Math 425 9/9/2011

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Teaching: 425

$$\frac{n!}{n_1! \cdot \ldots \cdot n_k!} = \binom{n}{n_1, n_2, \ldots, n_k}$$

$$n_1 + \ldots + n_k = n$$

The case $k=2$ has a slightly different notation (also, it's the most important)

$a+b=n$ $\quad$ ($a=n_1$, $b=n_2$)

$$\frac{n!}{a!\,b!}=\binom{n}{a}=\binom{n}{b}$$

Read: "n choose a"

"n choose b"

These are called combination numbers

Observe: $\binom{n}{a} = \frac{n!}{a!\,(n-a)!}$ is also equal to the number of a-element subsets of $\{1, \dots, n\}$. For, we can represent each such subset by an arrangement of numbers $0, 1$:

Example: $n = 5$ $\quad S = \{1, 3, 4\}$

$$\begin{array}{ccccc} S & S & S & S & S \\ \cup & \not\cup & \cup & \cup & \not\cup \\ 1 & 2 & 3 & 4 & 5 \end{array}$$

is represented by $(1\ 0\ 1\ 1\ 0)$

In general, write a sequence of numbers 0, 1 where the ith number is 1 if $i \in S$
0 if $i \notin S$.

If S has exactly $\underline{a}$ elements, we have exactly $\underline{a}$ ones. So we have $\underline{n}-\underline{a}$ zero's.

So the number of subsets S of $\{1, \dots n\}$ which have $\underline{a}$ elements is

$$\binom{n}{a} = \binom{n}{a,\ n-a} = \frac{n!}{a!\,(n-a)!}$$

<u>Remark</u>: These a-element subsets of $\{1, \dots, n\}$ used to be called <u>combinations</u>.

<u>Example</u>: How many committees of 3 people can be formed from a set of 20 people?

<u>Solution</u>: $\binom{20}{3} = \frac{20!}{3!\,17!} = \frac{\cancel{17!} \cdot 18 \cdot 19 \cdot 20}{3! \cdot \cancel{17!}} =$

$$= \frac{18 \cdot 19 \cdot 20}{1 \cdot 2 \cdot 3} = 1140. \quad \square$$

The binomial Theorem:

$$(x+y)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots$$

$$\cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n =$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^{k} \quad \left(= \sum_{k=0}^{n} \binom{n}{k} x^{k} y^{n-k} \right)$$

Proof:

$$(x+y)^n = \underbrace{(x+y) \cdot \ldots \cdot (x+y) \cdot (x+y)}_{n \text{ times}}$$

This is a sum of terms, we must choose either x or y (not both) in each parenthesis (that is one term). But these are precisely words made out of the letters x, y.

The number of terms which are equal to $x^a y^{n-a}$ is precisely the number of words which contain exactly $\underline{a}$ x's: This is $\binom{n}{a}$.

So the answer is
$$\sum_{a=0}^{n} \binom{n}{a} x^a y^{n-a}$$

This is the formula we are trying to prove. $\square$

$(\ldots = (x+y)^n)$

In general, we have the <u>multinomial theorem</u>:

$$(x_1 + \dots + x_k)^n = \sum_{\substack{n_1, \dots, n_k \geq 0 \\ n_1 + \dots + n_k = n}} \binom{n}{n_1, \dots, n_k} x_1^{n_1} \cdot \ldots \cdot x_k^{n_k}.$$

<u>Proof sketch</u>:

$$(x_1 + \dots + x_k)^n = \underbrace{(x_1 + \dots + x_k) \cdot \ldots \cdot (x_1 + \dots + x_k) \cdot (x_1 + \dots + x_k)}_{n \text{ times}}$$

We are choosing precisely one letter $x_1, x_2, \dots,$ or x_k

in each parenthesis, if we focus on the terms which have n_1 x_1's, n_2 x_2's ... n_k x_k's

we get $\binom{n}{n_1, \ldots, n_k}$ choices. So this is

the coefficient of $x_1^{n_1} \cdot \ldots \cdot x_k^{n_k}$. $\square$

Example:

$$(a+b+c)^3 = \binom{3}{3,0,0} a^3 + \binom{3}{2,1,0} a^2 b + \binom{3}{2,0,1} a^2 c +$$

$$+ \binom{3}{1,2,0} a b^2 + \binom{3}{1,1,1} abc + \binom{3}{1,0,2} a c^2 +$$

$$+ \binom{3}{0,3,0} b^3 + \binom{3}{0,2,1} b^2 c + \binom{3}{0,1,2} b c^2$$

$$+ \binom{3}{0,0,0} c^3$$

$$\binom{3}{3,0,0} = \frac{3!}{3!} = 1 \qquad \binom{3}{2,1,0} = \frac{3!}{2! \cdot 1!} = 3 \qquad \binom{3}{1,1,1} = \frac{3!}{1! \cdot 1! \cdot 1!} = 6$$

$$(a+b+c)^3 = a^3 + b^3 + c^3 + 3a^2b + 3a^2c + 3b^2a + 3b^2c$$

$$+ 3c^2a + 3c^2b + 6abc. \qquad \square$$

How many terms are there in the multinomial formula? How many monomials of order k in n variables are there?

add up the powers of all the factor variables

In the Example, $k = 3$, $n = 3$. Answer: 10

In general, what we need to compute is the number of choices of non-negative integers $m_1, \dots, m_k$ such that $\underbrace{m_1 + \dots + m_k = n}_{\text{an ordered partition of } \underline{n}}$.

<u>The trick</u>: Encode the choices by sequences of <u>1</u>'s and <u>0</u>'s.

$$m_1, m_2, \dots, m_k$$

$$(\underbrace{0,\dots,0}_{m_1},1,\underbrace{0,\dots,0}_{m_2},1,\dots,1,\underbrace{0,\dots,0}_{m_k}).$$

There will be $\underline{n}$ 0's, $\underline{k-1}$ 1's.

$$\binom{n+k-1}{n} = \binom{n+k-1}{k-1}$$

In the Example, $n = k = 3$

$$\binom{3+3-1}{3} = \binom{5}{3} = \frac{5 \cdot 4 \cdot \cancel{3}}{1 \cdot 2 \cdot \cancel{3}} = 10.$$

HW: (1) Theoretical Exercise 3 on p.18 (variations)

(2) Theoretical Exercise 13 on p.19

(3) Problem 26 on p.17

All HW due on
Wednesday!