Note Title 9/16/2011

Math 425 9/16/2011

Axioms of probability

- A set S = sample space
- There are certain measurable events subsets $E \subseteq S$.
- If E is a measurable event, we have a probability $P(E)$ (a real number)

Axiom 1: $0 \leq P(E) \leq 1$ for any measurable event

Axiom 2: S is a measurable event which has probability 1. $(P(S) = 1)$.

exclusive

Axiom 3: If $E_1, E_2, E_3, \ldots$ are disjoint measurable events

if $i \neq j$ then $E_i \cap E_j = \emptyset$

then $\bigcup_{i=1}^{\infty} E_i$ is measurable and

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

Axiom 4 : If E is a measurable event, so is E^c.

Explanation :

$\bigcup_{i=1}^{\infty} E_i$ means : all $x \in S$ where $x \in E_i$ for some $i = 1, 2, 3, \ldots$

$$\underbrace{\sum_{i=1}^{\infty} \underbrace{P(E_i)}_{\text{non-negative numbers}}}_{\text{an infinite series (Calculus 2)}} = \lim_{n\to\infty} \underbrace{\sum_{i=1}^{n} P(E_i)}_{\text{a sum of finitely many non-negative numbers}} \in [0, +\infty]$$

an increasing sequence

On measurability

B = unit ball in $\mathbb{R}^3$

Suppose there is vacuum in B

$S = B$ $\qquad E \subseteq S$: a particle (generated by an experiment, and thought of as a point $\in E$)

Suppose probability $P(E)$ is proportional to the volume.

$\mathrm{vol}(B) = \frac{4}{3}\pi$

Ball of radius r Calc. 3 has volume $\frac{4}{3}\pi r^3$

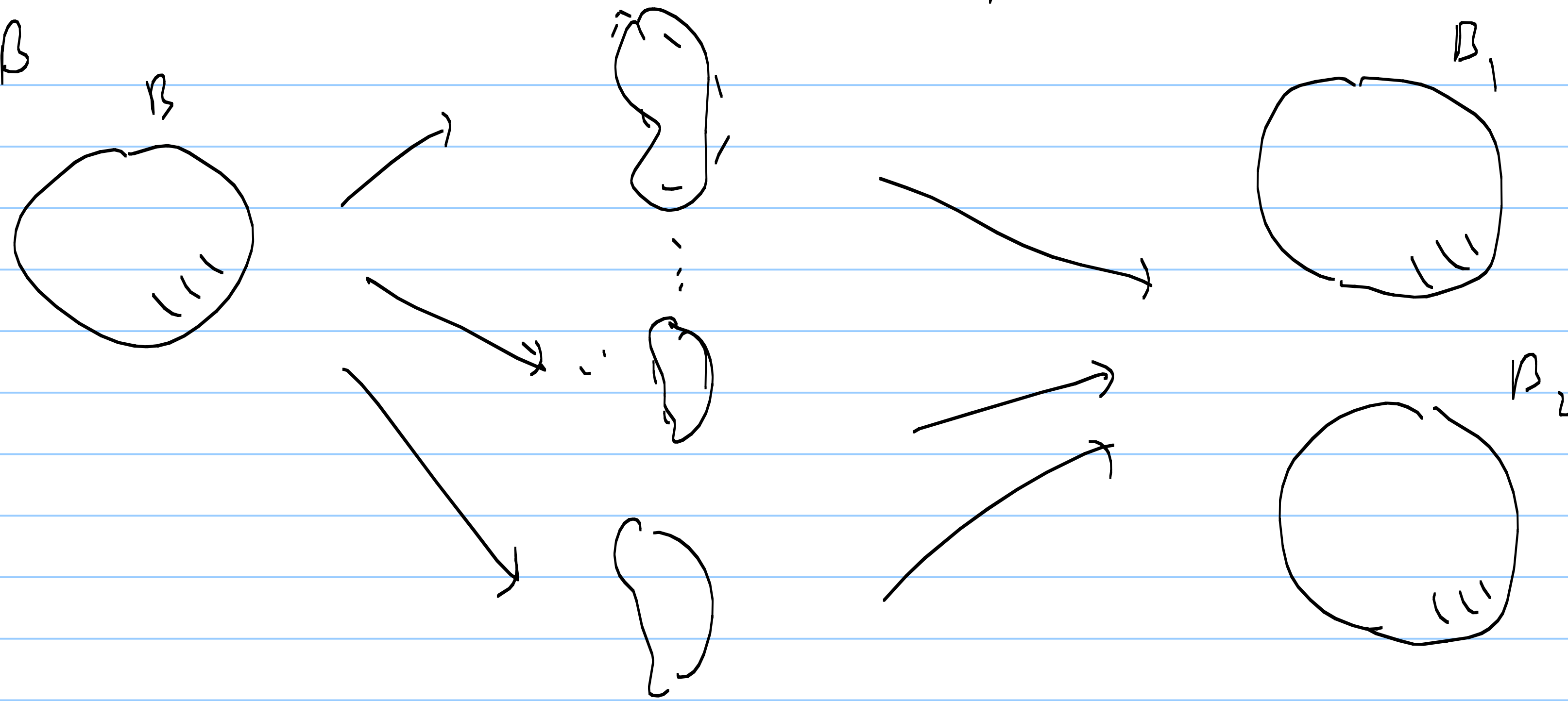
$P(E) = \frac{3}{4\pi}\,\mathrm{vol}(E)$.

