MATH 425 10/14/2011

Note Title 10/14/2011

A little more on Poisson distribution:

Recall mass probability function:

$$P\{X = k\} = e^{-\lambda} \frac{\lambda^k}{k!}$$

where $E(X) = \lambda$. (Limit Binomial distribution $n \to \infty$ $\boxed{np} = \lambda$)

The variance of the Poisson distribution

$$\mathrm{var}(X) = E(X^2) - (E(X))^2$$

$$E(X^2) = \sum_{k=0}^{\infty} k^2 \cdot \frac{e^{-\lambda} \lambda^k}{k!} = \sum_{k=1}^{\infty} k \cdot \frac{e^{-\lambda} \lambda^k}{(k-1)!} =$$

$$= \lambda \sum_{k=1}^{\infty} k \cdot \frac{e^{-\lambda} \cdot \lambda^{k-1}}{(k-1)!} = \lambda \sum_{j=0}^{\infty} (j+1) \frac{e^{-\lambda} \cdot \lambda^{j}}{j!} =$$

put $j = k-1$

$$= \lambda \Bigg(\underbrace{\sum_{j=0}^{\infty} j \frac{e^{-\lambda} \lambda^{j}}{j!}}_{E(X) = \lambda} + \underbrace{\sum_{j=0}^{\infty} \frac{e^{-\lambda} \lambda^{j}}{j!}}_{1} \Bigg) =$$

$$= \lambda(\lambda + 1)$$

$$\mathrm{var}(X) = E(X^2) - E(X)^2 = \lambda(\lambda+1) - \lambda^2 = \lambda$$

For the Poisson variable X with expected value λ,

$$\boxed{\mathrm{var}(X) = \lambda}$$

Where does the Poisson distribution apply?

an approximation of a large number of approximately independent (= weakly dependent) events with low probabilities whose probabilities add up to λ.

X = The number of these event that have occurred is ~ Poisson with expected value λ.

<u>Example</u>: An average number of phone calls to a credit card customer service number per day is 300. What is the probability exactly 25 people will call on 10/14/2011?

<u>Solution</u>: X = Poisson with expected value 300

$$P\{X = 25\} = e^{-300} \cdot \frac{(300)^{25}}{25!} = \dots$$

<u>An important case</u>: Events occurring over a length of time.

Suppose certain event occur in time,

and (1) an event occurring in a given time interval of length h has probability $\sim \lambda h$. (2) Also suppose that for very small h, the likelihood of two events occurring in the same time interval of length h is very small (compared to h).

(3) Also suppose that for two non-overlapping time intervals I, J, the event occurring in the interval I vs. in the interval J are independent.

<u>THEN</u>: The number X of events occurring in a time interval of length t is Poisson with $E(X) = \lambda t$.

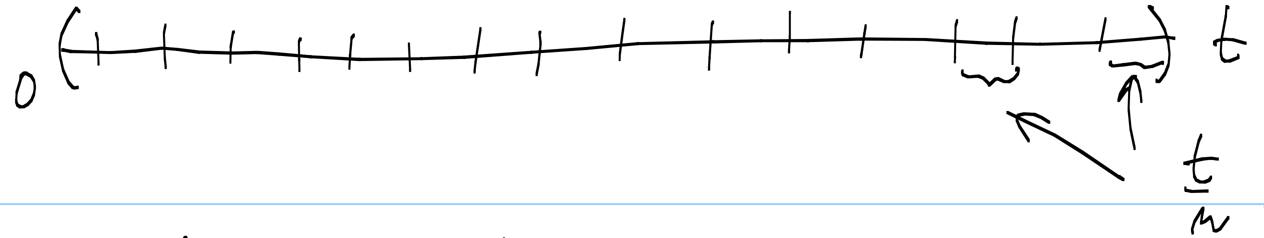

Fig. 1

Think of the events in each of the time intervals of length $\frac{t}{n}$ in Fig 1 as Bernoulli variables.
Take the limit as $n \to \infty$.

<u>Example</u>: Suppose an average of 10 lightning strikes occur in the city C. per week.
During any given year (52 weeks), what is the probability there will be no lightning strike in C?

Solution: Poisson with expected value

$$\boxed{0! = 1}$$

$$\lambda = E(X) = 10 \cdot 52 = 520$$

$$P\{X = 0\} \sim e^{-520}.$$

Poisson paradigm

OTHER DISTRIBUTIONS OF RANDOM VARIABLES

Geometric, negative binomial, hypergeometric

(exact formulas)

GEOMETRIC

Perform independent Bernoulli trials with probability p. X = number of trials needed for the first success.

$$P\{X = n\} = p(1-p)^{n-1} \qquad (P\{X=0\} = 0)$$

Recall: $1 + a + \dots + a^r = \dfrac{a^{r+1}-1}{a-1}$ ||

multiply by $a-1$, chain cancellation on the left.

$$1 + a + a^2 + \cdots = \sum_{k=0}^{\infty} a^k = \frac{1}{1-a}.$$

$|a| < 1$

$$P\{X \leq k\} = p(1-p)^0 + p(1-p)^1 + \cdots + p(1-p)^{k-1}$$

$$= \cancel{p}\, \frac{(1-p)^k - 1}{\underbrace{(1-p) - 1}_{-\cancel{p}}} = 1 - (1-p)^k.$$

$$E(X) = \sum_{k=1}^{\infty} k p (1-p)^{k-1} = p \sum_{k=1}^{\infty} k (1-p)^{k-1}$$

$$= p \frac{1}{(1-(1-p))^2} = \frac{p}{p^2} = \frac{1}{p}. \qquad \boxed{E(X) = \frac{1}{p}}$$

$$\sum_{k=1}^{\infty} \underbrace{k x^{k-1}}_{(x^k)'} = \left(\sum_{k=0}^{\infty} x^k \right)' = \left(\frac{1}{1-x} \right)' = \frac{1}{(1-x)^2}$$

$$\boxed{x = 1-p}$$

HW: 1 Suppose an average number of 2 people walk ~~into~~ a coffee shop per minute. What is the probability that at least 10 people will visit the shop in an hour? (Use the Poisson paradigm and a calculator).

(2) Suppose the success rate of getting into medical school is 20% in a year. Assuming the event of getting in in different years are independent, how many years would I expect to be applying before getting in?

(3) Theoretical problem 4.13 p. 180