MATH 396 PROBLEMS 12

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Regular problems:

1. Using calculus of variations, find all possible extrema of

$$\int_a^b (\phi'(x))^2 dx$$

over the space of all functions $\phi : [a,b] \to \mathbb{R}$ with fixed $\phi(a), \phi(b)$.

2. If the action of a variation problem is $F(x,y,z) = zf(y)$, prove that the Euler equation is satisfied by every (smooth) function. What does that say about

$$\mathfrak{F}(\phi) = \int_a^b F(x, \phi(x), \phi'(x))dx$$

in this case? Can you prove that conclusion directly (without using Euler equation)?

3. Let f be a function holomorphic in $B_\epsilon(a) = \{z |\ |z-a| < \epsilon\}$. Prove that if $f(a) = 0$, then $f(z) = (z-a)g(z)$ where g is holomorphic in $B_\epsilon(a)$. [Use the Taylor formula.]

4. Let f be a holomorphic function, and let L be the circle with radius 1 and center 0, oriented counter-clockwise. Let z be inside L, $z \neq 0$. Using the Cauchy formula, evaluate

$$\frac{1}{2\pi i} \int_L \frac{f(\zeta)\zeta}{\zeta - z} d\zeta.$$

Challenge problem:

5. Given numbers p, q, find an interval $[a, b]$ and a function $g : [a, b] \to \mathbb{R}$, $g(a) = p$, $g(b) = q$, such that the length of the graph of g is 1 and the area under the graph of g is maximal possible.

(a) This problem is slightly different than the typical variation problem we considered, because you don't know the interval $[a, b]$. Therefore, parametrize the graph of g not by $(x, g(x))$, but by $(\phi(t), \psi(t))$ $(\phi, \psi : [0, 1] \to \mathbb{R})$ where the length of the curve increases at a unit length (i.e. $(\phi')^2 + (\psi')^2 = 1$). The derivative of the area under the graph by t is $\phi'(t)\psi(t)$. Write a variation problem (Euler equation) for ψ.

(b) To solve the Euler equation, use the trick we used in class:

$$u' = v \Rightarrow (u^2)' = 2uv.$$