MATH 425 10/10/2011

Note Title 10/10/2011

Discrete random variables

Set S = sample space, measurable events (subsets)

Probability function.

$X : S \to \mathbb{R}$ } real function

which is measurable

For a real number x,

$\{ s \in S \mid X(s) \le x \}$ is a measurable event

$$\boxed{P\{X \le x\} = \text{the distribution function}}$$

A discrete random variable is one which can take on only finitely or countably many different values.

> The important point about countability: countably many (non-negative) numbers can be <u>added</u>.

Suppose X is a discrete random variable which can take on values $v_1, v_2, v_3, \ldots$

$P(X = r_n)$ is called the <u>probability mass function</u>.

The distribution function is given by

$$\boxed{P(X \le x) = \sum_{n : r_n \le x} P(X = r_n)}$$

<u>Example</u>: The probability mass function of a discrete random variable X is given by

In p. 124

$$p(i) = e^{-\lambda} \frac{\lambda^i}{i!} \qquad i = 0, 1, 2, \ldots$$

(Poisson distribution). Find $P(X=0)$ and $\underline{\underline{P(X>2)}}$.

$\left(\sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = e^{\lambda}, \text{ therefore } \sum_{i=0}^{\infty} p(i) = e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = e^{-\lambda} e^{\lambda} = 1. \right)$

OK.

<u>Solution:</u> $P(X=0) = e^{-\lambda}$.

$$P(X>2) = 1 - P(X \leq 2) =$$

$$= 1 - \left(e^{-\lambda} + e^{-\lambda} \frac{\lambda}{1!} + e^{-\lambda} \frac{\lambda^2}{2!} \right) =$$

$$= 1 - e^{-\lambda} \left(1 + \lambda + \frac{\lambda^2}{2} \right).$$

Expected value (expectation), variance, moments.

Expected value = average weighed by probability mass function

$$E(X) = \sum_{x : P\{X=x\} > 0} x \cdot P\{X=x\}$$

only countably many such x's.

(In the book, $p(x) = P\{X=x\}$.)

Example: The binomial distribution

(n independent experiment with outcomes $0, 1$

1 occurs with probability p ; X = Add the outcomes)

The probability mass function

$$p(k) \left(= P\{X = k\} \right) = \binom{n}{k} p^k (1-p)^{n-k}$$

What is the expected value?

$$E(X) = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

$$k \binom{n}{k} = \cancel{k} \frac{\boxed{n} \cdot (n-1) \cdot \ldots \cdot (n-k+1)}{1 \cdot 2 \cdot \ldots \cdot (k-1) \cdot \cancel{k}} = n \binom{n-1}{k-1}$$

$$E(X) = n \sum_{k=1}^{n} \binom{n-1}{k-1} p^k (1-p)^{n-k} =$$

$$l = k-1$$

$$= n \sum_{l=0}^{n-1} \binom{n-1}{l} p^{l+1} (1-p)^{(n-1)-l} =$$

$$= np \underbrace{\left(\sum_{l=0}^{n-1} \binom{n-1}{l} p^{l} (1-p)^{(n-1)-l} \right)}_{(p + (1-p))^{n-1} = 1} = \underline{\underline{np}}$$

(p. 139 4.6.1)

The k'th <u>moment</u> of a discrete random variable X

is $E(X^k)$ $k = 0, 1, 2, \dots$

The <u>variance</u>

$$\mathrm{var}(X) = E\Big(\underbrace{(X - E(X))^2}_{X^2 - 2E(X)X + E(X)^2}\Big) =$$

$$= E\Big(X^2 - 2\underbrace{E(X)}_{\text{numbers}}X + \underbrace{E(X)^2}_{\text{numbers}}\Big) = E(X^2) - \overbrace{2E(X)E(X)}^{E(X)^2} + E(X)^2$$

$$\boxed{\mathrm{var}(X) = E(X^2) - (E(X))^2}$$

Standard deviation:

$$\sigma(X) = \sqrt{\mathrm{var}(X)}.$$

<u>Example</u>: The moments of the binomial distribution $X_{n,p}$

number of experiments

probability of "1".

Find a formula for the moment.

Solution:

$i > 0$

$$E\left(X_{n,p}^k\right) = \sum_{i=0}^{n} i^k \binom{n}{i} p^i (1-p)^{n-i} =$$

$$\boxed{i\binom{n}{i} = n\binom{n-1}{i-1}}$$

$j = i - 1$

$$= n \sum_{i=1}^{n} i^{k-1} \binom{n-1}{i-1} p^i (1-p)^{n-i} =$$

$$= np \sum_{j=0}^{n-1} (j+1)^{k-1} \binom{n-1}{j} p^j (1-p)^{(n-1)-j} = np\, E\left[\left(X_{n-1,p}+1\right)^{k-1}\right]$$

HW 1 4.15 p. 173

Quiz Wednesday

2 4.18

3 4.16