0 & 5 & 0 \\ 5 & 0 & -2 \end{pmatrix}$.

2. Using the geometric interpretation of determinants, find the volume of the solid in $\mathbb{R}^3$ consisting of all (x, y, z) such that

$$(2x + 3y + z)^2 + (x - 2y + 3z)^2 + (x + y + z)^2 \leq 6.$$

[Find a linear transformation which maps the solid onto a ball, and consider the determinant of that transformation.]

3. For $x, y \in \mathbb{R}$, define

$$\begin{aligned} d_1(x,y) &= (x-y)^2, \\ d_2(x,y) &= \sqrt{|x-y|}, \\ d_3(x,y) &= |x^2 - y^2|, \\ d_4(x,y) &= |x - 2y|, \\ d_5(x,y) &= \tfrac{|x-y|}{1+|x-y|}. \end{aligned}$$

Determine, for each of these, whether it is a metric or not. [Recall that a metric is supposed to satisfy (1) $d(x, y) = 0$ if and only if $x = y$, (2) $d(x, y) = d(y, x)$ and (3) $d(x, z) \leq d(x, y) + d(y, z)$.]

4. Prove that if an invertible matrix is upper triangular ($a_{ij} = 0$ for $i > j$), then the inverse is also upper triangular.

Challenge problems:

5. Prove the following fact: Let n data pairs $(x_1, y_1),\dots, (x_n, y_n)$ be given such that the x_i are distinct. Then there exists a unique polynomial

$$p(x) = r_0 + r_1 x + \dots + r_{n-1} x^{n-1}$$

such that $p(x_i) = y_i$. [Hint: write a system of linear equations for the r_i's, solve using Cramer's rule and determinants.]

6. Let X be a metric space. Let $\bar{X}$ be the set of equivalence classes of all Cauchy sequences in X under the following equivalence relation: Cauchy sequences (a_n) and (b_n) are equivalent if the sequence $(a_1, b_1, a_2, b_2, \dots)$ is Cauchy (prove that this is an equivalence relation). Now if the metric on X is d, on $\bar{X}$, we can define

$$d((a_n), (b_n)) = \lim_{n\to\infty} d(a_n, b_n).$$

(a) Prove that this makes $\bar{X}$ into a metric space.
(b) Prove that the map $X \to \bar{X}$ given by $a \mapsto (a, a, \dots)$ preserves metric (=is an *isometry*), and is an embedding.
(c) Prove that $\bar{X}$ is a complete metric space (i.e. every Cauchy sequence in $\bar{X}$ converges).