MATH 425 10/26/2011

Note Title 10/26/2011

p. 192

Example: A stick of length 1 is split at a point U that is uniformly distributed in $(0,1)$. Determine the expected length of the piece that contains the point p, $0 \le p \le 1$.

Solution: Let $L_p(U)$ be the length of the piece which contains p.

$$L_p(U) = \begin{cases} U & U > p \\ 1-U & U < p \end{cases}$$

$$E(L_p(U)) = \int_0^1 L_p(u)\,du = \int_0^p (1-u)\,du + \int_p^1 u\,du =$$

$$= -\frac{1}{2}(1-u)^2\Big|_0^p + \frac{1}{2}u^2\Big|_p^1 =$$

$$= \frac{1}{2} - \underbrace{\frac{1}{2}(1-p)^2}_{-\frac{1}{2}(1+p^2-2p)} + \frac{1}{2} - \frac{1}{2}p^2 = \frac{1}{2} - p^2 + p \ . \ \square$$

NORMAL DISTRIBUTION

= GAUSSIAN DISTRIBUTION

GAUSS   Gaussian integral

$$\boxed{\int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi}}$$

$$\left(\int_{-\infty}^{\infty} e^{-x^2/2}\, dx\right)^2 = \int_{-\infty}^{\infty} e^{-x^2/2}\, dx \int_{-\infty}^{\infty} e^{-y^2/2}\, dy =$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\, dx\, dy = \int_{r=0}^{\infty}\int_{\theta=0}^{2\pi} r\, e^{-r^2/2}\, dr\, d\theta = 2\pi \int_0^{\infty} r e^{-r^2/2}\, dr$$

Polar coordinates : r , θ

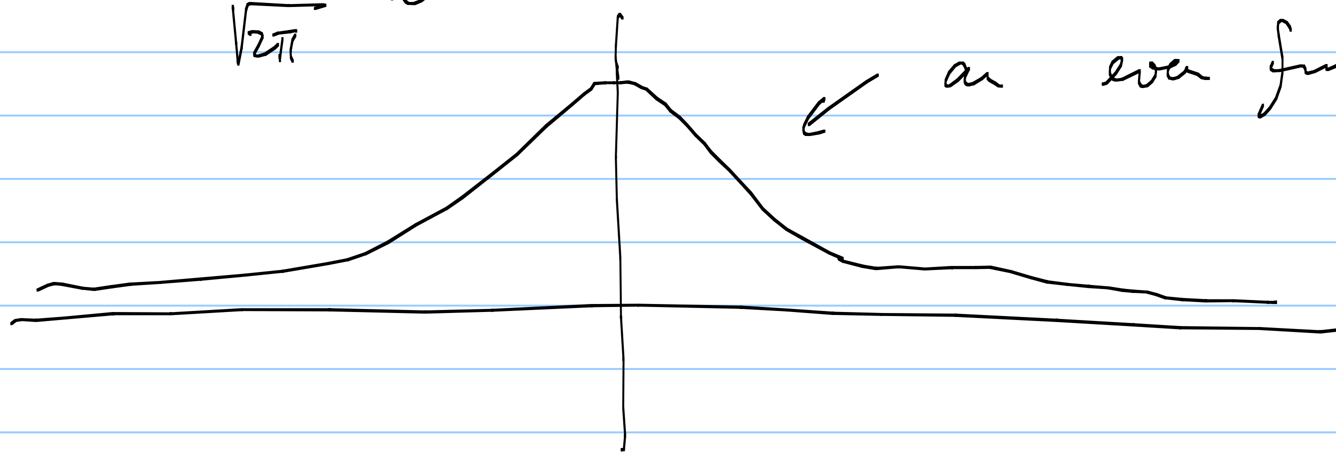
Jacobian : r (Stewart)

$$= 2\pi \left(-e^{-r^2/2}\right)\Big|_0^{\infty}$$

$$= 2\pi$$

The standard Gaussian variable X has probability density

$$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$



$\swarrow$ an even function

$$E(X) = 0$$

$$E(X^2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2/2}\, dx =$$

$u' = x e^{-x^2/2}$ $\quad v = x$

$u = -e^{-x^2/2}$ $\quad v' = 1$

$$= \underbrace{-\frac{1}{\sqrt{2\pi}} x e^{-x^2/2} \Big|_0^{\infty}}_{0-0} + \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\, dx}_{\sqrt{2\pi}} = 1$$

$$\boxed{E(X) = 0 \qquad \text{var}(X) = 1}$$ ← standard Gaussian variable.

You can get however <u>any</u> expected value and <u>any</u> variance (>0) by substituting for x.

<u>Example</u>: $E(X) = b$, $\text{var}(X) = 1$

Probability density $= \dfrac{1}{\sqrt{2\pi}} e^{-(x-b)^2/2}$

$$\int_{-\infty}^{\infty} e^{-(ax)^2/2}\, dx =$$

$u = ax$ $\quad a > 0$

$du = a\,dx$

$$= \frac{1}{a}\int_{-\infty}^{\infty} e^{-u^2/2}\, du = \frac{\sqrt{2\pi}}{a}$$

We have a Gaussian variable Y with $E(Y) = 0$ and probability density

$$\frac{a}{\sqrt{2\pi}}\, e^{-(ax)^2/2}.$$

We will compute its variance.

HW

1 5.17 p.225

2 5.12 p.225