MATH 395 PROBLEMS 5

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Regular problems:

1. Find all solutions to the differential equations
(a) $y' = yx^3$
(b) $y' = (x + 2y)^2$
(c) $y' = \sin(x)y + \cos(x)$.

2. Prove that if (X, ρ) is a metric space, then another metric d on X can be defined by $d(x, y) = \max\{1, \rho(x, y)\}$. Prove that the identity is a homeomorphism of (X, d) to (X, ρ). Prove also that (X, ρ) is complete if and only if (X, d) is complete.

3. Prove that there does not exist a differential equation

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)})$$

with f continuous and Lipschitz in the last n variables which would have the solutions

$$y = ax^2 + (4a + 2)x + a + 3.$$

for all $a \in \mathbb{R}$. [Use the existence and uniqueness theorem.]

4. Define $\mathbb{R}^\infty$ as the set of all infinite sequences

$$x_1, x_2, x_3, \ldots$$

of real numbers. Define

$$\rho((x_n), (y_n)) = \sum_{i=1}^{\infty} \frac{1}{2^i} \max\{1, |x_i - y_i|\}.$$

(see Problem 2). Is $\mathbb{R}^\infty$ a metric space? Is it complete?

Challenge problems:

5. Let $f(x,y)$ be a continuous function with continuous partial derivative by y such that $f(x_0,y_0)=0$, and

$$\frac{\partial f}{\partial y}(x_0,y_0)\neq 0.$$

Prove that there exists an $\epsilon>0$ and a unique function

$$g:(x_0-\epsilon,x_0+\epsilon)\to\mathbb{R}$$

such that

$$f(x,g(x))=0.$$

[Differentiate by x and, using the chain rule, write a differential equation for $y=g(x)$.]

6. Find a differential equation whose characteristics (=solutions defined on a maximal possible interval) have domains

(a) $(a,+\infty)$, $a\in\mathbb{R}$

(b) $(-a,a)$, $a>0$.