There is a mathematical construction decomposing B into finitely many disjoint sets

$E_1, \dots, E_n$

$E_1 \cup \dots \cup E_n = B \qquad E_i \cap E_j = \emptyset \quad i \neq j$

But after rotating or shifting, these sets can be reassembled into two exact copies of the ball B



The solution to this paradox is that these sets $E_1, \dots, E_n$ are not measurable.

However, in practical experiment we always expect to encounter measurable events.

Simple consequences of the axioms

① $P(\emptyset) = 0.$

Proof: $\emptyset = \emptyset \cup \cdots \cup \emptyset \cup \cdots$

$$\sum_{i=1}^{\infty} \underbrace{P(\emptyset)} = P(\emptyset)$$

must be 0, otherwise the sum would be ∞.

② For a finite number of disjoint sets $E_1, \dots, E_n$, $\underbrace{\text{disjoint}}_{E_i \cap E_j = \emptyset \text{ if } i \neq j}$

$$P(E_1 \cup \dots \cup E_n) = P(E_1) + \dots + P(E_n)$$

Proof: $E_1, \dots, E_n, \emptyset, \dots, \emptyset$

$$\sum_{i=1}^{n} P(E_i) + \underbrace{P(\emptyset) + P(\emptyset) + \dots}_{0} = P\left(\bigcup_{i=1}^{n} E_i\right)$$

$\square$

(3) $P(E^c) = 1 - P(E)$

Proof: E, E^c are disjoint, $E \cup E^c = S$.

$$P(E) + P(E^c) = P(S) = 1. \quad \square$$

(4) If $E \subseteq F$ then

$$P(E) \leq P(F).$$

Proof: $F \setminus E = F \cap E^c$

$$E \cap (F \setminus E) = \emptyset$$

$$E \cup (F \setminus E) = F$$

$$P(E) + \underbrace{P(F \setminus E)}_{\text{Axiom 1: } \geq 0} = P(F)$$

$$P(E) \leq P(F). \quad \square$$

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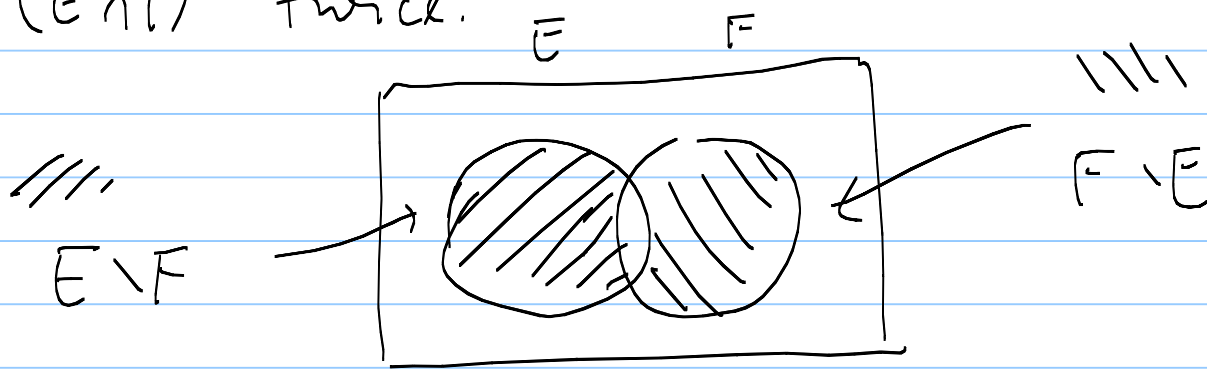
5 $$P(E \cup F) = P(E) + P(F) - P(E \cap F).$$

Proof:

$E \cup F$ ////

If we add $P(E)$ and $P(F)$, we have counted $P(E \cap F)$ twice.



$\boxed{\text{disjoint: } E \setminus F,\ E \cap F,\ F \setminus E}$

① $$P(E \setminus F) + P(E \cap F) = P(E)$$

② $$P(F \setminus E) + P(E \cap F) = P(F)$$

$(E \setminus F) \cup (E \cap F) = E$

$(F \setminus E) \cup (E \cap F) = F$

$$(E\setminus F)\cup(E\cap F)\cup(F\setminus E) = E\cup F$$

③ $P(E\setminus F)+P(E\cap F)+P(F\setminus E) = P(E\cup F)$

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①+② $P(E\setminus F)+P(F\setminus E)+2P(E\cap F) = P(E)+P(F)$

①+②−③ : $P(E\cap F) = P(E)+P(F)-P(E\cup F)$ $\square$

HW :

① A student will take ECON with probability 0.6, and will take MATH 425 with probability 0.3, and in fact will take both classes with probability 0.125. What is the probability that the student will take at least one of the two classes? What is the probability he/she will take none of them?

<u>Theoretical Exercise</u> 4 on p. 54

<u>Theoretical Exercise</u> 3 on p. 54