MATH 396 PROBLEMS 3

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Regular problems:

1. The orbits of the Earth and Jupiter are approximately circular: The distance of the Earth from the Sun is 92.9 million miles, the distance of Jupiter from the Sun is 483.6 miles. The minimum distance of Pluto from the Sun is 2.1 billion miles, but its maximum distance to the Sun is 4.6 billion miles. Calculate the approximate length of a Jupiter year and a Pluto year.

2. Calculate:

(a) $\int_S (1+x+y)^{-2} dxdy$ where S is the area bounded by the two coordinate axes and the line $x + y = 1$

(b) $\int_T (1+x+y+z)^{-3} dxdydz$ where T is the solid bounded by the three coordinate planes and the plane $x + y + z = 1$.

3. Calculate

$$\int_U \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right) dxdydz$$

where U is the solid bounded by the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

4. Consider the map $\mathbb{R}^2 \to \mathbb{R}^2$ given by the equations $x = u^2 - v^2$, $y = 2uv$.

(a) Compute the Jacobian determinant J of the map.

(b) Denote by T the rectangle in the uv-plane with vertices $(1,1)$, $(2,1)$, $(2,3)$, $(1,3)$. Describe, by means of a sketch, the image of S in the xy-plane.

(c) Let $C = \{(x,y) | x^2 + y^2 \leq 1\}$. Evaluate the integral

$$\int_C xydxdy$$

by making the substitution $x = u^2 - v^2$, $y = 2uv$.

Challenge problems:

5. Calculate the Lebesgue measure (="volume") of the n-dimensional ball

$$\{(x_1, ..., x_n) | \sum_{i=1}^{n} x_i^2 = r\}.$$

6. (a) An asteroid from outer space has come within the bounds of the Solar system. It is currently flying in a direction perpendicular to the line connecting it with the Sun. Using Kepler's laws, describe the possible behaviours of the asteroid depending on its speed.

(b) A TV satellite is orbitting the Earth on a circular orbit. As a result of a meteor shower, the satellite gets "knocked off" its orbit: its direction and speed does not change, but its distance to the Earth decreases slightly. Using Kepler's laws, predict what will happen: Will the satellite fall onto the Earth? Explain also why the prediction made by Kepler's law may not be reliable.