MATH 425 10/31/2011

Note Title 10/31/2011

The Gaussian limit of the binomial distribution $X_{n,p}$: n independent Bernoulli trials, success probability $= p$.

$$E(X_{n,p}) = np \longrightarrow b$$

$$\mathrm{var}(X_{n,p}) = np(1-p) \longrightarrow a^2$$

$$\lim_{n\to\infty} a\,\frac{X_{n,p} - np}{\sqrt{np(1-p)}} + b$$

} A Gaussian variable with expected value b and variance a^2

If $b = 0$, $a = 1$

$$\left.\lim_{n\to\infty} \frac{X_{n,p} - np}{\sqrt{np(1-p)}}\right\}$$ the standard Gaussian variable.

<u>Cummulative distribution</u>

$$F(x) = \int_{-\infty}^{x} \underbrace{f(t)}\, dt$$

probability density function

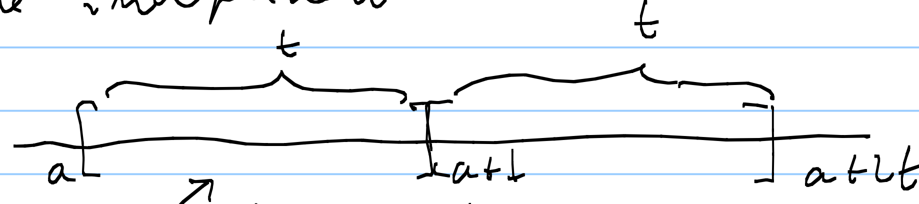
For the standard Gaussian variable:

$$F(x) = \int_{-\infty}^{x} e^{-t^2/2}\,dt$$ } there is no formula for this integral.

can figure numerically, use tables

The exponential distribution

Events happen at a constant average rate in time, and events happening at disjoint time intervals are independent.



$P\{$event does not happen in time interval $[a, a+t]\} = p$

$P\{$event does not happen in time interval $[a+t, a+2t]\} = p$

$P\{$event does not happen in time interval $[a, a+2t]\} = p^2$

$P\{$event does not happen in time interval of length $t\} = e^{-\lambda t}$ $\quad \lambda > 0.$

↑

must depend exponentially on t

$P\{$event does happen in time interval $[0, t]\} = 1 - e^{-\lambda t}$

$\parallel$

$$\int_0^t f(x)\, dx \qquad f(x) = \left(1 - e^{-\lambda t}\right)' = \lambda e^{-\lambda t}$$

The function

$$\boxed{f(t) = \begin{cases} 0 & \text{for } t < 0 \\ \lambda e^{-\lambda t} & \text{for } t \geq 0 \end{cases}}$$

is the probability density of the exponential distribution.

Computing the expected value

$$E(X) = \int_0^\infty t \lambda e^{-\lambda t} \, dt =$$

$u = t \qquad v' = \lambda e^{-\lambda t}$

$u' = 1 \qquad v = -e^{-\lambda t}$

$$= -te^{-\lambda t}\Big|_0^\infty + \int_0^\infty e^{-\lambda t}\,dt = -\frac{1}{\lambda}e^{-\lambda t}\Big|_0^\infty = \frac{1}{\lambda}.$$

Example: X = the number of years it takes for the next earthquake to occur. Suppose on average, the time interval between earthquakes at a certain location is 50 years. From a given time, what is the probability an earthquake at the location occurs within a 100 years?

Solution: $\frac{1}{\lambda} = 50.$ We have an exponential variable with parameter $\lambda = \frac{1}{50}$.

$$F(100) = 1 - e^{-100/50} = 1 - e^{-2}.$$

The variance of the exponential distribution

$$E(X^2) = \int_0^\infty x^2 \lambda e^{-\lambda x}\, dx = \qquad u = x^2 \quad v' = \lambda e^{-\lambda x} \qquad u' = 2x \quad v = -e^{-\lambda x}$$

$$= \underset{\text{from } uv}{0} + 2\int_0^\infty x\, e^{-\lambda x}\, dx \qquad u = x \quad v' = e^{-\lambda x} \qquad u' = 1 \quad v = -\frac{1}{\lambda} e^{-\lambda x}$$

from uv

$$= 0 + \frac{2}{\lambda}\int_0^\infty e^{-\lambda x}\,dx = \frac{2}{\lambda^2}$$

$$\mathrm{var}(X) = E(X^2) - E(X)^2 = \frac{1}{\lambda^2}.$$

The Hazard rate function for a random variable with probability density $f(x)$.

Cummulative distribution

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

"The probability that an item which has survived up to time t will survive for another time interval of dt" divided by dt

$$\lambda(t) = \frac{f(t)}{1 - F(t)}$$

A conditional probability (rate)

(We think of X = the time it will take for the item to fail.)

What happens for the exponential distribution:

$$\lambda(t) = \frac{\lambda e^{-\lambda t}}{1-(1-e^{-\lambda t})} = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda$$

Another way of characterizing the exponential distribution is by saying that the hazard rate is constant:

Recall the logarithmic derivative

$$\left(\ln G(x)\right)' = \frac{G'(x)}{G(x)}$$

We can apply this to $G(x) = 1 - F(x)$

$$-\lambda(x) = \left(\ln(1 - F(x))\right)' \qquad (*)$$

(Suppose $F(0) = 0$). Then

$$\ln(1 - F(x)) = -\int_0^x \lambda(t)\, dt$$

$$1 - F(x) = e^{-\int_0^x \lambda(t)\, dt}$$

$$\boxed{F(x) = 1 - e^{-\int_0^x \lambda(t)\, dt}}$$

(Calculating the cummulative distribution from the hazard rate function.)

Constant hazard rate $\Rightarrow$ exponential distribution.

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QUIZ WEDNESDAY

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(exponential vs. Poisson)