MATH 425 9/19/2011

Note Title 9/19/2011

The inclusion - exclusion principle

Suppose we have events $E_1, E_2, \ldots, E_n$.

$$P(E_1 \cup \ldots \cup E_n) = P(E_1) + \cdots + P(E_n) - P(E_1 \cap E_2) - P(E_1 \cap E_3) - \cdots$$
$$\cdots - P(E_{n-1} \cap E_n) + P(E_1 \cap E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_4) +$$
$$\cdots + \cdots + P(E_{n-2} \cap E_{n-1} \cap E_n) - \cdots$$
$$\cdots + (-1)^{n-1} P(E_1 \cap \cdots \cap E_n).$$

In the sum notation:

$$P(E_1 \cup \dots \cup E_n) = \sum_{i=1}^{n} P(E_i) - \sum_{1 \le i_1 < i_2 \le n} P(E_{i_1} \cap E_{i_2}) + \dots$$

$$\dots + (-1)^{r-1} \sum_{1 \le i_1 < i_2 < \dots < i_r \le n} P(E_{i_1} \cap \dots \cap E_{i_r}) + \dots$$

$$+ (-1)^{n-1} P(E_1 \cap \dots \cap E_n).$$

<u>Example</u>: $n = 3$. Ann buys three books. There is the probability 0.6 resp. 0.5 resp. 0.4 that she will book 1 resp. book 2 resp. book 3. There is the probability 0.3 that she will like Book 1 & Book 2

0.3 that she will like Book 1 & Book 3
0.2 that she will like Book 2 & Book 3
0.1 that she will like Book 1, Book 2 & Book 3.

What is the probability she will like at least one of the books?

$$P = 0.6 + 0.5 + 0.4 - 0.3 - 0.3 - 0.2 + 0.1$$
$$= 1.5 - 0.8 + 0.1 = \underline{\underline{0.8}}$$

Proof of the Inclusion and Exclusion Principle:

Look at all the samples which are contained in precisely $\underline{k}$ of the sets $E_1, \dots, E_n$. How many times do we count the probability of this event?

$$\binom{k}{1} - \binom{k}{2} + \dots + (-1)^{k-1}\binom{k}{k} \quad (*)$$

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} = 0$$

$$(x+y)^k = \sum_{i=0}^{k} \binom{k}{i} x^i y^{k-i}$$

Set $x = -1$
$y = 1$

$$0 = (-1+1)^k = \sum_{i=0}^{k} \binom{k}{i} (-1)^i$$

$$= \underline{\underline{\binom{k}{0}}} - \binom{k}{1} + \binom{k}{2} + \dots + (-1)^k \binom{k}{k}$$

$$\boxed{0! = 1} \qquad 1 - \underbrace{\binom{k}{1} + \binom{k}{2} + \cdots + (-1)^k \binom{k}{k}} = 0$$

$$1 = \binom{k}{1} - \binom{k}{2} + \cdots + (-1)^{k-1} \binom{k}{k}$$

(this is ⊛).

This proves the Inclusion-Exclusion Principle. □

The estimate version of the Inclusion-Exclusion Principle

$$P(E_1 \cup \cdots \cup E_m) \leq P(E_1) + \cdots + P(E_m)$$

$$P(E_1 \cup \cdots \cup E_n) \geq \sum_{i=1}^{n} P(E_i) - \sum_{i_1 < i_2} P(E_{i_1} \cap E_{i_2})$$

$$P(E_1 \cup \cdots \cup E_n) \leq \sum_{i=1}^{n} P(E_i) - \sum_{i_1 < i_2} P(E_{i_1} \cap E_{i_2}) +$$

$$+ \sum_{i_1 < i_2 < i_3} P(E_{i_1} \cap E_{i_2} \cap E_{i_3})$$

the inequalities alternate, as we stop counting at a given r.

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We want to investigate the sign of the

expression

$$\underbrace{\binom{k}{0} - \binom{k}{1} + \binom{k}{2} - \binom{k}{3} + \cdots + (-1)^r \binom{k}{r}}$$

≥ 0 when r is even

≤ 0 when r is odd

Claim: When $1 \leq i \leq \frac{k+1}{2}$ i is an integer then

$$\binom{k}{i-1} \leq \binom{k}{i}$$

$$\binom{k}{i} = \frac{k\cdot(k-1)\cdot\ldots\cdot(k-i+1)}{1\cdot 2\cdot\ldots\cdot i} = \underbrace{\frac{k-i+1}{i}}_{\geq 1}\binom{k}{i-1}$$

This proves the Claim.

$$\| $$

$$\frac{k!}{i!\,(k-i)!}$$

cancel out $(k-i)!$

$\square$

The Claim proves the sign alternation for $r \leq \frac{k+1}{2}$. For higher r's, use

$$\binom{k}{0} - \binom{k}{1} + \cdots + (-1)^k\binom{k}{k} = 0$$

$$\xleftarrow{\ \leq\ } \ r \ \xrightarrow{\ >r\ }$$

$$\binom{k}{0} - \binom{k}{1} + \cdots + (-1)^r \binom{k}{r} =$$

$$= \underbrace{-\binom{k}{r+1} + \binom{k}{r+2} + \cdots + (-1)^{k-1}\binom{k}{k}}$$

Remember $\binom{k}{i} = \binom{k}{k-i}$

$$(-1)^{k-1}\left(\underbrace{\binom{k}{0} - \binom{k}{1} + \cdots + (-1)^{k-r}\binom{k}{k-r-1}}\right)$$

This sign alternates with r by the Claim.

This proves the estimate version of the Inclusion Exclusion principle.

The most basic case of probability:

S = finite set of samples, } $\underline{n}$ samples
all outcomes are equally likely.

If E is an event which includes k samples, the probability will be $\frac{k}{n}$.

HW

1. Theoretical Exercise 11 on p. 55

2. Theoretical Exercise 13 on p. 55

3. Theoretical Exercise 10 on p. 55