

The inclusion-exclusion principle

Suppose we have events E_1, E_2, \dots, E_m .

$$\begin{aligned} P(E_1 \cup \dots \cup E_m) &= P(E_1) + \dots + P(E_m) - P(E_1 \cap E_2) - P(E_1 \cap E_3) - \dots \\ &\quad \dots - P(E_{m-1} \cap E_m) + P(E_1 \cap E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_4) + \\ &\quad \dots + \dots + P(E_{m-2} \cap E_{m-1} \cap E_m) - \dots \\ &\quad \dots + (-1)^{m-1} P(E_1 \cap \dots \cap E_m). \end{aligned}$$

In the sum notation:

$$P(E_1 \cup \dots \cup E_m) = \sum_{i=1}^m P(E_i) - \sum_{1 \leq i_1 < i_2 \leq m} P(E_{i_1} \cap E_{i_2}) + \dots$$

$$\dots + (-1)^{r-1} \sum_{1 \leq i_1 < i_2 < \dots < i_r \leq m} P(E_{i_1} \cap \dots \cap E_{i_r}) + \dots$$

$$+ (-1)^{m-1} P(E_1 \cap \dots \cap E_m).$$

Example : $m = 3$. Ann buys three books. There is the probability 0.6 resp. 0.5 resp. 0.4 that she will book 1 resp. book 2 resp. book 3. Then is the probability

0.3 that she will like Book 1 & Book 2

- 0.3 that she will like Book 1 & Book 3
 0.2 that she will like Book 2 & Book 3
 0.1 that she will like Book 1, Book 2 & Book 3.

What is the probability she will like at least one of the books?

$$\begin{aligned}
 P &= 0.6 + 0.5 + 0.4 - 0.3 - 0.3 - 0.2 + 0.1 \\
 &= 1.5 - 0.8 + 0.1 = \underline{\underline{0.8}}
 \end{aligned}$$

Proof of the Inclusion and Exclusion Principle :

Look at all the samples which are contained in precisely k of the sets E_1, \dots, E_n . How many

times do we count the probability of this event?

$$\binom{k}{1} - \binom{k}{2} + \dots + (-1)^{k-1} \binom{k}{k}$$



$$\sum_{i=0}^k (-1)^i \binom{k}{i} = 0$$

$$(x+y)^k = \sum_{i=0}^k \binom{k}{i} x^i y^{k-i}$$

Set $x = -1$

$y = 1$

$$0 = (-1+1)^k = \sum_{i=0}^k \binom{k}{i} (-1)^i$$

$$\binom{k}{0} - \binom{k}{1} + \binom{k}{2} - \dots + (-1)^k \binom{k}{k}$$

$$\boxed{0! = 1}$$

$$1 - \binom{k}{1} + \binom{k}{2} - \cdots + (-1)^k \binom{k}{k} = 0$$

}

$$0 = \binom{k}{1} - \binom{k}{2} + \cdots + (-1)^{k-1} \binom{k}{k}$$

↑ (this is \oplus).

This proves the Inclusion-Exclusion Principle. \square

The estimate version of the Inclusion-Exclusion Principle

$$P(E_1 \cup \cdots \cup E_m) \leq P(E_1) + \cdots + P(E_m)$$

$$P(E_1 \cup \dots \cup E_n) \geq \sum_{i=1}^n P(E_i) - \sum_{i_1 < i_2} P(E_{i_1} \cap E_{i_2})$$

$$P(E_1 \cup \dots \cup E_n) \leq \sum_{i=1}^n P(E_i) - \sum_{i_1 < i_2} P(E_{i_1} \cap E_{i_2}) +$$

the inequalities alternate,

as we stop

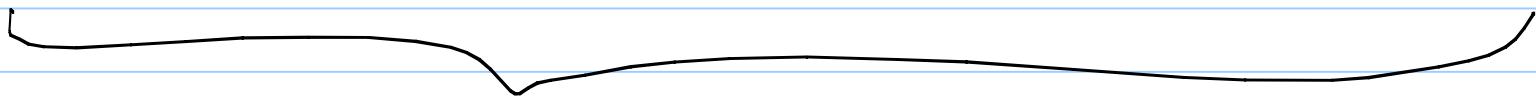
counting

at a given n .

We want to investigate the sign of the

expression

$$\binom{k}{0} - \binom{k}{1} + \binom{k}{2} - \binom{k}{3} + \cdots + (-1)^n \binom{k}{r}$$



≥ 0 when n is even

≤ 0 when n is odd

Claim: When $1 \leq i \leq \frac{k+1}{2}$ i is an integer

then

$$\binom{k}{i-1} \leq \binom{k}{i}$$

$$\binom{k}{i} = \frac{k \cdot (k-1) \cdot \dots \cdot (k-i+1)}{1 \cdot 2 \cdot \dots \cdot i} = \frac{k-i+1}{i} \binom{k}{i-1}$$

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$$\frac{k!}{i! (k-i)!}$$

cancel out $(k-i)!$

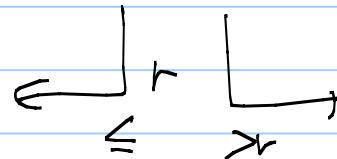
≥ 1 ↗ This proves the claim.

□

The Claim proves the sign alternation for

$r \leq \frac{k+1}{2}$. For higher r 's, we

$$\binom{k}{0} - \binom{k}{1} + \dots + (-1)^k \binom{k}{k} = 0$$



$$\binom{k}{0} - \binom{k}{1} + \cdots + (-1)^r \binom{k}{r} =$$

$$= -\binom{k}{r+1} + \binom{k}{r+2} + \cdots + (-1)^{k-1} \binom{k}{k}$$

Remember $\binom{k}{i} = \binom{k}{k-i}$

$$(-1)^{k-1} \left(\binom{k}{0} - \binom{k}{1} + \cdots + (-1)^{k-r} \binom{k}{k-r-1} \right)$$

This sign alternates with r by the claim.

This proves the estimate version of the Inclusion-Exclusion principle.

The most basic case of probability:

S = finite set of samples, } n samples
all outcomes are equally likely.

If E is an event which includes k samples,
the probability will be $\frac{k}{n}$.

(HW)

① Theoretical Exercise 11 on p. 55

② Theoretical Exercise 13 on p. 55

③ Theoretical Exercise 10 on p. 55