MATH 425 10/19/2011

Note Title 10/19/2011

Negative binomial variable

We keep making independent Bernoulli trials with probability of success $= p$, until we get r failures. How many successes? X

Probability mass function

$$P\{X = k\} = \binom{k+r-1}{k} p^k (1-p)^r$$

$$\underbrace{? \; - - - \ldots \; ?}_{k \text{ successes } \& \ (r-1) \text{ failures}} \; F$$

\# of possibilities is $\binom{k+r-1}{k}$

$$\binom{k+r-1}{k} = \frac{\overbrace{(k+r-1)(k+r-2)\cdot\ldots\cdot r}^{k \text{ factors}}}{k!} =$$

$$= \frac{(-r)\cdot(-r-1)\cdot\ldots\cdot(-r-k+1)}{k!}(-1)^k = (-1)^k\binom{-r}{k}$$

Negative binomial probability mass function

$$P\{X=k\} = \underbrace{\binom{-r}{k}(-p)^k(1-p)^r}$$ $\circledast$

What we learned: $a \in \mathbb{R}$, we can define

$$\binom{a}{k} = \frac{a\cdot(a-1)\cdot\ldots\cdot(a-k+1)}{k!} \qquad k \in \mathbb{N}_0 \qquad \binom{a}{0} = 1$$

Newton's formula (generalizes the binomial theorem)

$$(1+x)^a = \sum_{k=0}^{\infty} \binom{a}{k} x^k \qquad \text{for } |x|<1.$$

(Note: If $\alpha = n \in \mathbb{N}$, $k > n$ then $\binom{n}{k} = 0$.)

Let us verify $\sum_{k=0}^{\infty} P\{X = k\} = 1$:

From ⊗,

$$\sum_{k=0}^{\infty} P\{X=k\} = \sum_{k=0}^{\infty} \binom{-r}{k} (-p)^k (1-p)^r =$$

$$= (1-p)^r \sum_{k=0}^{\infty} \binom{-r}{k} (-p)^k = (1-p)^r (1-p)^{-r} = 1.$$

use Newton's formula

$x = -p \qquad 1+x = 1-p$

<u>Example</u>: A baseball pitcher walks a hitter with probability $\frac{1}{10}$. A manager decides to pull him when he has walked 5 hitters. What is the probability that he will have pitched to exactly 15 hitters before he is pulled?

<u>Solution</u>: $r = 5$ $\quad p = \frac{9}{10}$, $\quad k + r = 15$

$k = 10$

$$P\{X = 10\} = \binom{-5}{10} \cdot \left(-\frac{9}{10}\right)^{10} \left(\frac{1}{10}\right)^{5} = \ldots \qquad \square$$

Calculating the expected value and the variance of the negative binomial.

Expected value of the negative binomial

$$\sum_{k=0}^{\infty} k \cdot \binom{-r}{k} (-p)^k (1-p)^r = \sum_{k=1}^{\infty} (-r) \binom{-r-1}{k-1} (-p)^k (1-p)^r =$$

let $l = k-1$

$a = -r$

$$k \cdot \binom{a}{k} = \frac{a (a-1) \cdot \ldots \cdot (a-k+1)}{1 \cdot 2 \cdot \ldots (k-1) \cdot \cancel{k}} \cancel{k} = a \binom{a-1}{k-1}$$

$$= (-r)\sum_{\ell=0}^{\infty} \binom{-r-1}{\ell} (-p)^{\ell+1} (1-p)^{r} = (-p)(1-p)^{r}(-r)\sum_{\ell=0}^{\infty} \binom{-r-1}{\ell} (-p)^{\ell}$$

$$= (-r)(-p)(1-p)^{r}(1-p)^{-r-1} = \frac{pr}{1-p}$$

Caution: reparametrization in the book!

Second moment:

$$E(X^2) = \sum_{k=0}^{\infty} k^2 \binom{-r}{k} (-p)^{k} (1-p)^{r} =$$

$$=(-r)\sum_{k=1}^{\infty} k\binom{-r-1}{k-1}(-p)^{k}(1-p)^{r} = pr(1-p)^{r}\sum_{l=0}^{\infty}(l+1)\binom{-r-1}{l}(-p)^{l}$$

$$l=k-1$$

$$= pr(1-p)^{r}\left(\underbrace{\sum_{l=0}^{\infty} l\binom{-r-1}{l}(-p)^{l}}_{(r+1)p(1-p)^{-r-2}} + \underbrace{\sum_{l=0}^{\infty}\binom{-r-1}{l}(-p)^{l}}_{(1-p)^{-r-1}}\right)$$

$$\boxed{(1+x)^{a} = \sum_{k=0}^{\infty}\binom{a}{k}x^{k} \quad \Big| \quad a(1+x)^{a-1} = \sum_{k=0}^{\infty}k\binom{a}{k}x^{k-1}}$$

$$= pr\left((r+1)p(1-p)^{-2} + (1-p)^{-1}\right)$$

$$\mathrm{var}(X) = E(X^2) - E(X)^2 =$$

$$= pr\left(\underbrace{(r+1)p(1-p)^{-2}}_{rp(1-p)^{-2} + p(1-p)^{-2}} + (1-p)^{-1}\right) - p^2r^2(1-p)^{-2} =$$

$$= pr\left(p(1-p)^{-2} + \underbrace{(1-p)^{-1}}_{(1-p)(1-p)^{-2}}\right) = \frac{pr}{(1-p)^2}$$

HW

1. Reconcile my notation of the book and decide if I got the formulas for mean and variance right.

2. 4.75 p. 178

3. 4.28 p. 182