MATH 425 10/12/2011

Note Title 10/12/2011

$X_{n,p}$ = Binomial random variable,
sum of n trials outcomes 0, 1 probability
of 1 in a single trial is p.

= sum of n independent
Bernoulli variables

$$\boxed{\begin{aligned} P\{X=1\} &= p \\ P\{X=0\} &= 1-p \end{aligned}}$$

binomial with n = 1

$$\boxed{E\left(X_{n,p}^{k}\right) = np\, E\left(\left(1+X_{n-1,p}\right)^{k-1}\right)}$$

$$= np \sum_{j=0}^{k-1} \binom{k-1}{j} E\left(X_{n-1,p}^{j}\right)$$

$$E\left(X_{n,p}^{0}\right) = E(1) = 1$$

$$E\left(X_{n,p}^{1}\right) = np\, E\Big(\underbrace{\left(1+X_{n-1,p}\right)^{0}}_{1}\Big) = np$$

$$\boxed{E\left(X_{n,p}\right) = np}$$ ← expectation value of the binomial variable

Let's calculate $var(X_{n,p})$.

$$E(X^2_{n,p}) = np\Big(\underbrace{E(X^1_{n-1,p})}_{(n-1)p} + \underbrace{E(X^0_{n-1,p})}_{1}\Big) =$$

$$= np\big((n-1)p + 1\big)$$

$$var(X_{n,p}) = E(X^2_{n,p}) - \big(E(X_{n,p})\big)^2 =$$

$$= np\big(\underbrace{(n-1)p+1}_{np+1-p}\big) - n^2p^2 =$$

$= np(1-p)$

$$\boxed{\operatorname{var}(X_{n,p}) = np(1-p)}$$

$$\sigma(X_{n,p}) = \sqrt{np(1-p)}.$$

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p. 142 Computing the binomial distribution function

$$P\{X_{n,p} \le i\} = \sum_{k=0}^{i} \binom{n}{k} p^k (1-p)^{n-k}$$

<u>hard</u> to compute when $n$ gets large

The book suggests:

$$P\{X_{n,p} = k+1\} = \binom{n}{k+1} p^{k+1} (1-p)^{n-k-1}$$

$$= \frac{n-k}{k+1} \frac{p}{1-p} P\{X_{n,p} = k\}$$

faster than figuring from scratch

$$\binom{n}{k+1} = \frac{n \cdot (n-1) \cdot \ldots \cdot \boxed{(n-k)}}{1 \cdot 2 \cdot \ldots \cdot k \cdot \boxed{(k+1)}} = \frac{n-k}{k+1} \binom{n}{k}$$

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<u>How can one simplify the binomial distribution when $n$ is large?</u>

The Poisson variable is the limit of the Binomial variable as $n \to \infty$, and $pn = \lambda$ stays constant.

THE EXAMPLE: radioactive decay.

I have a sample of atoms. In a unit of time, an average number of atoms that decay is $\lambda$. What is the probability that exactly $k$ atoms decayed?

$$\text{"}\; X_\lambda = \lim_{\substack{n\to\infty \\ np=\lambda}} X_{n,p} \;\text{"}$$

$$p\{X_\lambda = k\} = \lim_{\substack{n\to\infty \\ np=\lambda}} \{X_{n,p} = k\} =$$

$$p = \frac{\lambda}{n}$$

$$= \lim_{\substack{n\to\infty \\ np=\lambda}} \binom{n}{k} p^k (1-p)^{n-k} =$$

$$(k \text{ constant})$$

$$= \lim_{n\to\infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k} = \circledast$$

$$\binom{n}{k} = \frac{n \cdot \overbrace{(n-1)}^{\sim n} \cdot \ldots \cdot \overbrace{(n-k+1)}^{\sim n}}{k!} \sim \frac{n^k}{k!}$$

$$\circledast = \frac{\lambda^k}{k!} \lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^n = \boxed{\frac{\lambda^k}{k!} e^{-\lambda}}$$

take derivatives

$$\lim_{n\to\infty} \ln\left(\left(1 - \frac{\lambda}{n}\right)^n\right) = \lim_{\substack{n\to\infty \\ n \in \mathbb{R}}} n \ln\left(1 - \frac{\lambda}{n}\right) = \lim_{n\to\infty} \frac{\ln\left(1 - \frac{\lambda}{n}\right)}{\frac{1}{n}} =$$

$$= \lim_{n\to\infty} \frac{\frac{1}{\left(1-\frac{\lambda}{n}\right)} \cdot \frac{\lambda}{n^2}}{-\frac{1}{n^2}} = -\lambda$$

If $X_\lambda$ is the Poisson variable with expectation $\lambda$, then

$$P\{X_\lambda = k\} = e^{-\lambda}\frac{\lambda^k}{k!}.$$

$\boxed{\text{HW}}$: (1) Compute the third moment of the binomial variable $X_{n,p}$.

(2) Performing $n$ tosses of fair dice (independent), what the expected value of the number of times a 6 comes up? What is the standard deviation?

3. Suppose in a sample of Cesium, on average of 1000 atoms decays per second. What is the probability at least 5 atoms will decay in a second.

**Do** use calculators!