MATH 396 PROBLEMS 7

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Regular problems:

1. Prove that $\mathbb{R}^3$ with the cross-product

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \times \begin{pmatrix} t \\ u \\ v \end{pmatrix} = det \begin{pmatrix} e_1 & x & t \\ e_2 & y & u \\ e_3 & z & v \end{pmatrix}$$

is a Lie algebra, i.e. satisfies

$$\vec{x} \times \vec{y} = -\vec{y} \times \vec{x},$$
$$(\vec{x} \times \vec{y}) \times \vec{z} + (\vec{y} \times \vec{z}) \times \vec{x} + (\vec{z} \times \vec{x}) \times \vec{y} = 0.$$

2. Using Green's formula, explain how you would find the potential of a function $f : U \rightarrow \mathbb{R}^2$ ($U \subseteq \mathbb{R}^2$ open), i.e. a function $F : U \rightarrow \mathbb{R}$ such that $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \partial F/\partial x \\ \partial F/\partial y \end{pmatrix}$, if one exists. Apply to the function f which assigns to a point $\begin{pmatrix} x \\ y \end{pmatrix}$ the vector which is a positive multiple of $\begin{pmatrix} x \\ y \end{pmatrix}$ and has length $1/(x^2+y^2)$.

3. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be given by $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} xe^{-y^2} \\ -x^2ye^{-y^2} + 1/(x^2+y^2) \end{pmatrix}$. Calculate $\int_L f$ where L is the boundary of the square with vertices $\begin{pmatrix} -a \\ -a \end{pmatrix}$, $\begin{pmatrix} a \\ -a \end{pmatrix}$, $\begin{pmatrix} a \\ a \end{pmatrix}$, $\begin{pmatrix} -a \\ a \end{pmatrix}$.

4. Calculate the area of the parametrized surface

$$\phi\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u\cos v \\ u \sin v \\ u^2 \end{pmatrix},$$

$0 \le u \le 4$, $0 \le v \le 2\pi$.

Challenge problem:

5. Prove (find an example) that the Green formula is false for a continuous function $f : \mathbb{R}^2 - \{0\} \to \mathbb{R}^2$ with continuous partial derivatives, if f is not required to be bounded. [Hint: recall some complex curve integrals.]