MATH 425 10/28/2011

Note Title 10/28/2011

Gaussian variables

Gauss integral : $\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}$.

Standard Gaussian variable X has probability density function

$$\frac{1}{\sqrt{2\pi}} e^{-x^2/2} \qquad E(X)=0 \qquad \mathrm{var}(X)=1$$

Gaussian variable with expected value b and variance 1 has probability density function

$$\frac{1}{\sqrt{2\pi}} e^{-(x-b)^2/2}$$

Changing the variance:

$u = ax \qquad du = a\,dx$

$$\int_{-\infty}^{\infty} e^{-a^2x^2/2}\,dx = \frac{1}{a}\int_{-\infty}^{\infty} e^{-u^2/2}\,du = \frac{\sqrt{2\pi}}{a}$$

We can make a random variable X with probability density function

$$\frac{a}{\sqrt{2\pi}} e^{-a^2x^2/2}$$

$$\mathrm{var}(X) = E(X^2) = \frac{a}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-a^2x^2/2} dx =$$

$u = ax$

$du = a\,dx$

$$= \frac{1}{(\sqrt{2\pi}) \cdot a^2} \underbrace{\int_{-\infty}^{\infty} u^2 e^{-u^2/2} du}_{\sqrt{2\pi}} = \frac{1}{a^2}$$

If I want a Gaussian variable X with

$$E(X) = b \qquad \mathrm{var}(X) = a^2 \qquad (\sigma(X) = a)$$

Probability density function:

$$\boxed{\frac{1}{a\sqrt{2\pi}} e^{-(x-b)^2/2a^2}}$$

← the general Gaussian (normal) <u>variable</u>

Looking ahead, suppose we have two random variables X, Y on the same sample space S

$$X : S \longrightarrow \mathbb{R}$$
$$Y : S \longrightarrow \mathbb{R}$$

(jointly distributed). We say that X, Y are independent random variables if for any measurable subsets $B \subseteq \mathbb{R}$, $C \subseteq \mathbb{R}$

$$P\{X \in B, Y \in C\} = P\{X \in B\} \cdot P\{Y \in C\}.$$

This is the same thing as: $X \in B$, $Y \in C$ are independent events.

<u>Example</u>: Throwing two dice, X = number on die 1
Y = number on die 2.

<u>Theorem</u>: If X, Y are independent random variables on a sample space S then

$$E(XY) = E(X)E(Y).$$

For any two random variables X, Y on a sample space S,

$$\boxed{E(X+Y) = E(X) + E(Y).}$$

If X, Y are independent

$$\text{subtract}\begin{cases} E((X+Y)^2) = E(X^2+2XY+Y^2) = E(X^2) + E(2XY) + E(Y^2) \\ \qquad = \underline{E(X^2) + 2E(X)E(Y) + E(Y^2)} \\ (E(X+Y))^2 = \underline{(E(X))^2 + 2E(X)E(Y) + (E(Y))^2} \end{cases}$$

$$\boxed{\mathrm{var}(X+Y) = \mathrm{var}(X) + \mathrm{var}(Y)}$$ $\Leftarrow$ provided X, Y are independent.

<u>Theorem</u>: The sum of two <u>independent</u> Gaussian variables X, Y on a sample space S is Gaussian.

<u>to be proved</u>

Example: The binomial distribution probability p and n trials is the sum of n independent random variables ($\in$ Bernoulli with probability p).

The claim is, for large n, the binomial variable is approximately a Gaussian variable.

independent Bernoulli

$$\underbrace{\underbrace{X_1 + \cdots + X_n}_{\sim \text{Gaussian}} + \underbrace{X_{n+1} + \cdots + X_{n+m}}_{\sim \text{Gaussian}}}_{\sim \text{Gaussian}}$$

Proof that a sum of two independent Gaussians X, Y is a Gaussian.

probability density function for X:

$$K e^{-a^2(x-b)^2/2}$$

constant

for Y:

$$L e^{-c^2(y-d)^2/2}$$

If the probability density function for X is $f(x)$
of Y is $g(y)$
and they are independent then the density

function for $X+Y$ is

$$h(z) = \int_{-\infty}^{\infty} f(x)\, g(z-x)\, dx$$

For Gaussians X, Y

$$\text{Constant} \int_{-\infty}^{\infty} e^{\underbrace{-\left(a^2(x-b)^2/2 \;+\; c^2(z-x-d)^2/2\right)}} \, dx \quad = h(z)$$

Idea: bring the exponent to perfect square by adding a linear function of z squared. $\leftarrow$ no x allowed

$$\underbrace{(\alpha z+\beta)^2 + \text{constant}}_{\text{a general quadratic in } z}$$

$$a^2(x-b)^2 + c^2(z-x-d)^2 = C\,x^2 + (\alpha z+\beta)\,x + \underbrace{\gamma z^2+\delta z+\varepsilon}$$

$$= C\left(x^2 + \frac{\alpha z+\beta}{C}\,x + \frac{\gamma z^2+\delta z+\varepsilon}{C}\right)$$

$$= C\left(\underbrace{\left(x + \frac{\alpha z+\beta}{2C}\right)^2}_{\int \cdots dx \quad \text{Gaussian integral}} + \underbrace{\text{quadratic in } z}_{\text{drops out if it's a Gaussian distribution!}}\right)$$

$\square$

How to normalise the binomial to get a Gaussian?

Take a limit $n \to \infty$, perform linear transformations to keep a fixed $\underbrace{\text{expected value}}_{b}$ and $\underbrace{\text{variance}}_{a}$.

$\left.\begin{array}{l}\text{For a binomial with probability } p \text{ and} \\ n \text{ (independent Bernoulli) trials}\end{array}\right\} X_{n,p}$

$E = np$

$\text{var} = np(1-p)$ NEXT TIME

HW
1. 5.21 p. 225 (table on p. 201?)
2. 5.24 p. 226