MATH 425 11/7/2011

Note Title 11/7/2011

REVIEW – General properties of a (absolutely) continuously distributed random variable.

Cummulative distribution

$$F(a) = \int_{-\infty}^{a} f(x)\,dx$$

$f(x)$ = probability density function $f(x) \geq 0 \quad \forall x \in \mathbb{R}$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

Focus on discrete random variables

- practice regular problems from the book.

The discrete random variable distributions

- probability mass function $f(x_i)$ $i = 1, 2, 3, \ldots$

$x_i \in \mathbb{R}$ all different

Cummulative distribution

$$F(a) = \sum_{i:\, x_i \leq a} f(x_i).$$

$$\boxed{\sum_{i=1}^{\infty} f(x_i) = 1}$$

① Binomial variable n independent Bernoulli trials with success probability p

X = How many successes? $\in \{0, \dots, n\}$

Probability mass function:

$$f(k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

$$\boxed{\text{A trick:} \quad k\binom{n}{k} = n\binom{n-1}{k-1}}$$

If we denote $X = X_{n,p}$

$$E(X_{n,p}^k) = np\, E\left((X_{n-1,p}+1)^{k-1}\right)$$

$$\boxed{E(X_{n,p}) = np \qquad \mathrm{var}(X_{n,p}) = np(1-p).}$$

2) <u>The Poisson distribution</u> $\lambda > 0$

$$X = \lim_{\substack{n\to\infty \\ np=\lambda}} X_{n,p} = \lim_{n\to\infty} X_{n,\lambda/n}.$$

$$\in \{0,1,2,3,\ldots\}$$

The probability mass function:

$$f(k) = e^{-\lambda}\frac{\lambda^k}{k!}$$

$$E(X) = \lambda \qquad \operatorname{var}(X) = \lambda$$

When does the Poisson variable apply?

– events over short periods of time are proportional in likeli-hood to the length of time

- over short periods of time, two events are extremely unlikely to occur.

- the number of misprints on a page
- the number of wrong telephone numbers dialed per day
- the number of customers entering a business in a unit of time

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③ <u>Negative binomial variables</u>

Perform independent Bernoulli trials success probability p, stop when we got r failures.

(book: add r to the variable $N = r+k$, $q = 1-p$ reverse success/failure)

Probability mass function

the standard way

$$f(k) = \binom{r+k-1}{k} p^k (1-p)^r$$

$$= \binom{-r}{k} (-p)^k (1-p)^r$$

A similar trick to computing higher moments

the book way

$$g(N) = \binom{N-1}{r-1} (1-q)^{N-r} q^r$$

$\boxed{\begin{array}{c}\text{the book's } p \\ = q \\ \text{the book's } n \\ = N\end{array}}$

$$E(X) = \frac{r}{1-p}$$

$$\text{var}(X) = \frac{rp}{(1-p)^2}$$

$$E(X) = \frac{r}{q}$$

$$\text{var}(X) = \frac{r(1-q)}{q^2}$$

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(4) Hypergeometric distribution Think about the example of binomial where you pick n balls out of N where m out of N are white. (binomial if you return the balls)

(hypergeometric if you don't return the balls)

X = number of white balls picked

$$P\{X=i\} = \frac{\binom{m}{i}\binom{N-m}{n-i}}{\binom{N}{n}} = \frac{\binom{n}{i}\binom{N-n}{m-i}}{\binom{N}{m}}$$

another notation for probability mass function

symmetry between "white ball" "chosen ball"

two subsets of size m, n respectively.

$$E(X) = E(X_{\text{binomial}}) = np.$$

(5) <u>Geometric</u> = Negative binomial with $r=1$.
in the book's notation

$$P\{X=n\} = (1-p)^{n-1}\, p$$

$$E(X) = \frac{1}{p} \qquad \mathrm{var}(X) = \frac{1-p}{p^2}$$

$$P\{X \geq n\} = (1-p)^{n-1}$$

$$P\{X \leq n\} = 1 - (1-p)^n.$$

This p is the book's p = the q in the above notation for negative binomial

Requested solutions of two HW problems:

$\boxed{4.85 \text{ (p. 179)}}$

This problem tested the additivity of expected value. Let

$$X_i = \left.\begin{cases} 0 & \text{if we didn't collect coupon } i \\ 1 & \text{if we did collect coupon } i \end{cases}\right\} \text{ while collecting the } n \text{ coupons}$$

Then $E(X_i) = 1 - (1-p_i)^n$.

The number of different kinds of coupons collected is

$$X = X_1 + \cdots + X_k,$$

so the answer is

$$E(X) = \sum_{i=1}^{k} E(X_i) = k - \sum_{i=1}^{k} (1-p_i)^n.$$

(I confess I only saw that after finding a much more complicated solution using generating functions.)

Theoretical Exercise 4.28 – the analytic method.

Use induction on n. The case $n < r$ follows from Newton's formula.

To get from $n-1$ to n, we must prove

$$-\binom{n-1}{r-1} p^r (1-p)^{n-r} = \sum_{i=0}^{r-1} \binom{n}{i} p^i (1-p)^{n-i}$$

(+)
$$- \sum_{i=0}^{r-1} \binom{n-1}{i} p^i (1-p)^{n-i-1}.$$

The second term on the right hand side (the one with the minus sign) is

$$\sum_{i=0}^{r-1} \binom{n-1}{i} p^i (1-p)^{n-i} + p \sum_{i=0}^{r-1} \binom{n-1}{i} p^i (1-p)^{n-i-1} =$$

$$= \sum_{i=0}^{r-1} \binom{n-1}{i} p^i (1-p)^{n-i} + \sum_{j=1}^{r} \binom{n-1}{j-1} p^j (1-p)^{n-j} =$$

$$= \sum_{i=0}^{r-1} \binom{n}{i} p^i (1-p)^{n-i} + \binom{n-1}{r-1} p^r (1-p)^{n-r}$$

by $\binom{n}{i} = \binom{n-1}{i} + \binom{n-1}{i-1}$.