MATH 425 9/23/2011

Note Title 9/23/2011

Example ① 5h p.37

A deck of 52 cards is dealt to 4 players.

ⓐ What is the probability one of the players receives all 13 spades?

ⓑ What is the probability each player receives one ace?

ⓐ **Solution**: E_1, E_2, E_3, E_4

E_i means: Player i has all 13 spades.

$$P(E_i) = \frac{1}{\binom{52}{13}}$$

$E_i \cap E_j = \emptyset$ if $i \neq j$

$$P(E_1 \cup E_2 \cup E_3 \cup E_4) = \frac{4}{\binom{52}{13}} \doteq 6.3 \cdot 10^{-12}$$

(b) Solution: How many ways are there to deal 52 cards among 4 players (each receives 13)?

$$\binom{52}{13\ 13\ 13\ 13} = \frac{52!}{13! \cdot 13! \cdot 13! \cdot 13!}$$

If we deal the cards without the aces the number is

$$\binom{48}{12 \quad 12 \quad 12 \quad 12}.$$

Now deal the aces one to each player

$$\binom{4}{1 \quad 1 \quad 1 \quad 1} = 4!$$

The answer

$$\frac{4! \dfrac{48!}{(12!)^4}}{\dfrac{52!}{(13!)^4}} = \frac{4! \cdot 13^4}{52 \cdot 51 \cdot 50 \cdot 49} \doteq 0.1055$$

Example ②: 5i p.38

If n people are in a classroom, what is the probability that no two celebrate their birthday on the same day of the year?

Solution: $|S| = (365)^n$.

$|E| = 365 \cdot 364 \cdot \ldots \cdot (365-n+1)$

$$P(E) = \frac{365 \cdot 364 \cdot \ldots \cdot (365-n+1)}{(365)^n}$$

Turns out, this is $< \frac{1}{2}$ when $n \geq 23$.

If $n = 50$, the probability that two people have their birthday on the same day of the year is ≈ 0.97

$n = 100$ $\qquad$ $3{,}000{,}000 : 1$

Example 3 : 5j on p. 38

Shuffle 52 cards and turn the cards face up one at a time until the first ace appears. Is it more likely the next card will be the ace of spades or the two of clubs?

<u>Solution</u>: $|S| = 52!$

$|E_{A\spadesuit}| = 51!$

Take the A$\spadesuit$ out. $\underbrace{\text{Shuffle the remaining cards}}_{51!\ \text{possibilities}}$

Now insert the A$\spadesuit$ randomly.
52 choices

There is precisely one choice in $E_{A\spadesuit}$: when A$\spadesuit$ is inserted after the first ace.

$$P(E_{A\spadesuit}) = \frac{1}{52} = P(E_{2\clubsuit}). \quad \square$$

Example 4: 5l p. 40 In a club,

36 members play tennis

28 play squash

18 badminton

22 tennis & squash

12 tennis & badminton

9 squash & badminton

4 play all three.

How many people play at least one of these sports?

Answer: $36 + 28 + 18 - 22 - 12 - 9 + 4 = 43$.

Example ③: N men at a party each lose their hat. At the end, each randomly selects a hat. What are the chances nobody gets their own hat?

Solution: $E_i = \{$ the ith person did get their own hat back $\}$

$$\left.\begin{array}{l} |S| = N! \\ |E_i| = (N-1)! \end{array}\right\} \quad P(E_i) = \frac{(N-1)!}{N!} \quad \Big\} \; N \quad i\text{'s}$$

$$\underbrace{i_1 < \cdots < i_k} \qquad |E_{i_1} \cap \cdots \cap E_{i_k}| = (N-k)!$$

$$\binom{N}{k} \qquad P(E_{i_1} \cap \cdots \cap E_{i_k}) = \left.\frac{(N-k)!}{N!}\right\} \binom{N}{k}$$

$$P(E_1 \cup \cdots \cup E_N) = N \cdot \frac{(N-1)!}{N!} - \binom{N}{2}\frac{(N-2)!}{N!} + \cdots$$

$$\cdots + (-1)^{k-1} \underbrace{\binom{N}{k}\frac{(N-k)!}{N!}} + \cdots + (-1)^{N-1}\binom{N}{N}\frac{0!}{N!}$$

$$\binom{N}{k}\frac{(N-k)!}{N!} = \frac{\cancel{N!}}{k!\,\cancel{(N-k)!}} \frac{\cancel{(N-k)!}}{\cancel{N!}} = \frac{1}{k!}$$

$$P(E_1 \cup \dots \cup E_N) = 1 - \frac{1}{2!} + \frac{1}{3!} - \dots + (-1)^{N-1}\frac{1}{N!}$$

$\underline{\text{Answer}}$: $1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^N \frac{1}{N!}$

$$\lim_{N\to\infty}\left(\quad \uparrow \quad \right) = e^{-1} \approx 0.37.$$

$$\boxed{\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^X}$$

$$e^X \cdot e^y = \left(\sum_{k=0}^{\infty} \frac{x^k}{k!}\right)\left(\sum_{l=0}^{\infty} \frac{y^l}{l!}\right) =$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{k!} \cdot \frac{1}{(n-k)!} x^k y^{n-k} =$$

$(n = k + \ell)$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}}_{(x+y)^n} = \sum_{n=0}^{\infty} \frac{1}{n!} (x+y)^n$$

$$= e^{x+y}$$

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